

Algebra

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ALGEBRA.

CHAPTER I.

DEFINITIONS AND SIGNS.

1. **Definition.** *Algebra* is the science given to the study of numbers by means of symbols used to represent them.

2. **Symbols.** A *symbol* is generally a mark, and, as such, serves as a substitute for certain purposes for a chosen thing or class of things. The symbols used in Algebra as such substitutes are the letters of the English and Greek alphabets, $a, b, c, \dots, m, n, \dots, x, y, z$; A, B, C , &c.; α (alpha), β (beta), γ (gamma), &c. Sometimes letters with accents and suffixes are used with a view to secure similarity of operations, *e.g.*, $a', b'', c''' \dots$ (read a dash, b two-dash, c three-dash,.....), a_1, b_2, c_3, \dots (read a one, b two, c three,.....).

N.B. The student should be cautioned against assuming any relation between a, a' and a_1 ; these must be considered as good as different letters or symbols, and will generally stand for different numbers.

3. **Quantity.** A *quantity* is usually a magnitude, that is, anything capable of division into parts; *e.g.*, the height of a tree, forty rupees, seven animals, &c. The *measure* of a quantity is obtained by determining the multiple, part or parts that the given quantity is of some standard quantity of its kind, which is called the *unit*. For instance, when a rupee is taken as the unit of money, the measure of the value of a ten-rupee note is 10; so when a foot is taken as the unit of length, the measure of the length of a pencil six inches long is $\frac{6}{12}$ or $\frac{1}{2}$.

N.B. In Algebra the word *quantity* is often restricted to mean only the number expressing its measure.

4. **Signs of Operation.** The student is supposed to be already acquainted with the signs $+$, $-$, \div , \times . These signs are called *signs of operation* or *operators*, and the numbers or quantities before which they stand are called *subjects* or *operands*.

They are also called *symbols of operation*, as distinguished from the *symbols of representation* referred to in ART. 2.

It should be noted that a dot often serves for \times . Thus $a.b$ means $a \times b$, 2.3 means 2×3 . In Algebra, when the dot is omitted, and simply ab is used as the equivalent of $a \times b$ or $a.b$. Thus $2abc$ stands for $2 \times a \times b \times c$.

There is a difference is apparent between the conventions in Algebra and Arithmetic. While xy means 'x into y,' 58 does not mean '5 into 8,' but '5 tens and 8.'

5. Quantities connected by + and -. $a+b+c$ means that b is to be added to a , and then to the sum obtained c is to be added; $a-b+c$ means that b is to be subtracted from a , and then c is to be added to the remainder. In short, the order of operations is from left to right. We shall return to this point in the chapter on Addition.

The student is supposed to know from Arithmetic that $4-3+7=4+7-3$, that is, the order of operations is indifferent, so far as the final result is concerned.

Ex. Find the numerical value of $a+b-x+\frac{y}{z}$.

when $a=6, b=7, x=9, y=10, z=4$.

Required value $= 6+7-9+\frac{10}{4} = 4+\frac{5}{2} = 4+2\frac{1}{2} = 6\frac{1}{2}$. *Ans.*

EXAMPLES 1.

When $a=2, b=7, c=12, d=1, m=3, f=11, h=21, x=7, y=7$, find the value of :

- $a+b+c$.
- $c-a+d-10$.
- $h+f-c+\frac{1}{2}12-b-d$.
- $2a+3f-h-\frac{1}{2}c$.
- $3a+6x-4y+\frac{1}{3}h$.
- $2b-7a+3f-x-3y$.
- $\frac{1}{2}b+\frac{1}{3}c+\frac{1}{4}d+\frac{1}{5}f$.
- $\frac{1}{2}b-\frac{1}{3}c+\frac{1}{4}d-\frac{1}{5}a+\frac{1}{6}f$.
- $72-\frac{60}{c}-h-b-c$.
- $69+\frac{38}{d}+\frac{e}{m}-\frac{c}{h}-\frac{b}{7}+\frac{7}{h}$.
- $\frac{4}{a}+\frac{y}{b}-\frac{5}{x}$.
- $1-m+\frac{x}{20}+31-\frac{3}{y}-\frac{7}{a}$.

6. Quantities connected by \times and \div (or their equivalents). $a \times b \times c$, or $a.b.c$, or abc means that a is to be multiplied by b , and the result by c .

Again, $b \div c \times d \div e$, means that b is to be divided by c , the quotient to be multiplied by d , and then the product thus obtained to be divided by e . Hence in these cases also, the order of operations is from left to right. It will be proved later on that, so far as

the final result is concerned, the order of operations is indifferent. For instance,

$$8 \times 7 \div 4 = 56 \div 4 = 14; \text{ and } 3 \div 4 \times 7 = 21 \div 4 = 5.25;$$

$$\therefore 8 \times 7 \div 4 \neq 8 \div 4 \times 7.$$

N. B. A distinction is to be drawn between the forms $a \div x \times y$ and $a \div xy$. The former, as has been said above, means that a is to be divided by x , and the quotient is to be multiplied by y ; but the latter means that a is to be divided by the product of x and y . Thus, for example, suppose $a=48$, $x=8$, $y=3$; then $a \div x = 48 \div 8 = 6$; $\therefore a \div x \times y = 6 \times 3 = 18$. But $xy = 8 \times 3 = 24$; $\therefore a \div xy = 48 \div 24 = 2$.

Similarly $10a \div 2b$ and $10a \div 2 \times b$ are different from each other. For, let $a=4$, $b=5$. Then, $10a=40$, $2b=10$, $\therefore 10a \div 2b = 40 \div 10 = 4$. Again $10a=40$, $\therefore 10a \div 2 \times b = 20 \times 5 = 100$.

The student should also distinguish between the forms $a \div \frac{1}{2} \times b$ and $a \div \frac{1}{2}$ of b ; in the latter the value of $\frac{1}{2}$ of b is to be found first; that is, the operator \div in $a \div \frac{1}{2}$ of b applies not to $\frac{1}{2}$ only, but to $\frac{1}{2}$ of b .

Ex. Find the numerical value of $a \times b \div c \times d \div ef$, when $a=4$, $b=\frac{1}{2}$, $c=5$, $d=\frac{1}{10}$, $e=2$, $f=\frac{1}{2}$.

$$a \times b = 4 \times \frac{1}{2} = 2; \therefore a \times b \div c = 2 \div 5 = \frac{2}{5};$$

$$\therefore a \times b \div c \times d = \frac{2}{5} \times \frac{1}{10} = \frac{2}{50}; \text{ also } ef = 2 \times \frac{1}{2} = 1;$$

$$\therefore a \times b \div c \times d \div ef = \frac{2}{50} \div 1 = \frac{2}{50}. \text{ Ans.}$$

EXAMPLES 2.

Find the value of :

1. $2 \times 4 \div 12$.

3. $4 \times 9 \div 7 \times 6$.

5. $4 \div 8 \times 9 \div \frac{1}{2} \times 6$.

2. $2 \div 12 \times 4$.

4. $4 \times 9 \div 6 \times 7$.

6. $4 \div 8 \times 9 \div \frac{1}{2}$ of 6.

When $a=b=5$, $c=\frac{1}{2}$, $d=\frac{3}{4}$, $x=6$, find the value of :

7. $ab \div c \times d$.

✓ 11. $12ac \div 5 \times x$.

15. $a \div bcd$.

8. $a \times b \div cd$.

✓ 12. $6a \div 5x \div 4c \div d$.

16. $a \div bcd$.

9. $12ac \div 5x$.

13. $a \div b \times c \times d$.

17. $b \div 4a \times c \div d \div 2x$.

10. $3abc \div 4dx$.

14. $3abc \div 4 \times dx$.

18. $3abc \div 4d \times x$.

7. Quantities connected by $+$, $-$, \times , \div . In such cases the following rule should be observed : *First perform the operations of multiplication and division, and next perform the operations of addition and subtraction.*

Ex. Find the value of $a \times b - c \div d + e \times f \div g$,

when $a=1$, $b=2$, $c=12$, $d=4$, $e=15$, $f=6$, $g=10$.

$$a \times b = 1 \times 2 = 2; \quad c \div d = 12 \div 4 = 3;$$

$$e \times f \div g = 15 \times 6 \div 10 = 9;$$

\therefore the value required $= 2 - 3 + 9 = 8$. Ans.

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EXAMPLES 3.

Evaluate :

1. $13 \times 2 - 4 \times 5$

4. $12 - 3 \times 6 \div 8 - 9 \div 2$

2. $27 - 4 \times 5$

5. $12 - 3 \times \frac{1}{2}$ of $5 - 4 \div \frac{1}{2}$ of 30.

3. $9 - 4 \div 5 + 1$

6. $3 \times 5 \div 6 - 2 \times 9 \div 3 + 4 \times \frac{1}{2} \div 6$

When $a=12$, $b=8$, $c=9$, $m=20$, $n=16$, $x=10$, $y=12\frac{1}{2}$, find the numerical value of :

7. $6a - b \times \frac{1}{2}$

11. $xy + ab \div c \times 90n \div 6x - 100$

8. $a \times b - c$

12. $\frac{2a - b + m}{2n + 4} - \frac{2x + 1}{a + \frac{1}{2}b + c} \div \frac{m - 1}{2y}$

9. $4ab - 2cm + 30 \div x$

13. $a \div b \div c + m \div 2n \div x - x \div 12y$

10. $\frac{bm}{4x} - am \div 3b \div x$

14. $\frac{a - b}{c - b} \div \frac{m - n}{n - x} - n + m + 20x \div \frac{1}{2}$ of y .

8. **Properties of 0.** The following properties of 0 should be noted :—

(1) The product of zero and any *finite* number is 0.

(2) Zero divided by any finite number gives 0.

EXAMPLES 4.

When $a=10$, $b=3$, $c=6$, $m=8$, $x=0$, find the value of :

1. abx

2. $b \div c \times x$

3. $x \div bc$

4. $b cx + ab + bx \div cm$

5. $11x + \frac{12x}{abcm} - mx \div bc$

6. $120x + 6m \div bc + 6m \div b \times c - \frac{mx}{b}$

9. **Factor, Coefficient.** (When a quantity is considered as the product of two or more quantities, each of these latter is called a *factor* of that quantity; thus $3abc$ has for its factors 3, a , b and c . When a quantity is looked upon as the product of two factors, each of these is called the *coefficient* (i.e., co-factor) of the other; thus, when $3abc$ is regarded as the product of 3 and abc , 3 is regarded as the coefficient of abc . We might also consider that $3a$ is the coefficient of bc , or that $3bc$ is the coefficient of a , $3ac$ that of b , a that of $3bc$, &c. It is usual, however, to put the coefficient first; thus in ab , a is the coefficient of b , and in ba , b is the coefficient of a ; $5x$, $9b$, &c. are never written as $x5$, $b9$, &c.

10. (When the coefficient is an actual number, it is called a *numerical coefficient*; thus in $5a$, 5 is the numerical coefficient of a . (When, however, the coefficient is in symbols, it is called a *literal coefficient*); thus a is a literal coefficient of x in ax .

N.B. abc may be regarded as $1. abc$, and x may be regarded as $1. x$; hence 1 is the coefficient of abc as well as of x .

When the coefficient is fractional and greater than unity, it is usually written as an improper fraction. Thus we write $1\frac{1}{2}a$ and not $3\frac{1}{2}a$.

11. Power, Index. If a quantity be multiplied by itself any number of times, the product is called a *power* of that quantity. Thus $a \times a$ is called the *second power* of a , and is written a^2 ; $a \times a \times a$ is the *third power* of a , and is written a^3 ; and so on.

The *index* or *exponent* of any power of a quantity is the number, given either explicitly or in symbols, which shows how often the quantity is to be taken as a factor in producing that power. This number is placed to the right of the quantity and above it.

Thus in a^{10} , 10 is the index of the power of a . a^{10} is read 'a raised to the tenth power,' or more shortly, 'a to the tenth.' So a^m is read 'a to the mth,' or 'a to the power m.'

a^2 is, however, generally read 'a squared,' and a^3 is generally read 'a cubed.'

Note. In accordance with the above forms, a^1 should be called the *first power* of a , and mean that a is taken only once. Thus $a^1 = a$, so that any quantity may be regarded as the first power of itself.

It is to be borne in mind that any finite power of $1 = 1$, and that any finite power of $0 = 0$.

Thus $1^5 = 1 \times 1 \times 1 \times 1 \times 1 = 1$, and $0^3 = 0 \times 0 \times 0 = 0$.

12. Bracket, Vinculum. The signs $()$, $\{ \}$, $[]$ are called *brackets*. $(a+b) \times c$ means that $a+b$ is to be found first, and then the sum is to be multiplied by c . Thus the operation $\times c$ applies not to a or b singly, but to $a+b$ as a whole or collectively, i.e., to the entire value of $a+b$. The difference between $(a+b) \times c$ and $a+b \times c$ is now quite clear. For, suppose $a=1$, $b=2$, $c=3$.

Then, $a+b=1+2=3$, $\therefore (a+b) \times c = 3 \times c = 3 \times 3 = 9$;

also $b \times c = 2 \times 3 = 6$, $\therefore a+b \times c = 1+6 = 7$.

The student should therefore see that when any number of quantities connected by the ordinary signs of operation are enclosed in a bracket, they are to be looked upon as forming a single quantity, and their value as a whole is to be used in all operations indicated by the signs that precede or succeed the bracket.

Sometimes a line serves as a bracket. Thus $a - \overline{b-c}$ means the same thing as $a - (b-c)$, and is therefore different from $a - b - c$. The line so used is called a *vinculum*.

N.B. The three kinds of brackets $()$, $\{ \}$ and $[]$ are distinguished as *parentheses*, *braces* and *crotchets* respectively.

Ex. 1. Find the value of a^m , when $a=4$, $m=3$.

$$a^m = 4^3 = 4 \times 4 \times 4 = 64.$$

Ex. 2. Find the value of $\left(\frac{3a+bx}{a+b}\right)^x$, when $a=1$, $b=2$, $x=3$.

$$\frac{3a+bx}{a+b} = \frac{3 \cdot 1 + 2 \cdot 3}{1+2} = \frac{3+6}{3} = \frac{9}{3} = 3;$$

$$\therefore \left(\frac{3a+bx}{a+b}\right)^x = 3^3 = 3 \times 3 \times 3 = 27. \text{ Ans.}$$

EXAMPLES 5.

When $a=1$, $b=2$, $c=3$, $x=7$, $y=5$, $z=4$, $m=0$, $n=\frac{1}{2}$, find the value of:—

1. 3^5 .

5. b^3c^4 .

9. $\frac{5}{3}(x^2 \div by)^{0+c}$.

2. $(12)^c$.

6. $\frac{1}{2}(by)^{2b}$.

10. $\frac{2}{3}\left(\frac{1}{2}x - \frac{1}{3}yz\right)^{0+2a}$.

3. $(11)^{2b}$.

7. $(3az)^{2nb}$.

11. $5a^2x^2 - 2(ay^3 - bz^2) - m^2$.

4. $(bc)^{3a}$.

8. $3a^{2nb}$.

12. $2x^2y^2 \div (c^2 + z^2) - 2m^2$.

13. $\frac{16+2(2c^3)b^3}{5^x}$.

14. $9\left(\frac{2x^2 - 3y^2}{c^2 - m^2}\right)^b - \left(\frac{bx}{3c}\right)^b 3c$.

13. Square root, Cube root, &c. The *square root* of a given quantity is that quantity whose *square* or *second power* gives back the proposed quantity. Thus the square root of 25 is 5, because $5^2=25$.

The *cube*, *fourth*, *fifth*, &c., *root* of any quantity is that quantity whose third, fourth, fifth, &c., power gives back the given quantity.

The square root of a is denoted by \sqrt{a} , or more simply by \sqrt{a} . The cube, fourth, fifth, &c., roots are denoted by the symbols $\sqrt[3]{a}$, $\sqrt[4]{a}$, $\sqrt[5]{a}$, &c. Similarly $\sqrt[n]{a}$ denotes the n th root of a , and is that quantity whose n th power is a .

The root symbol $\sqrt{}$ is called the *radical sign*, and is a corruption of the initial letter r of the Latin word *radix* which means a root.

Note. The form $\sqrt{a+b}$ is often used as an equivalent of $\sqrt{(a+b)}$ and means the square root of the quantity $a+b$ taken as a whole. Here the radical sign and the vinculum have been joined together to form one symbol $\sqrt{}$. Thus $\sqrt[3]{2a^2xy}$ is equivalent to $\sqrt[3]{(2a^2xy)}$, and means the n th root of the whole quantity $2a^2xy$. We must therefore distinguish between the forms $\sqrt{9a}$ and $\sqrt{9 \cdot a}$; $\sqrt{9a}$ means the square root of $9a$, while $\sqrt{9 \cdot a}$ means the product of the square root of 9 and a , which of course is different from the other. Suppose $a=4$; then

$$\sqrt{9a} = \sqrt{9 \times 4} = \sqrt{36} = 6;$$

$$\text{but } \sqrt{9 \cdot a} = 3a = 3 \times 4 = 12.$$

Ex. Find the value of $2\sqrt{abc} - \sqrt[3]{(e^2 + f^2)} + \frac{1}{2}xz$,
 when $a=2, b=6, c=12, e=10, f=5, x=16, z=\frac{1}{2}$.
 $\sqrt{abc} = \sqrt{2 \times 6 \times 12} = \sqrt{12 \times 12} = 12$;
 $\sqrt[3]{(e^2 + f^2)} = \sqrt[3]{10^2 + 5^2} = \sqrt[3]{125} = 5$;
 and $\frac{1}{2}xz = \frac{1}{2} \times 16 \times \frac{1}{2} = 2 \times \frac{1}{2} = 1$;
 \therefore the given expression $= 2 \times 12 - 5 + 1 = 20$. *Ans.*

EXAMPLES 6.

When $a=6, b=9, c=8, d=5, n=3$, find the numerical value of ;

- ✓ 1. $\frac{2\sqrt{3abc}}{n}$.
- ✓ 3. $\frac{\sqrt{2c+b}}{d}$.
- ✓ 5. $\frac{b}{2} \sqrt[3]{\left(\frac{1}{a^2b^3c^2}\right)}$.
- ✓ 2. $\frac{a}{9} \sqrt[4]{16a^3b^2c^2}$.
- ✓ 4. $\frac{\sqrt{2c+b}}{d}$.
- ✓ 6. $\frac{\sqrt[3]{3b}}{\sqrt{(2a^2c)}}$.
7. $\sqrt{ab+c^2+a+4d} \div n^2$.
8. $\sqrt{7(a^2+b^2-c^2-d^2)} \div 7 \sqrt{(4d^2-a^2)}$.
9. $\frac{\sqrt[3]{(c^3d^2+abd^3)}}{7b+11d} \div \frac{1}{\sqrt{(4a^2+d^2)}} - \sqrt[3]{\left(\frac{a^2+d^2}{b^2+c^2-2abdc-36}\right)}$.

14. Unit, Measure. Any quantity by means of which another of the *same kind* is measured is called its *unit*.

Every quantity is measured by means of another of its kind. When the weight of a body is given as 5 maunds, what is meant is that the body is five-times as heavy as another whose weight we agree to call one maund. Here the standard of measurement is one maund, and the weight of the body is given by expressing how often that maund is contained in it. When, again, we say that the weight of a body is half a maund, we mean that the body is just half as heavy as another of the standard weight, one maund.

Again, when we say that the length of a room is 5 feet, we are supposed to have taken for our unit a standard length, which we agree to call 1 foot, the length of the room having been found by ascertaining what multiple (or part) this length is of the standard length.

Of course, the length of a room is measured by *the length of a rod*, and not by *the weight of a body*; in other words, a *quantity is measured by another of its own kind*.

The *measure* of a quantity is the number that expresses what multiple or part, this quantity is of the unit previously agreed upon. Thus if one foot be chosen as the unit, the measure of the height of a tree 7 feet high is 7.

When we say that the measure of a sum of money is $5\frac{1}{2}$, we mean that the sum is $5\frac{1}{2}$ times the unit of money; if a rupee be taken as the unit, the sum of money = $5\frac{1}{2}$ rupees. Thus,

Any quantity = its measure \times the unit.

Ex. 1. Find the measure of Rs. 50, the unit of money being Rs. 10.

We are here to find what multiple Rs. 50 are of Rs. 10.

\therefore the measure required = Rs. 50 \div Rs. 10 = 5.

The question is similar to this :—how many currency notes of Rs. 10 each, can I buy with Rs. 50 ?

Ex. 2. How heavy is a body whose weight is represented by $5\frac{1}{2}$, when the unit of weight is 2 seers 6 chhataks ?

The weight of the body = 2 seers 6 chks. $\times 5\frac{1}{2}$ = 13 seers 1 chk.

Ex. 3. What is the unit of area, when the area of a plot of land measuring one acre is represented by 363 ?

By the question, the measure of 1 acre is 363 ;

\therefore 1 acre = 363 times the unit of area ;

\therefore the unit of area = 1 acre \div 363

$$= 4840 \text{ sq. yds. } \div 363$$

$$= 13^0 \text{ sq. yds.}$$

$$= \underline{13 \text{ sq. yds } 3 \text{ sq. ft}}$$

EXAMPLES 7.

1. What is the measure of 5 hours 24 minutes and 24 seconds, when the unit of time is 2 minutes and 5 seconds ?

2. How will 2 acres 5 sq. yds. 6 sq. ft. be represented, when 1 sq. yd. 1 sq. ft. is taken as the unit of area ?

3. The measure of a certain sum of money is $10\frac{1}{2}$; if the coin denoting the unit of money is worth £1. 5s. 6d., find the sum in pounds, &c.

4. How heavy is a body whose weight is represented by $24\frac{1}{2}$, when the unit of weight is $2\frac{1}{2}$ lbs. ?

5. A mountain is $3\frac{1}{2}$ miles high ; what must be the units of measurement, when its height is represented by 1320 and 176 respectively ?

15. Convention of positive and negative quantities.
Suppose a man, *A*, to walk from a certain place 4 miles eastwards, and suppose another man, *B*, to walk from the same place 4 miles

westwards. Now, each has walked the same distance, though in opposite directions. To preserve the distinction in *direction*, we shall have to repeat the words 'eastwards' and 'westwards' as often as we have to deal with the distances walked. But in Algebra we adopt a shorter mode of expressing the distinction just referred to. The signs $+$ and $-$ are made to do duty for the words 'eastwards' and 'westwards.' Any one of the directions, east or west, is *chosen* as the positive direction, and then the *opposite direction* is regarded as negative. Thus if we agree to take the east as the positive direction, then we say that

the distance in miles walked by $A = +4$,

and " " " " " " $B = -4$.

In dealing with two quantities of *opposite character*, we make it a general rule to regard one of the quantities as positive, while the other is regarded as negative, and is therefore written with the sign $-$ prefixed. This is, of course, a perfectly *conventional mode of expressing two quantities that are opposite in character*, that is, one agreed upon for the sake of convenience; but we shall find it of the greatest service, especially in higher Mathematics. Let us take another illustration here. Suppose a man to enter on a speculation with Rs. 1000, and suppose that he gains Rs. 400. How much is he worth now? How much should he have, if he lost Rs. 400 instead of gaining? To work out the two answers, the student will assume some such pair of rules:

(1) The sum required = capital *plus* gain;

(2) The sum required = capital *minus* loss.

But let us see how rule (1) can be made to cover both the cases. Consider his gain a positive quantity, and then his loss would be a negative quantity, and would appear affected with the negative sign. Thus, in Algebraical language, a *gain of -400 rupees* would be equivalent to what is meant in common language by a *loss of 400 rupees*. In Algebra, therefore, the signs $+$ and $-$ are made to tell simply the story of the man's luck in business. Thus,

in case of *gain*, his *gain* = $+400$ rupees;

in case of *loss*, his *gain* = -400 rupees.

\therefore from rule (1), in case of gain, his money = Rs. 1000 + Rs. 400 = Rs. 1400; and in case of loss, from the same rule, his money = Rs. 1000 - Rs. 400 = Rs. 600.

Now take the case of a trader who lays out Rs. 200, but loses Rs. 300; what is he worth at the end? He is, as we readily see, *in debt* to the extent of 100 rupees. This fact we algebraically express by saying that his worth in rupees = $200 - 300 = -100$, and this mode of representing his financial position is in keeping with our tacit assumption that he had $+200$ rupees to begin with.

As will be clear from the above illustrations, the use of the signs $+$ and $-$ in preference to others to indicate quantities of opposite character is justified by their pointing at once to the *additive* and *subtractive* natures of those quantities. The consequence is that equal quantities of opposite character destroy one another, when taken together. Thus a gain of Rs. 200 followed by a loss of Rs. 200 makes the resulting gain Rs. 200 - Rs. 200, *i.e.*, *nil*. This point will be dealt with at length in the Chapter on Addition.

A very nice use of the convention of signs here explained has been made in thermometers. When the Centigrade thermometer is placed in melting ice, the level of the mercury is against the mark *zero* on the stem. If a greater cold is applied, the level falls *below zero*, and when heat is applied, the level rises *above zero*. Suppose at a certain time the level of the mercury is against the mark *4 above zero*. Then the temperature is read *4 degrees* (which is the same as $+4$ degrees). Now suppose that on account of an intense cold the mercurial column falls an equal distance *below zero*. How should we designate this new temperature? Certainly not as -4 degrees. It is usually expressed as -4 degrees. All readings *above zero* are regarded as *positive*, while those *below zero* are regarded as *negative*, and have the *negative* sign prefixed. To say that the temperature is -7 degrees is just as much as to say that it is *7 degrees below zero*.

16. Uses of $+$ and $-$. Thus there are two distinct purposes for which the signs $+$ and $-$ are used :

(1) Ordinarily, as in Arithmetic, they are used to signify the work of addition or subtraction, and when so used they are regarded as *signs of operation*. Thus $4+3$ means that 3 is to be added to 4.

(2) They are also used to denote the nature of a quantity without any reference to the operation of addition or subtraction. Thus when we say that the temperature is -4 degrees, we do not mean that 4 degrees are to be subtracted. In such cases, the signs $+$ and $-$ are called *signs of affection*, and only serve to indicate a comparison between quantities of opposite character.

17. Quantities of the same character should be affected with the same sign. It should be carefully remembered that *in the same piece of work the same sign must always be prefixed to quantities of the same character*. Thus if a speculator first gains Rs. 500, and next Rs. 300, and then loses at two different times Rs. 200 and Rs. 400, then, in calculating his net gain, we take the gains with the positive sign, and the losses with the negative sign, and so the net gain in rupees $= 500 + 300 - 200 - 400 = 200$.

It should be noticed, moreover, that in dealing with two quantities of opposite character we are *free to choose any one of the two as positive*; but that when the choice has once been made, the other must be regarded as negative. Thus if a distance due north-east from a place be taken as positive, any distance due south-west must be affected negatively; we may, however, consider the latter distance positive, and then the first distance must be affected negatively.

EXAMPLES 8.

1. A man travels 10 miles towards the east, and then rides back 15 miles. How far has he advanced towards the east?

2. A man gains £1. 10s. in his first venture, and loses £2 in the second. Find his ultimate *gain*. What is his loss?

3. How much *longer* is a rod of five yards than one of 7 yards? How much shorter?

4. If a yard measure is -5 in. too long, what is its length?

5. If a weight is -10 seers short of a maund, how heavy is it?

6. What times are -15 minutes *past* and *to* 10 o'clock?

7. A thermometer, reading 5°C , subsequently falls 10° . What is the final reading?

8. If a man is $-a$ years older than another whose age is b years, how old is he? How old, if $-a$ years younger?

18. **Essential and apparent characters of algebraical quantities** $+a$, or more simply a , may, *as we please*, stand for 200 or -200 . a being only a symbol, it depends entirely upon our choice to use it as a substitute for 200 or -200 . In other words, we do not bind ourselves to always taking a or $+a$ as the substitute for a positive number; we may, if we like, use it as the substitute for a number affected negatively. When $+a$ stands for a *positive quantity*, it is *apparently* as well as *essentially* positive; but when $+a$ stands for a negative quantity, it is said to be *apparently* positive but *essentially* negative. It is, however, usual to speak of $+a$ as a *positive quantity* and $-a$ as a *negative quantity*, simply because the one is expressed with the $+$ sign, and the other with the $-$ sign.

19. **Absolute value.** The magnitude of a quantity irrespective of its sign is called its *absolute value*. Thus 3 and -3 have got the same absolute value, as also a and $-a$.

20. **Historical note.** The ancient Hindus are generally credited with having invented the Science of Algebra. Arabic mathematicians derived their knowledge of it from the Hindus;

and its introduction into Europe, which took place about the beginning of the thirteenth century, was from the works of these Arabic writers. In fact the name, Algebra, is merely a corruption of *al jabr al makatalah*, meaning *restoration and reduction*.

Examination on Chapter I.

1. Define Algebra. With whom did it originate? How did it find its way into Europe?

2. Define the following terms, giving examples in each case: Symbol, Quantity, Coefficient, Vinculum, Power of a Quantity, Unit and Measure.

3. Distinguish between Literal and Numerical Coefficients; Power and Index; Parentheses, Braces and Crotchets.

4. What are the Signs of Operation? What purposes other than as operators do the signs + and - serve? By reference to the thermometer, show the usefulness of the convention of signs in Algebra.

5. Does +a always denote a positive quantity? What is meant by the absolute value of a quantity? Illustrate your answer by examples.

EXAMPLES 9.

When $x=1$, find the value of:

1. $4x^4 - 9x^3 + 6x - 1$.

3. $(8x-6)(10-4x) + (5-x)(11-2x)$.

2. $7x^5 - 2x^2 + 3$.

4. $(4+3x)(4-3x) + 12x^2$.

When $x=10$, what number is represented by

5. $7x^2 + 2x + 1$?

7. $4x^4 + 2x^2 + 7$?

6. $9x^3 + 2x^2 + 5x + 3$?

8. $7x^4 + 5x + 3$?

If $a=3$, $b=4$, $c=5$, and $d=6$, find the value of:

9. $20c^2b$.

10. $a^2b^3 + c^2d^2 - 4acd$.

11. $2a + 3b^3 + 4c^2$.

12. $3(d-a)^2 - 4(c-a)^2 + 5(d-b)^2$.

13. $a^2b - b^2c + c^2d - d^2a + bc^2$.

14. $a^2(c-b) + b^2(c-a) + c^2(b-a)$.

15. $a^2b^3 + b^2c^3 + 3c^{a-b}d^{a-b}$.

16. $a^2b^3c \div (3a^2 + 9b^3 + 2c^2 + 5)$.

17. $(a+b)(b+c)(c+a) - 2abc$.

18. $(a+b)^2 + 2(a+b)(b+c) + (b+c)^2$.

If $a=8$, $b=3$, $c=0$, $k=9$, $x=4$, $y=1$, evaluate

19. $9a^3xy \div 4k$.

22. $\frac{1}{2}akh^2y^3$.

20. $\frac{a^4 + c^2 + x^2 - y^2}{ac + k + xy}$.

23. $\frac{b^{2x} + a^{2y}}{k^x - c^{4x}}$.

21. $a \div k + k \div x - 1 \div 3y$.

24. $a \div b^2 \times k \div c \div k^2 \times x^y - a \div b^2k$.

$$25. x^2 + a \times b - x^2 + ab + b \div b \div x + 4 \times 3a.$$

$$26. 2b^2a^2 - a^2b - c + a^2(b - c).$$

If $m=3$, $n=7$, $p=11$, $t=4$, and $v=0$, find the value of :

$$27. \frac{\sqrt{(m^2 - 6n + 14p)}}{\sqrt{9t + 7v}}. \quad 28. \frac{5\sqrt{3mt} - \sqrt{bp + t} + \sqrt{5vt}}{\sqrt{27m + t/64t}}.$$

$$29. \sqrt{\left\{ \frac{m^2 + t^2 + v^2}{2np - (t + 2m)} \right\}}. \quad 30. 35 \sqrt{\frac{m^2t^2}{12n^2} - \frac{3n + 2m + 73}{\sqrt{4p + 2t + 4m}} \frac{t}{n}}.$$

31. A farmer takes x sheep to market, sells off y of them, and then purchases z sheep. Express algebraically the number of sheep he now has. Give the result, if $x=40$, $y=25$, and $z=16$.

32. A book containing 3 chapters has a pages in each chapter, b lines in each page, and c words in each line. How many words are there altogether? What is the answer, when $a=30$, $b=45$, and $c=12$?

33. The length and breadth of the floor of a room are y feet each. What is the area of the floor in sq. ft.? Express the result also in sq. yds.

34. A snail creeps up a vertical pole a inches, then slips down b^2 inches, next goes up c^3 inches, and again slips down d^4 inches. How far up the pole is it at last? If $a=9$, $b=4$, $c=8$, $d=2$, what is the answer?

35. A bag contains a sovereigns, b half-sovereigns, c crowns, d florins, e shillings, and f pence. Express the sum in *shillings*.

CHAPTER II.

ADDITION.

21. **Addition.** Addition is the process of finding in the simplest form the result of taking several quantities together. These several quantities are called *addends* or *summands*, and the single quantity is called their *sum*.

22. **Expression.** An algebraical expression is a collection of algebraical symbols connected by the signs of operation; the parts of an expression which are separated from each other by the signs $+$ and $-$ being called its *terms*. Thus $5x - 3y + 7z^2 + ma^2b$ is an expression consisting of the four terms, $5x$, $-3y$, $+7z^2$, and $+ma^2b$.

23. **Classification of Expressions.** Expressions are either *simple* or *compound* according as they consist of *one* or of *more* terms.

A **simple expression**, or a *monomial*, consists of one term only ; as $9ab^2x$. **Compound expressions** are such as consist of more than one term.

A *binomial expression*, or briefly a *binomial*, consists of two terms ; as, $5x^2 - ab^2$.

A *trinomial* consists of three terms ; as, $9a^2 - 3abc + 5xy$.

A *polynomial expression* consists of more than three terms ; as, $7a^2 - 9xyz + a^2bc + 7x^2y \div mxy$.

24. Like and Unlike terms. When terms do not differ, or differ only in their numerical co-efficients, they are called *like* ; otherwise they are called *unlike*. Thus $5b, -3b, 7b$ are like terms, as also are $\frac{1}{2}x^2yz, -4x^2yz, 29x^2yz$; but $5b$ and $9c$ are unlike terms, as also are $\frac{1}{2}x^2y^2, 19x^2y^3, 13a^2bx$.

25. Sign of a term. The *sign of a term* usually means the sign + or - prefixed to it.

Hence when we are told to *change the sign* of a term, we have to change + into -, or - into +.

Note. In the expression $7xy + z^2 - 9x$, the sign of $7xy$ is understood to be + ; so in the expression $1a^2b - c^2a, 4a^2b$ is really a form of $+4a^2b$.

26. Addition of like terms having the same sign. *Add the numerical co-efficients and prefix their sum to the common letter or letters.*

Ex. 1. Find the value of $7x + 9x$.

We are here told to add 9 like things, each of value x , to 7 like things of the same kind, each of these being also of the same value x . The sum is therefore $(7+9)$ or 16 such things ; $\therefore 7x + 9x = 16x$.

Ex. 2. Add together $-x, -7x, -12x$.

Suppose a man to walk successively x miles, $7x$ miles and $12x$ miles, to the *south* of his starting place. How far has he walked ? Of course $x + 7x + 12x$ or $20x$ miles to the *south*. Now suppose distances measured to the *south* are negative, the *north* being agreed upon as the positive direction ; then, according to the convention of signs, the man has walked $-x$ miles, $-7x$ miles and $-12x$ miles. But we have seen that the total distance walked is equal to $20x$ miles reckoned to the *south*, i.e., to $-20x$ miles. Therefore the sum of $-x, -7x, -12x = -20x$.

We may look upon the above summation in another light. If we subtract from a certain quantity x , then $7x$, and again $12x$, how much do we subtract on the whole ? Evidently $20x$ is subtracted. Thus the sum of $-x, -7x, -12x = -20x$.

Note. It should be remembered that $x = 1.x$, and $-x = -1.x$.

27. Addition of like terms not all of the same sign.

RULE. *Add together separately the numerical coefficients of all the positive and all the negative terms; find the difference of these two sums, prefix to it the sign of the greater sum, and place the result thus obtained before the common letter or letters.*

Ex. 1. Add together $9a$ and $-4a$.

Here we are to take together 9 like things of one character and 4 like things of the *opposite* character. Now we have seen in ART. 15 that to add -4 to 9 in Algebra is equivalent to subtracting 4 from 9 in Arithmetic. Thus, since $9a = +9a$, we proceed thus :

$+9a$	coefficient of the positive term = 9,
$\underline{-4a}$	" " negative " = 4,
$+5a$	difference = 5,

or simply $5a$. and the greater coefficient has the sign +.

Note here that algebraical addition may produce a *decrease*, whereas addition in the ordinary arithmetical sense always produces an *increase*.

Ex. 2. Find the sum of $7abx$, $-3abx$, $-10abx$, $4abx$, $-11abx$.

The sum of the coefficients of the positive terms $= 7 + 4 = 11$;
the sum of the coefficients of the negative terms $= 3 + 10 + 11 = 24$.

Also $24 - 11 = 13$, and the greater sum 24 belongs to the negative terms ;

\therefore the required sum $= -13abx$. *Ans.*

N.B. The process shown *above* is known as that of *collecting terms*.

Note. We may take the coefficients in any order we like, attending only to the operations indicated by the signs. Thus, in the above Example, we may mentally proceed thus :

$$7 - 3 = 4 ; 4 - 10 = -6 ; -6 + 4 = -2 ; -2 - 11 = -13.$$

Hence the sum $= -13abx$.

EXAMPLES 10.

Add together :

- | | |
|-----------------------------------|--|
| 1. $4a, 3a, 5a$. | 3. $6a^2bc, a^2bc, 7a^2bc, 9a^2bc$. |
| 2. $4pqr, 7pqr, 8pqr$. | 4. $4a^2x, a^2x, 19a^2x, 22a^2x, 3a^2x$. |
| 5. $-9a, -3a, -6a$. | 7. $-xyz, -11xyz, -10xyz, -24xyz$. |
| 6. $-3mp^2, -9mp^2, -12mp^2$. | 8. $-bx^2, -15bx^2, -11bx^2, -12bx^2$. |
| 9. $27abc, -27abc$. | 11. $29a^2b, -5a^2b, 2a^2b, -12a^2b, -4a^2b$. |
| 10. $3a^2l^2, -4a^2l^2, a^2l^2$. | 12. $-21a^2b, -5a^2b, 6a^2b, 4a^2b, +5a^2b$. |

$$\begin{array}{r} 13. \quad -mnr \\ -6mnr \\ 15mnr \\ 2mnr \\ \hline 13mnr \end{array}$$

$$\begin{array}{r} 14. \quad -12a^2b \\ +27a^2b \\ 13a^2b \\ -21a^2b \\ \hline -43a^2b \end{array}$$

$$\begin{array}{r} 15. \quad 21l^2m^3n^4 \\ -l^2m^3n^4 \\ 3l^2m^3n^4 \\ -10l^2m^3n^4 \\ \hline -13l^2m^3n^4 \end{array}$$

$$16. \quad \frac{x^2}{4}, -\frac{x^2}{3}, -\frac{x^2}{5}$$

$$17. \quad -\frac{\sqrt{xy}}{6}, -\frac{\sqrt{xy}}{9}, \frac{\sqrt{xy}}{12}, \frac{\sqrt{xy}}{18}, -\frac{\sqrt{xy}}{2}$$

Find the value of :

$$18. \quad \frac{1}{2}p^3 - \frac{1}{3}p^3 + \frac{1}{6}p^3 \quad 19. \quad -\frac{1}{2}abcd - \frac{2}{3}abed + 4abcd - aqed.$$

$$20. \quad -5x^2y^3 + \frac{2}{3}x^2y^3 - 3x^2y^3 - 8x^2y^3.$$

$$21. \quad 24l^2m^3 - 2l^2m^3 + 4l^2m^3 - \frac{1}{2}l^2m^3 + l^2m^3 - \frac{1}{3}l^2m^3.$$

$$22. \quad \frac{x^2}{2} - \frac{x^2}{3} + \frac{x^2}{5} - \frac{x^2}{5} \quad 23. \quad \frac{\sqrt{a^2b}}{5} - \frac{\sqrt{a^2b}}{6} - \frac{\sqrt{a^2b}}{8} + \frac{\sqrt{a^2b}}{9} - \frac{\sqrt{a^2b}}{12}.$$

28. Addition of unlike terms. When we have to add a and b , it remains only to connect them by the sign + ; and leave the result in the form $a+b$.

29. The forms $a+(+b)$, and $a+(-b)$. In the form $a+(+b)$, the sign + before b contained within the brackets is simply a sign of *affection*, and denotes the character of the quantity to be added to a . Hence

$$a+(+b)=a+b; \quad 4+(+5)=4+5=9.$$

In like manner, in the form $a+(-b)$, the sign - before b within the brackets only serves to indicate a comparison between the two addenda. We have already seen that adding $5a$ and $-3a$ means *subtracting* $3a$ from $5a$. Thus to add $-b$ to a must be the same as to *subtract* b from a , and we therefore have

$$a+(-b)=a-b; \quad 5+(-9)=5-9=-4.$$

30. Equivalence of $a-b$ and $-b+a$. The student should be on his guard against supposing any *substantial* difference between the results expressed by $a-b$ and $-b+a$

Suppose a tradesman with $\text{Rs } 500$ in his box *gains* $\text{Rs } 30$, and then *loses* $\text{Rs } 10$; evidently his money is at the end equal to $\text{Rs } 530 - \text{Rs } 10$, or $\text{Rs } 520$. Suppose, however, that he had first *lost* $\text{Rs } 10$, and afterwards *gained* $\text{Rs } 30$. Then also his money would at the end be $\text{Rs } 490 + \text{Rs } 30$, or $\text{Rs } 520$; *i.e.*, exactly the same as before. Now in the former case his gain should be algebraically expressed by $\text{Rs } 30 - \text{Rs } 10$, and in the latter by $-\text{Rs } 10 + \text{Rs } 30$; therefore $\text{Rs } 30 - \text{Rs } 10 = -\text{Rs } 10 + \text{Rs } 30$;

or, generally, $a-b = -b+a$.

31. Unlike terms. Terms differing in letters or in the powers of the same letter are regarded as *unlike*. Thus x and y , $5a$ and $3b$, a^3 and a^2 , $5bc$ and $5ac$, $7a^2b$ and $5ab$ are pairs of unlike terms.

32. Arrangement of terms. When an expression involves one letter only, the terms are generally arranged in the order of *descending* powers of that letter, beginning with the highest. Thus $5a^5 + 27a^2 - 3a^4 - a + 2 + a^3$ is usually put in the form $5a^5 - 3a^4 + a^3 + 27a^2 - a + 2$.

The terms may, however, also be arranged in the order of *ascending* powers of the letter used; as $2 - a + 27a^2 + a^3 - 3a^4 + 5a^5$.

When two letters are involved, the terms are arranged in the order of the ascending powers of one of them and the descending powers of the other. Thus $7x^2y + 5x^3 - y^3 - 3xy^2$ is usually written $5x^3 + 7x^2y - 3xy^2 - y^3$, i.e., in the order of the descending powers of x and ascending powers of y .

Note. When only different letters are involved, we may be guided by the order of the letters in the alphabet; e.g., $2a - 5b + c - d$.

Ex. Arrange properly the sum of a^3 , $2a^3$, $-\frac{1}{2}a$, -1 .

$$\begin{aligned}\text{The sum} &= a^3 + 2a^3 - \frac{1}{2}a - 1 \\ &= \underline{3a^3 + a^2 - \frac{1}{2}a - 1} \text{ Ans.}\end{aligned}$$

EXAMPLES 11.

Add together, arranging in due order :

- | | |
|---|---|
| 1. $x, y.$ | 2. $2x, 3y, z.$ |
| 3. $\frac{2}{3}x, -7y, z$ | 4. $14a^3, -15b^2, -\frac{1}{2}c/m, 12a^2.$ |
| 5. $2a^4, a^2, -a,$ | 6. $x^2y^3, x^2y^2, -xy, -2.$ |
| 7. $y^5, -16y^7, 6y^2, -\frac{1}{2}y, \frac{1}{3}y^3, 1.$ | 8. $11b^5, \frac{2}{3}b^{10}, \frac{1}{4}b^7, 5, -3b^3, b.$ |
| 9. $4b^3, -7c^3, 5bc^2, -b^3c$ | 10. $7l^4m^4, 4l^3m^5, -9lm^7, -4l^2m^6.$ |
| 11. $-4x^2y^2, -3xy^4, -5x^2y^3, -7y^5, -5x^5, 27x^4y.$ | |

Remove the brackets from, and simplify, when possible :

- | | |
|--|---|
| 12. $2a + (+3b).$ | 13. $20a^2 + (+\frac{1}{2}bc) + (+9b^2).$ |
| 14. $3x^3 + (-9y^3).$ | 15. $3x^2y^2 + (-3xy^4) + (-7y^5).$ |
| 16. $9a^3b^3 + (-19ab^5) + (21b^5)$ | 17. $-\frac{1}{2}a^2c^3 + (-\frac{1}{3}b^3) + (-\frac{1}{4}xyz).$ |
| 18. $5xyz + (-9xyz) + (-\frac{1}{2}xyz).$ | 19. $-5a^3m + (-\frac{1}{3}a^2m) + (+\frac{1}{4}a^2m).$ |
| 20. $2x^5 + (-3x^3y) + (4x^4y) + (-x^2y) + (-x^4y) + (-27).$ | |

33. Addition of Compound Expressions. The solution of the following example will illustrate the method of adding compound expressions, and lead to the Rule by which this is usually done.

Ex. 1. Find the sum of $2a+3b-c$, $3c-6b-d$, $2d-a-2c$.

$$\begin{aligned}\text{The sum} &= 2a+3b-c+3c-6b-d+2d-a-2c \\ &= 2a-a+3b-6b-c+3c-2c-d+2d, \quad (\text{re-arranging terms}). \\ &= \underline{a-3b+d}, \text{ collecting like terms.}\end{aligned}$$

The addition may, therefore, be conveniently done by the following rule:

Rule. Arrange the expressions in lines one under the other, so that the like terms may be in the same vertical columns, and next add each column.

$2a+3b-c$		Beginning at the left,
$-6b+3c-d$		the sum of the terms in the 1st column = a .
$-a-2c+2d$		" " " 2nd " = $-3b$,
<u>$a-3b+d$</u>		" " " 3rd " = 0 ,
		" " " 4th " = $+d$.

Ex. 2. Add together $3x^4-5x^2+1$, $6x^3+4x^2-6$, $-x^3+x^2-4x$, and $-2x^4-5x^2+4$.

$3x^4-5x^2+1$		The terms are arranged according
$+6x^3+4x^2-6$		to descending powers of x .
$-x^3+x^2-4x$		The required sum
$-2x^4-5x^2+4$		= x^4-4x-1 .
<u>x^4-4x-1</u>		

Ex. 3. Add together: $4a^4b-5a^3b^2-a^2b^3$, $3ab^4-3a^5-2a^2b^3$, $7b^3a^2+b^5-8ab^4$, and $3b^4a-2a^2b^3+2a^5$.

$+4a^4b-a^2b^3-5a^3b^2$		The terms are first arranged
$-3a^5$		according to the descending
$-2a^3b^2+3ab^4$		powers of a and ascending
$+7a^2b^3-8ab^4+b^5$		powers of b .
$+2a^5$		
<u>$-a^5+4a^4b-a^3b^2-2a^2b^3-2ab^4+b^5$</u>		= the required sum.

Note. An expression is generally commenced with a positive term, and the answer may accordingly be put as

$$b^5-2ab^4-2a^2b^3-a^3b^2+4a^4b-a^5.$$

EXAMPLES 12.

Add together :

- | | |
|----------------------------|----------------------------------|
| 1. $a-b$, $a+b$. | 2. $2a+5b$, $4a-7b$, $3b-6a$. |
| 3. $a-b$, $b-c$, $c-a$. | 4. $x+y-z$, $y+z-x$, $z+x-y$. |

6. $4x + 9y - 7z$, $2y - 3z + 11x$, $20x - 13y + \frac{1}{2}z$.
8. $9a - 7b^2 + 3c - 4d$, $6a^2 - 3a + 9c + 2d$, $11a - 2b + 2c + 2d$, and $a - 3c + 3b + 6a$.
7. $x^2 - y^2 - z^2$, $y^2 - z^2 - x^2$, $z^2 - x^2 - y^2$, $x^2 + y^2 + z^2$.
8. $2a^4 + 2b^4 - c^4$, $3b^4 + c^4 - a^4$, $14c^4 - 5a^4 - a^4$, $2a^4 - a^4 - b^4$.
9. $2a^5 + 2b^5 + 3a^5$, $3b^5 + 4c^5 + 5a^5$, $5c^5 - 7b^5 + 12a^5 - a^5$, and $13a^5 - 5b^5 + 11a^5$.
10. $a^3 - 9a + 7$, $2a^3 + 3a + 9$, $4 - 11a - 3a^3$.
11. $14b^3 + 6b - 12$, $13 - 11b + b^3$, $2 - 3b - b^3$.
12. $2x^2 - 4xy + 2y^2$, $-4x^2 + 7xy - y^2$, $-3x^2 + 2xy + 7y^2$.
13. $4y^2 - 7xy - 5x^2$, $12xy - y^2 - 2x^2$, $5xy - 3y^2$.
14. $2x^3 - 12x^2 + 3x - 9$, $4x^3 - 2x^2 - 3$, $-10x^3 + 3x + 2$.
15. $13x^4 - 2x^2 + x + 3$, $21x^3 - 3x^2 + 2x$, $4x^4 + 7x^3 + 11x^2 - 10x + 1$, $7x^4 + 5x^3 - 6x$, $1 - 20x^4 + 17x^3 - 23x + 13x^3$.
16. $5x^4 - 6x^3 + 9x^2 + 6x - 2$, $3 - x - x^4$, $12x^3 + x^2 + x - 2$, $2x^4 + 6$, $5 - x - 6x^3$, $6x^3 - 11x + 9x^2 - 10$.
17. $5 + y - y^3 - 4y^3$, $6y + 2y^3 - 9$, $7y^3 - 4 - 5y^3 - y$, $3y^3 - 6 - 2y^3$, $4y^2 - 6 - 5y$, $2y^3 - 7$, $6y^3 - 5 - 7y^3 + 3y$.
18. $4x^3 + 3x^2y - 7xy^2 + y^3$, $9x^2y - 2x^3 - 3xy^2 - y^3$, $4x^3 - 3x^2y - 4y^2x - y^3$.
19. $3a^4 - 4a^2b + 5a^2b^2 - 2ab^3 - b^4$, $-a^4 - 7a^3b - 4a^2b^2 + 5b^4$, $4a^4 + 3b^4$, $2b^4 + 4ab^3 - 3a^3b - 11a^4$.
20. $2a^2x^3 - a^2x^2 + 6a^4x$, $4a^2x^4 - 3a^2x^3 - 4a^5$, $21^4 - 6ax^4 + 4a^2x^3$, and $3a^5 + a^4x - 10a^2x^3$.
21. $2a^3 - 3ac + 2ab - 4bc - c^3$, $b^2 + 4ab - bc + ac - a^2$, $-2c^2 + 5bc$, and $a^2 + 2ac = 7ab - 2b^3$.
22. $5a^2 - 4ab + 3b^2 + 7a - 2b + 5$, $9 - 8a + 2b - 4a^2 - 9ab - b^2$, and $5a^2 - 6ab + 4b^2 - 4a - 2$.
23. $x^3 + 2x^2y - 3x^2z + 4xy^2 - 2xyx$, $3x^2z - 4xy^2 + 2xz^2 - 2y^2z + 3y^2$, $-x^3y - 2xz^3 + 2xyz + 3y^2z + 7x^2$.
24. $\frac{2}{3}x^4 - \frac{1}{2}x^3 + \frac{1}{3}x^2 - 1$, $\frac{1}{2}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x + 1$, $\frac{1}{3}x^3 - x^2 + \frac{1}{2}x - \frac{1}{3}$.
25. $\frac{2}{3}a^2 + \frac{1}{2}ab - \frac{1}{3}b^2$, $\frac{1}{3}a^2 - \frac{2}{3}ab + \frac{2}{3}b^2$, $\frac{1}{3}a^2 + \frac{1}{3}ab + b^2$.
26. $\frac{1}{2}x - \frac{1}{3}y$, $\frac{1}{3}x + \frac{1}{2}y$, $\frac{1}{2}y - \frac{1}{3}x$.
27. $\frac{1}{2}a + \frac{1}{3}b - \frac{1}{4}c$, $\frac{1}{3}b + \frac{1}{2}c - \frac{1}{2}a$, $\frac{1}{4}b - \frac{1}{2}c - \frac{1}{2}a$.
28. $\frac{1}{2}x^2 - \frac{1}{3}xy + \frac{2}{3}y^2$, $\frac{1}{2}xy - \frac{2}{3}y^2 - \frac{1}{3}x^2$, $\frac{1}{3}x^2 - \frac{1}{2}xy + \frac{1}{3}y^2$.
29. $\frac{\sqrt{ab}}{2} - \frac{\sqrt{bc}}{6}$, $\frac{\sqrt{bc}}{9} - \frac{\sqrt{ca}}{7}$, $\frac{\sqrt{ca}}{14} - \frac{\sqrt{ab}}{12}$.

30. If $x = 5a - 2b + 3c$, $y = 5a + 6b - 9c$, and $z = 7c - 3b - 9a$,
shew that $x + y + z = a + b + c$.

31. If $a = 5x^2 + 3yz + 4xy - 7z^2$,
 $b = 8y^2 - 5xy - 10xz - 4x^2$,
 $c = 8z^2 - 4yz + 9xz - 7y^2$,

prove that $a + b + c = x^2 + y^2 + z^2 - yz - zx - xy$.

32. A thermometer shows $-y$ degrees in the morning, but at noon the temperature rises by x degrees. What is the reading at noon? If $x = 5$, and $y = 6$, what is the result, and how do you interpret it?

33. A man walks $2a$ miles towards the north, and then $3b$ miles towards the south. How far is he at last from the starting-point? If $a = 4$, and $b = 3$, interpret the result.

34. What expression is greater than the sum of

$$a^2 - 2b^2 + c^2 - 2d^2 \text{ and } a^2 + b^2 - c^2 + d^2 \text{ by } a^2 + b^2 + c^2 + d^2?$$

CHAPTER III.

SUBTRACTION.

34. **Simple Brackets.** We already know that $a + (b + c)$ means that the sum of b and c is to be added to a . The result evidently is the same if we first add b , and next add c to the sum obtained. Thus we have

$$a + (b + c) = a + b + c.$$

Again, $a + (b - c)$ means that b diminished by c is to be added to a . If we add only b to a , thus getting $a + b$, we add c too much, and so exceed the desired result by c ; therefore the desired result must be equal to $a + b - c$.

Thus
$$a + (b - c) = a + b - c.$$

In like manner, $a - b + (c - d - e + f) = a - b + c - d - e + f$.

Hence the following rule :

Rule When an expression within brackets is preceded by the sign $+$, the brackets may be removed without affecting the value of the expression.

Note. The student already knows that $a + (-b) = a - b$, (ART. 29), and will easily see how the above rule applies to this case.

If we subtract b from a , and again c from the result, we evidently subtract $b + c$ on the whole. Thus

$$a - (b + c) = a - b - c.$$

Take now the expression $a - (b - c)$. It means that the excess of b over c is to be subtracted from a . Hence if we take away b , we take away c too much; therefore, to obtain the desired result, we should add c to $a - b$.

$$\therefore a - (b - c) = a - b + c.$$

In like manner, $a - b - (c - d - e + f) = a - b - c + d + e - f$.

Hence the following rule:

Rule. When an expression within brackets is preceded by the sign $-$, the brackets may be removed on changing the sign of every term within the brackets, $+$ into $-$, and $-$ into $+$.

Hence we infer that $a - (-b) = a + b$.

Suppose a thermometer at one time shows -3° , and next shows 5° ; what is the rise of temperature? The rise is evidently the excess of 5° over -3° , and so should be algebraically represented as $5^\circ - (-3^\circ)$. Now, we know that -3° means 3° below zero, so that a rise of 3° brings up the reading to 0° , and a further rise of 5° gives the final reading. Hence the total rise is $3^\circ + 5^\circ$ or 8° . Thus $5^\circ - (-3^\circ) = 5^\circ + 3^\circ$.

35. Definition. Subtraction is the inverse of addition; the quantity subtracted is called the **subtrahend**, and that from which it is subtracted is called the **minuend**.

The student should carefully note what we mean by the definition of subtraction given above. In ordinary Arithmetic we do not subtract 5 from 3, but subtract 3 from 5. But in Algebra we may propose to subtract 5 from 3, expressing our work as $3 - 5 = -2$. The student will note that $3 = 5 - 2 = 5 + (-2)$. Thus -2 added to 5 (the subtrahend) gives 3 (the minuend). Hence in finding the value of $3 - 5$, we find out what must be added to 5 to produce 3. When we are given two quantities, we can find their sum, and, conversely, given the sum and one of the two addenda, we find the other by subtraction; in this latter view, subtraction is regarded as the inverse of addition. Thus in obtaining $a - (-b)$, we seek that quantity which must be added to $-b$ to give a . Now, we know that the sum of $a + b$ and $-b = a + b - b = a$; i.e., $a + b$ is to be added to $-b$ (subtrahend) to produce a (minuend).

\therefore by definition, $a - (-b) = a + b$.

Similarly, in finding the value of $(a - b) - (c - d - e + f)$, we note that $(c - d - e + f) + (a - b - c + d + e - f) = a - b$;

$$\text{hence } (a - b) - (c - d - e + f) = a - b - c + d + e - f.$$

The following results are now apparent:

$$6a - 4a = 2a; \quad -6a - 4a = -10a; \quad -6a - (-4a) = -6a + 4a = -2a;$$

$$4a - 6a = -2a; \quad 6a - (-4a) = 6a + 4a = 10a.$$

Ex. 1. Subtract $c - d$ from $a - b$.

The required result $= a - b - (c - d)$,

$$= a - b - c + d, \text{ changing signs within brackets.}$$

Ex. 2. Subtract $7m - 6l - 4n$ from $8l - 2m + 5n$.

The answer $= 8l - 2m + 5n - (7m - 6l - 4n)$,

$$\begin{aligned}
 &= 8l - 2m + 5n - 7m + 6l + 4n, \text{ changing the signs} \\
 &\quad \text{within the brackets,} \\
 &= 8l + 6l - 2m - 7m + 5n + 4n, \text{ re-arranging the terms,} \\
 &= \underline{14l - 9m + 9n}, \text{ collecting like terms.}
 \end{aligned}$$

EXAMPLES 13.

Subtract.

- | | | |
|--|--|--------------------------------|
| 1. $15a$ from $20a$. | 2. $\frac{1}{2}a$ from $\frac{1}{3}a$. | 3. $\frac{3}{4}a$ from $7a$ |
| 4. $-\frac{2}{3}a$ from $7a$. | 5. $-\frac{2}{3}a$ from $-7a$. | 6. $\frac{3}{4}a$ from $-7a$. |
| 7. $3b$ from $2a$ | 8. $-6x$ from $9y$. | 9. $-20x$ from $-9y$. |
| 10. $a - b$ from $c + d$ | 11. $2a + 3b$ from $\frac{1}{2}c$. | 12. $3c - a$ from $-11b$. |
| 13. $a - b$ from $a + b$. | 14. $a - 2b - c$ from $a + b + 2c$. | |
| 15. $\frac{1}{3}a - \frac{2}{3}b$ from $\frac{1}{2}x - \frac{2}{3}b + c$. | 16. $\frac{1}{3}x - \frac{1}{4}x - \frac{1}{5}y$ from $\frac{1}{2}x - \frac{1}{10}y + \frac{1}{15}z$ | |

Simplify

- ✓ 17. $2x + (-3y) - (-y)$. ✓ 18. $4x - (-7x) - (-y)$.
- ✓ 19. $-x + 3y - 4z - (4x + 7z)$. ✓ 20. $-x + 3y - 4z - (-4x + 7z)$.
- ✓ 21. $-x + 3y - 4z - (-4x - 7z)$. ✓ 22. $3a - 3d + (b - 2a) - (c - d)$.
23. $2x^2 - (3x^2 - 4) + (7x - x^2) - x - (4 - x - 2x^2) + (x^2 - 12x)$.
24. $11x^3 + x^2 - (x^2 - x + 2) + (9x^2 - x) - (x^3 + 2x^2 - 3x - 2)$.
25. $7x^4 - (3x^2y^2 - y^4) - (x^4 + 2x^2y^2) + (2x^4 + 7x^2y^2 + 3y^4)$.
26. $yz - (2yz - xz + 3xy) - (4xy - 7zx) + yz + xz + xy$.

36. Like terms in a column. To subtract $7x - 5y + z$ from $2x - 3y - 4z$, we proceed thus:

$$\begin{aligned}
 \text{The required result} &= 2x - 3y - 4z - (7x - 5y + z) \\
 &= 2x - 3y - 4z - 7x + 5y - z.
 \end{aligned}$$

The latter part of the work may be done by placing like terms in a vertical column, as in addition. Thus

$$\begin{array}{r}
 \text{to} \quad 2x - 3y - 4z \\
 \text{add} \quad -7x + 5y - z, \text{ which we get by changing the sign of} \\
 \quad \underline{-5x + 2y - 5z} \quad \text{every term of } 7x - 5y + z.
 \end{array}$$

Hence the following rule for subtraction:

Rule. Change the sign of every term of the expression to be subtracted, and add the result to the other expression.

Note It is not usual to actually change the signs of the subtrahend; the operation of changing the signs is performed mentally.

Ex. From $2x^3 - 3x^2 + x$ take $-x^3 - x^2$.

$$\begin{array}{r} 2x^3 - 3x^2 + x \\ -x^3 - x^2 \\ \hline 3x^3 - 2x^2 + x \end{array}$$
 In the 1st column, we add mentally $2x^3$ and $+x^3$, and get $3x^3$; in the 2nd column, we add mentally $-3x^2$ and $+x^2$, and get $-2x^2$.

EXAMPLES 14.

From

- $2a^3 - 2 + 1$ take $3a^3 - 1$.
- $7a^3 - 9$ „ $5a^3 - 9a + 3$
- $5x^3 + 4xy + 2y^3$ take $9x^3 - 11xy + y^3$.
- $yx + zx + xy$ „ $-yz + zx - xy$.
- $\frac{3}{2}a^3 - \frac{1}{2}ab + \frac{1}{2}b^3$ „ $\frac{1}{2}a^3 - \frac{1}{2}ab - \frac{1}{2}b^3$.
- $x^4 + 2a^2x^2 - a^4$ „ $-2x^4 - 2ax^3 + a^3x$.
- $2x^3 + ax - bx - a^3$ take $x^3 + ax - a^3$.
- $ax^3 - ax - bx - b^3$ „ $2ax^3 - bx - a^3$.
- $-x^4 + 2a^2x^2 - a^4$ „ $-2x^4 - ax^3 + a^2x^3$.
- $a^3 - b^3 + c^3 + bc - 2ca$ take $c^3 + a^3 - b^3 - bc - 2ab$.
- $x^4 + 3x^3y - 3x^2y^2 + 4xy^3 - y^4$ take $y^4 + 3xy^3 - 4x^2y^2 - x^3y - x^4$.
- $2x^4 + 7x^3y + xy^3 + y^4 + 3x^2y^2$ „ $3x^4 - y^4 + 2x^3y^2 - 7x^2y - 5xy^3$.
- $3x^5 + 4x^4y - x^3y^2 + xy^4 - y^5$ „ $12x^5 - 10x^4y^3 + xy^4 + 2y^5 - x^4y$.
- $9x^6 + 5x^5y - 2y^6 - 3xy^5 + x^3y^3$ „ $11x^6 - 9x^4y^3 - 7x^5y - 3y^6 + xy^6$.
- $\frac{1}{2}a^3 - \frac{3}{2}a^2 + \frac{1}{2}a - 1$ take $\frac{1}{2}a^3 + \frac{1}{2}a^2 - \frac{1}{2}a - 2$.
- $\frac{3}{2}p^3 + ap^2 - 1$ take $-\frac{1}{2}p^3 - ap^2 - ap + 1$.
- $4x^3 - 7x^2 + 4$ „ $3x^3 + 4x^2 - 2x + 2$.
- $4x^3 + 5x^2 - 3x^2$ „ $-3x^3 - 3x^2$, and $x^3 - 2x^3$.
- $7a^3 + \frac{3}{2}a^2 - \frac{1}{2}a^3$ „ $2b^3 - c^3 + 2f^3 - \frac{1}{2}c^3$.
- What must be added to $x^3 - 2y^3 + z^3$ in order that the sum may be $3x^3 - xy - z^3$?
- To what expression must $1 - 3a + 7a^2 - 2a^3 - a^4$ be added in order that the sum may be zero?
- By how much does the sum of $ax^3, 2a^2x^2 - 3ax^3, -4x$, and $-ax^3 + 2ax - 6$ fall short of zero?
- What must be subtracted from $qr - 4rs$ in order that the remainder may be $-3pq + 7qr - 4rs$?
- Subtract $2p^3 - 3pq^2 - 2q^3$ from 1, and $1 - 2p^3$ from 0, and add the results.

CHAPTER IV.

MULTIPLICATION.

37. Extended sense of Multiplication. Multiplication, in its primary sense, is a shorter method of finding the sum of a number repeated any number of times. Thus 14×5 means $14 + 14 + 14 + 14 + 14$. But even in ordinary Arithmetic this definition fails when we are told to multiply a number by a fraction, *e.g.*, to find $4 \times \frac{3}{5}$; 4 may be repeated once, twice &c, but it is meaningless to say that 4 is repeated $\frac{3}{5}$ times. Still in Arithmetic we admit such results, as $4 \times \frac{3}{5} = \frac{4 \times 3}{5}$. For such cases a new idea of multiplication is necessary.

We get $1 \times \frac{3}{5}$, or, more shortly, $\frac{3}{5}$ by dividing unity into 5 equal parts and taking 3 such parts. Now let us do to 4 what we have done to unity to get $\frac{3}{5}$; that is, divide 4 into 5 equal parts, and take 3 such parts. In this way we get $\frac{4}{5} + \frac{4}{5} + \frac{4}{5}$, or $\frac{4+4+4}{5}$ or $\frac{4 \times 3}{5}$, which, we know, is the value of $4 \times \frac{3}{5}$. We therefore take $4 \times \frac{3}{5}$ to signify that we are to do to 4 what we have done to unity to get $\frac{3}{5}$. In like manner, to find $\frac{3}{5} \times \frac{4}{7}$, we are to do to $\frac{3}{5}$ what we do to unity to get $\frac{4}{7}$. Now $\frac{4}{7}$ is obtained by taking the seventh part of unity four times. Hence to find $\frac{3}{5} \times \frac{4}{7}$ we take the seventh part of $\frac{3}{5}$ four times. Thus $\frac{3}{5} \times \frac{4}{7} = \frac{3}{5} \times \frac{4}{7} + \frac{3}{5} \times \frac{4}{7} + \frac{3}{5} \times \frac{4}{7} + \frac{3}{5} \times \frac{4}{7} = \frac{3+3+3+3}{5 \times 7} = \frac{3 \times 4}{5 \times 7}$, which gives the usual result in Arithmetic.

This extended sense of multiplication will also cover the three following cases not admitted in Arithmetic :

$$(-4) \times 3 = -4 \times 3 = -12,$$

$$4 \times (-3) = -4 \times 3 = -12,$$

$$(-4) \times (-3) = + (4 \times 3) = +12.$$

Let us consider the last result, $(-4) \times (-3) = +4 \times 3$. We have here to do to -4 what has to be done to unity to get -3. Now we obtain -3 by repeating unity thrice and finally changing its sign; hence to find $(-4) \times (-3)$ we have to take -4 thrice, and change the sign of the result. Thus

$$(-4) \times (-3) = \{-4 + (-4) + (-4)\} \text{ with the sign changed,}$$

$$= -12 \text{ with the sign changed,}$$

$$= +12,$$

$$= +4 \times 3.$$

The student should now note the following results :

$$(+4) \times (+3) = +4 \times 3; \quad (-4) \times (+3) = -4 \times 3;$$

$$(+4) \times (-3) = -4 \times 3; \quad (-4) \times (-3) = +4 \times 3.$$

Note that here we might have taken fractional numbers as well.

In algebraical symbols the corresponding results will be :

$$\begin{aligned} (+a) \times (+b) &= +ab ; & (-a) \times (+b) &= -ab ; \\ (+a) \times (-b) &= -ab ; & (-a) \times (-b) &= +ab. \end{aligned}$$

N. B. In the abovesymbols a and b may be integral or fractional.

We now lay down the following definition of Multiplication.

Def. To multiply a by b is to do to a what has to be done to unity to obtain b .

Note. a and b may be positive or negative, integral or fractional.

38. Law of signs in Multiplication. It may be briefly expressed thus :

(1) Like signs produce +, and (2) unlike signs produce -.

39. To prove that $a \times b = b \times a$.

We shall examine in order the following cases :

- (1) When a and b are both positive integers ;
- (2) when either of them or both are positive fractions ;
- (3) when either of them is negative ;
- (4) when both are negative.

(1) Let a and b be both positive integers, e. g., 5 and 7.

$$5 = 1 + 1 + 1 + 1 + 1,$$

and since 5×7 means 5 repeated 7 times, we are to take

$(1 + 1 + 1 + 1 + 1)$ repeated 7 times ; hence

$$\begin{aligned} 5 \times 7 &= 1 + 1 + 1 + 1 + 1 \\ &\quad + 1 + 1 + 1 + 1 + 1 \\ &\quad + 1 + 1 + 1 + 1 + 1 \\ &\quad + 1 + 1 + 1 + 1 + 1 \\ &\quad + 1 + 1 + 1 + 1 + 1 \\ &\quad + 1 + 1 + 1 + 1 + 1 \\ &\quad + 1 + 1 + 1 + 1 + 1 \end{aligned}$$

Now we easily see that there are 7 units in each vertical column, and there are 5 such columns. Therefore the sum of the 5 columns may be represented as 7×5 ;

$$\therefore 5 \times 7 = 7 \times 5.$$

Thus when a and b are positive integers, $a \times b = b \times a$.

(2) Let a and b be fractional numbers; e.g., let $a = \frac{2}{3}$, and $b = \frac{4}{5}$. Now, we have seen that $\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$, and we can also find that $\frac{4}{5} \times \frac{2}{3} = \frac{4 \times 2}{5 \times 3} = \frac{8}{15}$:

$$\therefore \frac{2}{3} \times \frac{4}{5} = \frac{4}{5} \times \frac{2}{3}.$$

Similarly, $5 \times \frac{2}{3} = \frac{2}{3} \times 5$.

Hence, in this case also, $a \times b = b \times a$.

(3) Let $a = -m$, and $b = n$.

$$a \times b = (-m) \times n = -mn,$$

$$b \times a = n \times (-m) = -nm = -mn;$$

$$\therefore a \times b = b \times a.$$

(4) Let $a = -m$, and $b = -n$.

$$a \times b = (-m) \times (-n) = +mn;$$

$$b \times a = (-n) \times (-m) = +nm = +mn.$$

$$\therefore a \times b = b \times a.$$

Hence, universally, $a \times b = b \times a$.

40. The order of operation in a chain of multiplication is indifferent. For instance,

$$abc = ab \times c = bac, \because ab = ba, \text{ Art. 39,}$$

$$= bca, \because ac = ca, \text{ \&c.}$$

Similarly, $abcd = bcad = cabd = dacb, \text{ \&c.}$

41. Product of powers of the same quantity.

We know that $a^3 = a \times a \times a$,

$$\text{and } a^4 = a \times a \times a \times a;$$

$$\therefore a^3 \times a^4 = a \times a \times a \times a \times a \times a \times a, .$$

$$= a^7,$$

$$= a^{3+4}.$$

In like manner, if m and n be positive integers,

$$a^m \times a^n = a^{m+n}$$

For, $a^m = a \times a \times a \times \dots$... to m factors,

and $a^n = a \times a \times a \times \dots$... to n factors;

$\therefore a^m \times a^n = a \times a \times a \times \dots$... to m factors

$\times a \times a \times a \times \dots$... to n factors,

$= a \times a \times a \times \dots$... to $m+n$ factors,

$$= a^{m+n}.$$

\therefore also $a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{m+n+p}$, and so on.

Rule. *Add the indices of the powers.*

N.B. The indices should be added mentally.

42. Simple factors with numerical coefficients.

$$\begin{aligned}\text{Ex. 1. } 7x^2yz \times 4x^3y^2z &= 7 \times 4 \times x^2 \times x^3 \times y \times y^2 \times z \times z, \text{ Art. 40,} \\ &= 28x^{2+3} \times y^{1+2} \times z^{1+1}, \text{ Art. 41,} \\ &= 28x^5y^3z^2.\end{aligned}$$

$$\begin{aligned}\text{Ex. 2. } (-\frac{2}{3}a^2b^3c^2) \times (-\frac{1}{2}a^3bc^3) \\ &= (-\frac{2}{3}) \times (-\frac{1}{2}) \times a^2 \times a^3 \times b^3 \times b \times c^2 \times c^3 \\ &= +\frac{2}{3} \times \frac{1}{2} \times a^{2+3} \times b^{3+1} \times c^{2+3}, \text{ Art. 41,} \\ &= \frac{1}{3}a^5b^4c^5.\end{aligned}$$

Rule. *First multiply the coefficients together as in Arithmetic, and then prefix this product to the product of the letters, having regard to the rule of signs.*

EXAMPLES 15.

Multiply

1. $2a$ by $3b$.
2. $5a$ by $5a$.
3. $7ab$ by $2a$.
4. $9a^2$ by $5a$.
5. $7x^3$ by $-3x$.
6. $3a^2b$ by $-ab$.
7. $3xy$ by $-4x^2y^3$.
8. $-4x^2yz$ by $-5xy^2z^3$.
9. $-9a^4b^5c^6$ by $-6a^3b^2c^2$.
10. $5a^2bc$ by $-\frac{1}{2}a^2c$.
11. $-\frac{2}{3}x^2yz$ by $\frac{1}{4}ax$.
12. $\frac{4}{7}a^2x$ by $\frac{3}{7}a^2x^3$.
13. $-\frac{2}{3}a^m b^n$ by $-\frac{1}{4}a^m b^n$.
14. $\frac{9}{7}a^4y^m z^n$ by $\frac{1}{7}y^m z^n$.
15. $\frac{1}{3}a^2b^3$ by $-\frac{2}{3}ab^2c^2$.

Find

16. $(-a)^2$.
17. $(-ab)^2$.
18. $(-a^2x)^2$.
19. $(-2x^2yz)^2$.
20. $(5ab^2c^3x^2y^5)^2$.
21. $(-2a^2x^2y^4)^2$.
22. $(-\frac{1}{2}x^2y^3z^2w^{10})^2$.
23. $(\frac{1}{3}a^2b^3c)^2$.

43. Multiplication of several monomials.

Ex. 1. Multiply together ad , $-3ac$, $-2abc$, and $-4a^2bd^2$.

$$ad \times (-3ac) = -a \times 3a \times c \times d = -3a^2cd;$$

$$\begin{aligned}\therefore ad \times (-3ac) \times (-2abc) &= (-3a^2cd) \times (-2abc) \\ &= +3 \times 2 \times a^2 \times a \times b \times c \times c \times d \\ &= 6a^3b^2c^2d;\end{aligned}$$

$$\begin{aligned}\therefore ad \times (-3ac) \times (-2abc) \times (-4a^2bd^2) &= 6a^3b^2c^2d \times (-4a^2bd^2) \\ &= -6 \times 4 \times a^3 \times a^2 \times b \times b \times c^2 \times d \times d^2 \\ &= -24a^5b^3c^2d^3.\end{aligned}$$

The following Rules are now evident :—

(a) The sign of the product is $-$, only when the sign $-$ occurs an odd number of times, but is otherwise $+$.

(b) The factors of the product are

- (1) the product of the numerical coefficients ;
- (2) each letter raised to the power whose index is equal to the sum of the indices of that letter.

EX. 2. Find the continued product of $-x$, $\frac{2}{3}x^2y$, $-\frac{2}{3}x^3yz^2$, $-\frac{1}{2}x^2y^4z^6w^5$, and $-9yzw^2$.

Since the sign $-$ occurs an even number (4) of times, the sign of the product is $+$.

The product of the numerical coefficients $= 1 \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{2} \times 9 = 1$.

The sum of the indices of the powers of $x = 1 + 2 + 3 + 2 = 8$;

" " " " " " " " " " $y = 1 + 3 + 4 + 1 = 9$;

" " " " " " " " " " $z = 3 + 6 + 1 = 10$;

" " " " " " " " " " $w = 5 + 2 = 7$.

\therefore the required product $= + 1 \cdot x^8 y^9 z^{10} w^7 = x^8 y^9 z^{10} w^7$.

EXAMPLES 16.

Multiply together

1. $3a^3, -2b^2, -3c^2$.

2. $-2a, 2b, -3c^2$.

3. $9a^3, -2b, -4c^2, -2d$.

4. $-2p, -2q, -2r, -3s, -t$.

5. ab, a^2, b^2, c^2 .

6. $2xy, 3yz, 4x^2, zx$.

7. $-\frac{2}{3}ab, -\frac{2}{3}ac, 9bc$.

8. $\frac{2}{3}a^2xy, -5x^2yz, 3fg, -\frac{2}{3}abxy$.

9. $\frac{2}{3}lmn, 7^{\frac{1}{2}}m^2, \frac{1}{2}m^2n^2, n^2l^2$

10. $\frac{2}{3}p^2q^2r^2, -\frac{2}{3}p^3q^2r^3, -\frac{2}{3}p^4q^4r^4, -10p^5q^5r^5$.

11. $(ab)^2, (bc)^2, (ca)^2, (abc)^2$ 12. $(-abxy)^2, -(bcyz)^2, (caxz)^2$.

13. $\frac{1}{2}a^2b^2, -5a^2b^2c, -\frac{2}{3}a^2b^2c, 9c^2$ 14. $\frac{1}{2}a^2b^2, -4b^2c^2, -c^2a^2$.

15. $a^m b^m, -b^m c^m, -c^m a^m, a^m b^m c^m a^m$.

16. $xyz, -4x^m y^m z^2, -3ax^2 y^2, b)^2 z^2, -cx^m y^m$

17. $a^2 b^m, b^m c^n, c^n a^2, a^2 b^m c^n$ 18. $a^{1+m}, b^{m+n}, c^{n+1}, c^{1+m}, c^{m+n}, a^{n+1}$.

Find the value of

19. $(a^2)^3$ 20. $(-ab)^2$ 21. $(-a^2 b^3 c^4)^3$ 22. $(-2a^2 x^3)^5$.

23. $(3ab)^4$ 24. $(-\frac{2}{3}a^2 b^2 c)^4$ 25. $(-\frac{1}{2}m^2 n^2)^5$ 26. $(-2xy^2 z^3 w^2)^6$.

Find the continued product of

27. $ab, (-ab)^2, (-ab)^3$ 28. $(-xyz)^2, -(xyz)^2, (-xyz)^3$.

$$29. -(ab)^2, -(-bc)^2, -(-ca)^2. \quad 30. (2ax)^4, (-3xy)^4, (-4xz)^4.$$

$$31. \frac{1}{2}(lm)^2, 3(mn)^2, -4l^2m^2n^2.$$

$$32. \frac{1}{2}(x^2yzw)^2, -(\frac{1}{2}xy^2zw)^2, (-\frac{1}{2}xyz^2w)^2.$$

44. Multiplication of a polynomial by a monomial.

When a, b, m are positive, and m an integer,

$$(a+b)m = (a+b) + (a+b) \text{ \&c., i.e., } a+b \text{ repeated } m \text{ times,}$$

$$= a \text{ taken } m \text{ times } + b \text{ taken } m \text{ times,}$$

$$= am + bm, \text{ i.e., sum of the products of each term of the binomial by } m.$$

This result is true, even when m is fractional; for we can easily verify from Arithmetic that $(5 + \frac{2}{3}) \times \frac{3}{2} = 5 \times \frac{3}{2} + \frac{2}{3} \times \frac{3}{2}$.

$$\text{Also observe that } m(a+b) = (a+b)m, \text{ (Art. 39)} = am + bm.$$

In like manner,

$$(a-b)m = (a-b) + (a-b) + \text{\&c., i.e., } a-b \text{ repeated } m \text{ times,}$$

$$= a \text{ repeated } m \text{ times } - b \text{ repeated } m \text{ times,}$$

$$= am - bm.$$

If we now regard $-bm$ as the product of $-b$ and m , we may also here state the result as the sum of the products of each term of the binomial $a-b$ by m ; i.e.,

$$(a-b)m = a \times m + (-b) \times m = am - bm.$$

There is thus a distinct advantage in assuming $(-b) \times m = -bm$, for thereby both the products $(a+b)m$ and $(a-b)m$ are brought under the same law. We shall very soon see the advantage of assuming $(-b) \times (-m) = +bm$.

We shall, with these assumptions, have

$$(a+b) \times (-m) = a \times (-m) + b(-m) = -am - bm;$$

$$(a-b) \times (-m) = a \times (-m) + (-b) \times (-m) = -am + bm.$$

If we are to multiply $a-b+c$ by m , we may regard $a-b$ as a single quantity and equal to p . Thus

$$(a-b+c)m = (p+c)m, \text{ putting } p \text{ for } a-b,$$

$$= pm + cm,$$

$$= (a-b)m + cm, \text{ putting } a-b \text{ for } p,$$

$$= am - bm + cm, \text{ i.e., sum of the products of every}$$

term of the polynomial by m .

Similarly it may be shewn that

$$(a-b+c+d-e)m = am - bm + cm + dm - em.$$

Hence we have the following rule :

Rule. Collect the partial products of each term of the polynomial by the monomial.

Ex. Multiply $x^2y - x^2z + 4yz^2$ by $2x^2yz^2$.

$$\begin{aligned}(x^2y - x^2z + 4yz^2)2x^2yz^2 &= x^2y \cdot 2x^2yz^2 - x^2z \cdot 2x^2yz^2 + 4yz^2 \cdot 2x^2yz^2 \\ &= 2x^4y^2z^2 - 2x^4yz^3 + 8x^2y^2z^4.\end{aligned}$$

This operation is also shown thus :

$$\begin{array}{r}x^2y - x^2z + 4yz^2 \\ 2x^2yz^2 \\ \hline 2x^4y^2z^2 - 2x^4yz^3 + 8x^2y^2z^4.\end{array}$$

Note. Since $(x^2y - x^2z + 4yz^2)2x^2yz^2 = 2x^2yz^2(x^2y - x^2z + 4yz^2)$, Art. 39, the product of a monomial by a polynomial comes under the same rule as the product of a polynomial by a monomial.

EXAMPLES 17.

Multiply :—

1. $x + 4$ by 3.
2. $2x - 3y$ by $4x$.
3. $ab - ac$ by a^2c .
4. $7a^3 - 3b^3$ by $10a^2bc^3$.
5. $2x^3 - 3y^3$ by ab .
6. $5p^2q - 2pq^2 + 3$ by $2pq$.
7. $5a^2b - 7ab + 2a$ by $9a^3bc$.
8. $-\frac{2}{3}xyz^2$ by $6x^2y^2z^4 + 2xy^2z^5 - \frac{1}{2}y^7$.
9. $-\frac{1}{2}a^2bcd$ by $\frac{1}{3}a^2bc - \frac{5}{4}b^2cd - \frac{3}{2}a^2bd - abcd$.
10. $-3abcd$ by $4a^2b^3cd - 7a^3b^2c^2d + 5abcd^4$.
11. $7x^3 - 3x(2x^2 - 1) + 4x^2(5x - 1)$.
12. $(2ax^2 - 12x + 1)x^2 - 3x(4x^2 - 2x^2 + 6) - 4(2x^3 - 3)$.
13. $3bx(b - 1) - 3b(x - 1) - 7b(b - x)$.
14. $a(b - c) + b(c - a) + c(a - b)$.
15. $ab(a - b) + bc(b - c) - a^2(b - c) - b^2(c - a)$.

45. General case. We know from Art. 44 that

$$(a + b)m = am + bm.$$

$$\text{Let } m = c + d.$$

$$\text{Then } (a + b)(c + d) = a(c + d) + b(c + d),$$

$= ac + ad + bc + bd$, i.e., sum of the products of every term of one factor by every term of the other.

If $c - d$ be put for m in the equality, $(a + b)m = am + bm$,

$$(a + b)(c - d) = a(c - d) + b(c - d),$$

$= ac - ad + bc - bd$, which may be regarded as the sum of the products of every term of one factor by every term of the other, assuming that $a \times (-d) = -ad$, and $b \times (-d) = -bd$. See Art. 37.

Again, we know that $(a-b)m = am - bm$. Put $c-d$ for m .

Then $(a-b)(c-d) = a(c-d) - b(c-d)$

$$= ac - ad - (bc - bd)$$

$$= ac - ad - bc + bd, \text{ which result may be interpreted as before, if we are prepared to assume that}$$

$a \times (-d) = -ad$, $(-b) \times c = -bc$, and $(-b) \times (-d) = +bd$.

$$a \times (-d) = -ad, (-b) \times c = -bc, \text{ and } (-b) \times (-d) = +bd.$$

The advantage of the extended sense of multiplication with which we began is now apparent, as it readily leads to the following general rule:

The product of two factors is the sum of the partial products of every term of one factor by every term of the other, like signs producing +, and unlike signs producing -

Ex 1. Multiply $x+a$ by $x+b$.

$$(x+a)(x+b) = \text{product by } x \text{ of each term of } x+a$$

$$+ \text{product by } b \text{ of each term of } x+a,$$

$$= (x+a)x + (x+a)b,$$

$$= x^2 + ax + bx + ab,$$

$$= x^2 + (a+b)x + ab.$$

The preceding operation is usually shown thus:

	$x + a$	
	$x + b$	
The like terms, ax ,	$x^2 + ax$	= product by x of $x+a$.
bx , should be in	$bx + ab$	= product by b of $x+a$.
the same column	<u>$x^2 + (a+b)x + ab$</u>	= whole product.

Ex. 2. Simplify $(2x-3)(3x-4)$.

$$(2x-3)(3x-4) = (2x-3)3x - (2x-3)4$$

$$= 6x^2 - 9x - (8x - 12)$$

$$= 6x^2 - 9x - 8x + 12$$

$$= \underline{6x^2 - 17x + 12.} \text{ Ans.}$$

The work may also be shown thus:

	$2x - 3$	
	$3x - 4$	
	$6x^2 - 9x$	= product by $3x$.
	$-8x + 12$	= product by -4 .
	<u>$6x^2 - 17x + 12$</u>	= whole product.

Ex. 3. Multiply $a^3 - ab + b^3$ by $a + b$.

$$\begin{aligned}(a^3 - ab + b^3)(a + b) &= (a^3 - ab + b^3)a + (a^3 - ab + b^3)b \\ &= a^4 - a^2b + ab^3 + a^3b - ab^2 + b^4 \\ &= \underline{a^4 + b^4}. \quad \text{Ans.}\end{aligned}$$

The work may also be shown thus :

$$\begin{array}{r} a^3 - ab + b^3 \\ a + b \\ \hline a^4 - a^2b + ab^3 \\ + a^3b - ab^2 + b^4 \\ \hline a^4 + b^4 \end{array}$$

Like terms are placed in the same column.

N. B. Note here the arrangement in descending powers of a and ascending powers of b .

Ex. 4. Multiply $2x^4 + 3x^3y - 4x^2y^2 - 5xy^3 + y^4$ by $x^2 - 2y^2$.

$$\begin{array}{r} 2x^4 + 3x^3y - 4x^2y^2 - 5xy^3 + y^4 \\ x^2 - 2y^2 \\ \hline 2x^6 + 3x^5y - 4x^4y^2 - 5x^3y^3 + x^2y^4 \\ - 4x^4y^3 - 6x^3y^4 + 8x^2y^5 + 10xy^6 - 2y^6 \\ \hline 2x^6 + 3x^5y - 8x^4y^3 - 11x^3y^4 + 9x^2y^5 + 10xy^6 - 2y^6 \end{array}$$

EXAMPLES 18.

Find the product of

- | | |
|---|--|
| 1. $x + 1$ and $x + 7$. | 11. $a^2 + 5a + 4$ and $a + 3$. |
| 2. $x + 2a$ and $x + 3a$. | 12. $a^2 - 2a + 3$ and $3a + 5$. |
| 3. $x + 2a$ and $x - 3a$. | 13. $a^3 - 7a + 5$ and $2a - 7$. |
| 4. $x - 2a$ and $x - 3a$. | 14. $x^2 + 4xy + 4y^2$ and $x + 2y$. |
| 5. $6 - a$ and $a + 3$. | 15. $x^3 - 2x + 5$ and $x^2 - 2$. |
| 6. $7 - x$ and $-x - 9$. | 16. $x^4 - x^3y + x^2y^2 - xy^3 + y^4$ and $x + y$. |
| 7. $ax + by$ and $ax - by$. | 17. $20x^2 - 32xy + 24y^2$ and $5y - 4x$. |
| 8. $a + b$ and $a - b$. | 18. $x^2 - b + ax$ and $ax - b$. |
| 9. $a + b$ and $a + b$. | 19. $a^5 - b^3 + 2a^3b$ and $a^2 - 2b$. |
| 10. $\frac{1}{2}ab - 2cd$ and $\frac{2}{3}ab + 3cd$. | 20. $x^3 + x + x^2 + 1$ by $x - 1$. |

Simplify

- | | |
|-------------------|---|
| 21. $(a + b)^2$. | 23. $(2a - 13)(a + 7)$. |
| 22. $(a - b)^2$. | 24. $(\frac{1}{2}x - 3y)(x - \frac{2}{3}y)$. |

25. $(2x-1)(2x-3)-(x-3)(x-7)$.
 26. $a^2(a-2) + (2a-1)(a+1)a + a(a^2-7)$.
 27. $(x^2+x+1)(x-1) - (x^2-x+1)(x^2+1)$.
 28. $(x^2-4xy+4y^2)(x+2y) - (x^2+4xy+4y^2)(x-2y)$.

EXAMPLES WORKED OUT.

Ex. 1. Multiply $x^3 - ax + a^2$ by $x^2 + ax + a^2$.

$$\begin{array}{r}
 x^3 - ax + a^2 \\
 \times \quad x^2 + ax + a^2 \\
 \hline
 x^5 - ax^3 + a^2x^2 \\
 + ax^3 - a^2x^2 + a^3x \quad = \text{product by } x^2 \\
 + a^2x^2 - a^2x + a^4 \quad = \text{product by } ax \\
 \hline
 x^5 \quad + a^3x^2 \quad + a^4 = \text{product by } a^2 \\
 \hline
 x^5 \quad + a^3x^2 \quad + a^4 = \text{whole product.}
 \end{array}$$

Ex. 2. Multiply $8x^3 - 3x - 2x^2$ by $3x^2 + 1 - 5x$.

Re-arrange the terms in descending powers of x .

$$\begin{array}{r}
 8x^3 - 2x^2 - 3x \\
 \times \quad 3x^2 - 5x + 1 \\
 \hline
 24x^5 - 6x^4 - 9x^3 \\
 - 40x^4 + 10x^3 + 15x^2 \\
 + 8x^3 - 2x^2 - 3x \\
 \hline
 24x^5 - 46x^4 + 9x^3 + 13x^2 - 3x = \text{required product.}
 \end{array}$$

N.B. The re-arrangement of terms, though not necessary, is very convenient, as it renders the collection of like terms more easy.

EXAMPLES 19.

Multiply

1. $2x^2 + 2x + 1$ by $2x^3 - 2x + 1$, and $3x^2 - 2x - 2$.
2. $x^3 - 4x^2 + 5x - 24$ by $x^2 - 4x - 5$, and $12x^2 - 1$.
3. $4a^2 - 3bx + 5abx^2$ by $a^2 + bx - a^2x^2$, and $1 - 2ax + x^2$.
4. $x^2 - 4xy^2 + 2x^2y$ by $xy - 3y^2 + 2x^2$, and $y^2 - 4x^2$.
5. $a^3 - ab + 2b^2$ by $b^2 + ab - 2a^2$, and $2a^2 + 3b^2 - 4ab$.
6. $ax^2 - bx + 2c$ by $ax + b$, and $px + 2q$.
7. $3x^2y - 2x^2y^2 + xy^2$ by $x^2 + y^2$, and $2x^2 - xy - 5y^2$.
8. $ax^2 - bx + c$ by $x^2 - x + 1$, and $lx^2 + mx + n$.
9. $5x - 3x^2 - 2x^3 - 4x^4$ by $x - 4$, and $2 - 3x + 4x^2$.
10. $a^3 + 4a^2b + 2ab^2 + b^3$ by $a^2 - 4a^2b + 2ab^2 - b^2$.

11. $a^2 - b^2 - c^2 - 2bc$ by $a^2 + b^2 + 2bc - c^2$.
 12. $\sqrt{5xy - x^2 - y^2}$ by $\sqrt{5xy + x^2 + y^2}$, and $\sqrt{5xy - x^2 + y^2}$.
 13. $5m^2 + 3 - 4m$ by $m^2 - 1$, and $5 - 4m + 3m^2$.
 14. $a^2 + b^2 + c^2 + ab + ac - bc$ by $a - b - c$.
 15. $a^m + b^m$ by $a^m + b^m$, and $x^n + y^n$ by $x^n - y^n$.

Simplify

16. $(x^2 - x^2 + x - 1)(x^2 + x^2 + x + 1)$
 17. $(a + b)(a^2 - ab + b^2) + (a - b)(a^2 + ab + b^2)$.
 18. $(b - c)(a - b - c) + (c - a)(b - c - a) + (a - b)(c - a - b)$.
 19. $(b + c + d - a)(a - b + c + d) + (a + b - c + d)(a + b + c - d)$.

Find the product of

20. $x^4 + \sqrt{6x^2y^2 + y^4}$ and $x^4 - \sqrt{6x^2y^2 + y^4}$.
 21. $x^3 + 4x^2 - 5x - 3$ and $x^3 - 2x - 3$
 22. $2x^4 - 3x^3 - x + 1$ and $4x^3 - 3x + 2$.
 23. $ax^4 - bx^2 + c$ and $ax^2 + bx + c$.
 24. $2a^4 - a^2 + a^2 - a + 3$ and $2a^2 - a - 1$.
 25. $a^3 + b^3 + c^3 + bc - ca + ab$ and $a - b + c$.
 26. $a^3 + 4b^3 + 9c^3 + 6bc - 3ca + 2ab$ and $a - 2b + 3c$.
 27. $x^2 - xy + y^2 + 2x + 2y + 4$ and $x + y - 2$.
 28. $x^3 + y^3 + x^2y - xy^2$ and $x^2 - y^2 + x^2y + xy^2$.
 29. $x^4 - 4x^2y^2 + 4y^4$ and $x^2 + 2xy + 4y^2$.
 30. $1 - 3x + 5x^2 - 3x^3 - 11x^4 + 15x^5$ and $1 + 3x + 4x^2$.
 31. $a - bx + cx^2 + dx^3 - ex^4$ and $1 - x + x^2 - x^3$.
 32. $a^7 - a^6b + a^5b^2 - a^4b^3 + a^3b^4 - a^2b^5 + ab^6 - b^7$ by $a + b$.
 33. $x^3 - 99x^2 + x - 29$ by $x^5 - 17x^4 + 105x^3 - 19x^2 + 23x - 41$.
 34. $\frac{x}{2} - \frac{2}{3}$ by $2x - 4$, $\frac{x}{2} - \frac{3}{2}$, and $2x - \frac{3}{2}$.
 35. $\frac{1}{2}x^2 + \frac{1}{2}x + 1$ by $\frac{1}{2}x + \frac{1}{2}$, and $\frac{1}{2}x - \frac{1}{2}$.
 36. $\frac{1}{2}x^3 - 3xy + 9y^2$ by $\frac{1}{2}x - 3y$, and $x^2 + \frac{1}{2}xy - \frac{3}{2}y^2$.
 37. $\frac{1}{2}x^3 - \frac{1}{2}x^2y + \frac{1}{2}xy^2$ by $\frac{1}{2}xy + \frac{1}{2}x^2$.
 38. $\frac{1}{2}a^4x^4 + \frac{1}{2}a^3bx^2y + \frac{1}{2}a^2b^2x^2y^2 + \frac{1}{2}ab^3xy^3$ by $\frac{1}{2}ax - \frac{1}{2}by$.
 39. $a^2x^3 - abxy + b^2y^2$ by $2x^2 - 3xy + 4y^2$.
 40. $a^3b^3 - 2a^2b^2c^2 + 3ab^2c^2 - 4c^4$ by $a^3 - ab + c^2$.
 41. $a^4l^3 - 5a^3l^2 + 6a^2l^2 - 4al^3 - 2l^7$ by $a^3 - al + l^2$.
 42. $a^2x^2y^2 - axy^2 - 2y^4 + 3a^4$ by $x^2y^2 - axy + a^2$.

43. $2x^6 - 3x^5 + 7x^4 - 2x^3 - x - 1$ by $1 - 3x - 4x^2 + 7x^4$.
 44. $3x^6 - 11x^4 + 16x^3 - 5x - 8$ by $2x^3 - 3x^2 - 4x + 4$.
 45. $4x^4 - 3x^3 + 9x^2 - 11x - 15$ by $-4x^3 + 3x^2 - 1$.
 46. $9a^5 - 11a^3 + 21a^2 - 16$ by $2a^3 - a^2 - 5$.
 47. $42x^3 - 19x^2y - 21xy^2 + 10y^3$ by $10y^2 - 21x^2 - xy$.
 48. $3x^3 - 14x^2 + 17x - 6$ by $2x^2 + x^2 - 2x + 8$.
 49. $b^2c^2 + c^2a^2 - a^2b^2 - 2abc^2$ by $bc - ca - ab$.
 50. $\frac{x^2}{3} + \frac{2x^2}{5} - x - 2$ by $\frac{x^2}{3} - \frac{2x^2}{5} + x - 2$.
 51. $\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{2}$ by $x^2 - \frac{2}{3}x^2 - \frac{1}{3}x + 1$.
 52. $\frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{1}{4}x^2 - \frac{1}{2}$ by $3x^4 - 2x^3 + \frac{2}{3}x^2 - 3$.
 53. $\frac{a^4}{5} - \frac{a^2b}{4} - \frac{a^2b^2}{3} + 1$ by $10 + \frac{10a^2b^2}{3} + \frac{5a^2b}{2} + 2a^4$.
 54. $l^3m^3 - 2l^2m^3 + lm$ by $l^2m^2 + 2lm + 1$.
 55. $x^3y^3 - 2x^2y^3 + xy$ by $x^2y^3 + 2xy - 1$.
 56. $x^6 - y^6 + 3x^2y - 4xy^5$ by $2x^2 - 3xy + y^2$.
 57. $7x^{14} - 4x^{10} + 3x^6 - x^3 + 3$ by $4x^2 - 5x + 1$.
 58. $2x^{30} - 3x^{15} - 11x^{10} + 9x^6 - 7$ by $3x^4 - 7x^2 - 13$.

46. Some important results The following results of multiplication should be committed to memory.

$(a+b)^2$	$= a^2 + 2ab + b^2$...	i.
$(a-b)^2$	$= a^2 - 2ab + b^2$...	ii.
$(a+b)(a-b)$	$= a^2 - b^2$...	iii.
$(a+b)(a^2 - ab + b^2)$	$= a^3 + b^3$...	iv.
$(a-b)(a^2 + ab + b^2)$	$= a^3 - b^3$...	v.

Ex. 1. Find 1512^2 and 1695^2 .

$$\begin{aligned} 1512^2 &= (1500 + 12)^2 = 1500^2 + 2 \times 1500 \times 12 + 12^2 \\ &= 2250000 + 36000 + 144 \\ &= 2286144. \text{ Ans.} \end{aligned}$$

$$\begin{aligned} 1695^2 &= (1700 - 5)^2 = 1700^2 - 2 \times 5 \times 1700 + 5^2 \\ &= 2890000 - 17000 + 25 \\ &= 2873025. \text{ Ans.} \end{aligned}$$

Ex. 2. Expand $(x+y-z)^2$.

Let a stand for $x+y$; then $x+y-z = a-z$.

$$\begin{aligned} \therefore (x+y-z)^2 &= (a-z)^2 = a^2 - 2az + z^2, & \text{Formula ii,} \\ &= (x+y)^2 - 2(x+y)z + z^2, & \because a = x+y, \\ &= (x^2 + 2xy + y^2) - 2xz - 2yz + z^2, & \text{Formula i,} \\ &= x^2 + y^2 + z^2 - 2yz - 2zx + 2xy. & \text{Ans.} \end{aligned}$$

N.B. In practice we omit the first step and begin at once with $\{(x+y)-z\}^2 = (x+y)^2 - 2(x+y)z + z^2$, &c.

Ex. 3. Find the product of $x-y+z$ and $x+y-z$.

$$\circ \quad x-y+z = x-(y-z), \text{ and } x+y-z = x+(y-z).$$

If, then, we put a for $y-z$, we have

$$x-y+z = x-a, \text{ and } x+y-z = x+a.$$

$$\begin{aligned} \therefore (x-y+z)(x+y-z) &= (x-a)(x+a), \\ &= x^2 - a^2, & \text{Formula iii,} \\ &= x^2 - (y-z)^2, & \because a = y-z, \\ &= x^2 - (y^2 - 2yz + z^2), & \text{Formula ii} \\ &= x^2 - y^2 - z^2 + 2yz. & \text{Ans.} \end{aligned}$$

N.B. In practice we begin at once thus :

$$(x-y+z)(x+y-z) = \{x-(y-z)\}\{x+(y-z)\} = x^2 - (y-z)^2 = \&c.$$

EXAMPLES 20.

Multiply out or otherwise simplify

1. $(x+y)^2$; $(a+2b)^2$.
2. $(2a+3b)^2$; $(4a+5x)^2$.
3. $(7x+8y)^2$; $(ax+by)^2$.
4. $(x-2y)^2$; $(3x-4y)^2$.
5. $(10a-3)^2$; $(1-12x)^2$.
6. $(6l-5m)^2$; $(2la-5mb)^2$.
7. $(x+a)^2 + (x-a)^2$.
8. $(p+2q)(p-2q)$.
9. $(4a+5b)(4a-5b)$.
10. $(7x+3y)(7x-3y)$.
11. $\{(a^2+b^2)-ab\}\{(a^2+b^2)+ab\}$.
12. $(a-b+c)(a+b-c)$.
13. $(a+b+c+d)(a+b-c-d)$.
14. $(a-b+c-d)(a+b+c+d)$.
15. $(a-b-c+d)(a+b-c-d)$.
16. $(a-b+c-d)(a-b-c+d)$.
17. $(x+\sqrt{xy}+y)(x-\sqrt{xy}+y)$.
18. $(2x+3y)(2x-3y)+9y^2$.
19. $(5x^2+3x-1)(5x^2+3x-1)$.
20. $(5x^2+3x-1)(5x^2+3x+1)$.
21. $(5x^2+3x+1)(5x^2-3x-1)$.
22. $(5x^2+3x-1)(5x^2-3x+1)$.
23. $(x^5-3ax^2+3a^2x-a^3)(x^5+3ax^2+3a^2x+a^3)$.
24. $(x+y)^2 + (x-y)^2 + 2(x+y)(x-y)$.
25. $(x+y)^2 + (x-y)^2 - 2(x+y)(x-y) - 4y^2$.
26. $(a-b)^2 + (b+c)^2 + 2(a-b)(b+c) - (a-c)^2$.
27. $(x+y+z)^2 + (x-y+z)^2 - 2\{(x+z)^2 - y^2\}$.

Find the product of

28. $(a+b)x - (a-b)y$ by $(a-b)x + (d+b)y$.

29. $x^2 + \sqrt{2xy} + y^2$ by $x^2 - \sqrt{2xy} + y^2$.

30. $x^2 + \sqrt{2xy} + y^2$ by $x^2 + \sqrt{2xy} - y^2$.

31. $a^2b^2 - 2ab\sqrt{pq} + pq$ by $a^2b^2 + 2ab\sqrt{pq} + pq$.

32. $\frac{l^2}{5} + \frac{l}{3} - \frac{1}{2}$ by $\frac{l^2}{5} - \frac{l}{3} + \frac{1}{2}$, and $\frac{l^2}{5} - \frac{l}{3} - \frac{1}{2}$.

33. $\frac{a^2b^2}{4} + \frac{ab}{3} - 1$ by $\frac{a^2b^2}{4} - \frac{ab}{3} + 1$.

34. $x^2 - xy + y^2$ by $x + y$.

35. $x^2 + xy + y^2$ by $x - y$.

36. $a^2b^2 - abc^2 + c^4$ by $ab + c^2$.

37. $a^2b^2 + ab^2c^2 + c^4$ by $ab - c^2$.

38. $a^4x^2y^2 - a^2xyz^2 + z^4$ by $a^2xy + z^2$.

39. $a^4b^4 + a^2b^2mn + m^2n^2$ by $a^2b^2 - mn$.

40. $a^4 - a^2b^2 + b^4$ by $a^2 + b^2$.

41. $a^4b^4p^2q^2 + a^2b^2pq + 1$ by $a^2b^2pq - 1$.

42. $(a^3 - a + 1)(a + 1)$ by $(a^3 + a + 1)(a - 1)$.

43. $(a^2 - ab + b^2)(a^2 + ab + b^2)$ by $(a + b)(a - b)$.

Simplify

44. $(a + b - c)(a - b) - (a - b + c)(a + b)$.

45. $(a + b)(a^2 - ab + b^2) + (a - b)(a^2 + ab + b^2)$.

46. $(a^3 + b^3 - c^3)(a^3 - b^3 + c^3) + (b - c)^2(b + c)^2$.

47. $(a^2 - 4a + 4)(a^2 + 4a + 4) - a^2(a^2 - 8)$.

48. $(x^4 + a^2x^2 + a^4)(x^4 - a^2x^2 + a^4) - (x^4 + a^4)(x^4 - a^4)$.

49. $(x^4 - y^4)(x^4 + x^2y^2 + y^4) - (x^4 + y^4)(x^4 - x^2y^2 + y^4)$.

Find the squares of

50. 46 ; 405 ; 97000.

51. 9998 ; 7983 ; 19987 ; 899983.

47. Continued product. The following examples with their solutions will illustrate the way in which the continued product of any number of given factors is found.

Ex. 1. Find the continued product of $x - a$, $x - b$, $x - c$.

First find the product of any two of the given factors, say $x - a$ and $x - b$; this product multiplied by the third factor $x - c$ will, of course, give the final result.

$$\begin{array}{r}
 x - a \\
 \underline{x - b} \\
 x^2 - ax \\
 \quad - bx + ab \\
 \hline
 \text{Multiply this} \quad x^2 - (a+b)x + ab \\
 \text{by} \quad \underline{x - c} \\
 x^3 - (a+b)x^2 + abx \\
 \quad - cx^2 + (ac+bc)x - abc \\
 \hline
 \text{Final product} = \underline{x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc}
 \end{array}$$

Ex 2. Multiply together $a-b$, $a+b$, a^2+b^2 , a^4+b^4 .

$$\begin{array}{r}
 a-b \\
 \underline{a+b} \\
 a^2-ab \\
 \quad +ab-b^2 \\
 \hline
 \text{Multiply this} \quad a^2-b^2 \\
 \text{by} \quad \underline{a^2+b^2} \\
 a^4-a^2b^2 \\
 \quad +a^2b^2-b^4 \\
 \hline
 \text{Multiply this} \quad a^4-b^4 \\
 \text{by} \quad \underline{a^4+b^4} \\
 a^8-a^4b^4 \\
 \quad +a^4b^4-b^8 \\
 \hline
 a^8-b^8 \text{ required product.}
 \end{array}
 \quad \begin{array}{l}
 \text{Shortly thus :} \\
 (a-b)(a+b) = a^2-b^2 \\
 \therefore (a-b)(a+b)(a^2+b^2) \\
 = (a^2-b^2)(a^2+b^2) \\
 = (a^2)^2 - (b^2)^2 \\
 = a^4-b^4 \\
 \therefore (a-b)(a+b)(a^2+b^2)(a^4+b^4) \\
 = (a^4-b^4)(a^4+b^4) \\
 = (a^4)^2 - (b^4)^2 \\
 = a^8-b^8.
 \end{array}$$

Ex. 3. Find $(a+b)^3$ in powers of a and b .

$$\begin{array}{r}
 a+b \\
 \underline{a+b} \\
 a^2+ab \\
 \quad +ab+b^2 \\
 \hline
 \text{Multiply this} \quad a^2+2ab+b^2 \\
 \text{by} \quad \underline{a+b} \\
 a^3+2a^2b+ab^2 \\
 \quad +a^2b+2ab^2+b^3 \\
 \hline
 a^3+3a^2b+3ab^2+b^3 = \text{reqd. expression.}
 \end{array}
 \quad \begin{array}{l}
 \text{Observe that} \\
 (a+b)^3 = (a+b)(a+b)(a+b), \\
 \text{and not } = a^3+b^3.
 \end{array}$$

* Similarly, $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

Note. The above results should all be carefully remembered.

EXAMPLES 21.

Find the product :

1. $(ax + by)(ax - by)(a^2x^2 + b^2y^2)(a^4x^4 + b^4y^4)$.
2. $(a + b)(a - b)(a + 2b)(a - 2b)$.
3. $(3x - 4y)(x - 2y)(3x + 4y)(x + 2y)$.
4. $(x - a)(x + a)(x^2 + a^2)(x^4 + a^4)(x^8 + a^8)$.
5. $(1 - ax)(1 - bx)(1 - cx)$. 6. $(1 - x + x^2)(1 + x + x^2)(1 - x^3 + x^4)$.
7. $(a^2 - ax + x^2)(a^2 + ax + x^2)(a^4 - a^2x^2 + x^4)$.
8. $(a + b)(b + c)(c + a)$. 9. $(x + 2)(x + 3)(4 - x)(5 - x)$.

Find the continued product of

10. $x - a, x - b, x - c$, and $x - d$. 11. $a + 1, a + 2, a + 3$ and $a + 4$.
12. $a - b, b - c$ and $c - a$. 13. $a + b, b + c, c + a$.
14. Expand $(3x + 2y)^2, (ax - by)^2$, and $(x - y)^2(x + y)^2$.

The following multiplications should be performed mentally.

- | | |
|----------------------------|--------------------------------|
| 15. $(x + 1)(x + 2)$. | 27. $(15a + 7)(11a + 6)$ |
| 16. $(x + 7a)(x + 5a)$ | 28. $(9x + 20)(12x + 7)$. |
| 17. $(x + 8)(x - 1)$. | 29. $(8y - 21)(5y + 6)$. |
| 18. $(x - 5y)(x + 2y)$. | 30. $(12x - 5y)(14x + 9y)$. |
| 19. $(x - 7)(x - 11)$. | 31. $(6a + 11b)(11a - 25b)$. |
| 20. $(x - a)(x + b)$. | 32. $(3y - 5x)(14y - 15x)$. |
| 21. $(x - 5a)(x - 7b)$. | 33. $(8x + 13y)(13x + 8y)$. |
| 22. $(2x + 3)(3x + 5)$. | 34. $(21 - x)(x + 8)$. |
| 23. $(7x + 9)(5x + 3)$. | 35. $(7 - 2a)(5a + 9)$. |
| 24. $(9x - 11)(8x - 7)$. | 36. $(ax + b)(cx + d)$. |
| 25. $(5x + 11)(6x + 5)$ | 37. $(a - bx)(c - dx)$. |
| 26. $(11x - 4)(13x + 3)$. | 38. $(2ax - 3by)(4ax + 5by)$. |

CHAPTER V.

DIVISION.

48. Def. Division is the operation which is the inverse of **Multiplication**. In multiplication, we are given two factors and are required to find their product; in division, we are given the product of two factors as well as one of them, and have got to find the other. The given product is called **dividend**, the given factor **divisor**, and the required factor **quotient**.

49. Rule of signs Since ab is the product of a and b ,

$$\therefore ab \div a = b.$$

$$\text{Also, } -ab = (-a) \times b; \quad \therefore -ab \div (-a) = b.$$

Thus when the dividend and divisor are of the same sign, the sign of the quotient is +:

$$\text{Again, since } -ab = a \times (-b), \quad \therefore -ab \div a = -b;$$

$$\text{and since } ab = (-a) \times (-b), \quad \therefore ab \div (-a) = -b$$

Thus when the dividend and divisor are of opposite signs, the sign of the quotient is -.

Hence we have the same rule for signs in multiplication as well as in division:

Like signs produce +, and unlike signs produce -.

50. Powers of the same quantity. When the dividend and divisor are powers of the same quantity, the quotient is also a power of that quantity, and the index of the power in the quotient = that in the dividend minus that in the divisor.

$$\therefore a^5 = a^3 \times a^2, \quad \therefore a^5 \div a^3 = a^2 = a^{5-3}.$$

More generally, since $a^m = a^{m-n} \times a^n$, $\therefore a^m \div a^n = a^{m-n}$; here m being the index of the power in the dividend, and n that in the divisor, $m-n$ is the index of the power in the quotient.

It should be noted that $a^m \div a^m = 1$. But according to the above rule, $a^m \div a^m = a^{m-m} = a^0$; hence we get the curious result $a^0 = 1$. This result will be dealt with more fully afterwards.

51. Division of one monomial by another.

Rule. The index of the power of each letter in the quotient is found by subtracting the index of the power of that letter in the divisor from the same in the dividend. To the result thus obtained, prefix with its proper sign the quotient of the coefficient of the dividend by that of the divisor.

Note. Observe that the quotient $a \div b$ is generally denoted as $\frac{a}{b}$.

Ex. 1. Divide $-49a^6b^0x^7$ by $-7a^2b^4x^6$.

$$\text{The quotient} = \frac{-49a^6b^0x^7}{-7a^2b^4x^6} = 7a^{6-2}b^{0-4}x^{7-6} = 7a^4b^4x.$$

Ex. 2. Divide $-32a^5b^7cx^3$ by $5a^3b^5x^3$.

$$\text{The quotient} = \frac{-32a^5b^7cx^3}{5a^3b^5x^3} = -\frac{32}{5}a^2b^2c, \quad \because \frac{x^3}{x^3} = 1.$$

EXAMPLES 22.

Divide

1. x by $-x$.
2. $-a^3$ by a .
3. $4a^3$ by 4 , and $-4a$.
4. $\frac{1}{2}x^5$ by $\frac{1}{3}x^3$.
5. $\frac{1}{3}x^7$ by $\frac{1}{2}x^4$.
6. $-4a^3$ by $-a^2$ and $4a$.
7. $-35x^4y$ by $3x^2$.
8. $4ab$ by $-a$, and by $-ab$.
9. $25a^7b^5c^2$ by $15a^2bc$.
10. $-24p^2q$ by $-20p^2$, and by $16pq$.
11. $-168a^3x^3y^7z^{10}$ by $\frac{1}{2}ax^2s^5$.
12. $-132abcx^mym^ms$ by $-55abx^ny^n$.
13. $-25x^3y^4z^7$ by $5x^2y^4z^7$, $-10x^3s^3$, $-15x^3y^3$, $27xy^2s^4$, and $30x^2s^7$.
14. $2la^2b^3c^3$ by $la^2b^3c^3$, $-a^2b^3$, $-2^2a^2c^2s$, $2ma^2b^3$, and $-mb^3c^3$.
15. $p^{12}m^8n^{48}$ by $q^{12}r^{12}m^{24}n^{24}$, $r^{12}n^{24}$, and $s(lm)^3$.
16. $(2ax)^3(by)^3(-cx)^3$ by $a^3b^3c^3$, $-2xy^2z^2$, $-4abxyz$, and $(xyz)^3$.
17. $-(lx)^3(my)^3(nz)^3$ by $-\frac{2}{3}l^3m^3n^3$, $-\frac{2}{3}lmnxyz$, and $\frac{2}{3}(lmnxyz)^3$.
18. $(\frac{1}{2}ab)^3(\frac{2}{3}bc)^3(\frac{3}{2}ca)^3$ by $(\frac{1}{2}ab)^3(\frac{2}{3}bc)^3(\frac{3}{2}ca)^3$, and $-\frac{1}{2}ab^3\frac{2}{3}bc^3\frac{3}{2}ca^3$.

52. Division of a polynomial by a monomial.

We know that $(a+b-c+d-e)x = ax+bx-cx+dx-ex$;

$$\therefore \frac{ax+bx-cx+dx-ex}{x} = a+b-c+d-e.$$

$$\text{But } \frac{ax}{x} + \frac{bx}{x} + \frac{-cx}{x} + \frac{dx}{x} + \frac{-ex}{x} = a+b-c+d-e;$$

$$\therefore \frac{ax+bx-cx+dx-ex}{x} = \frac{ax}{x} + \frac{bx}{x} + \frac{-cx}{x} + \frac{dx}{x} + \frac{-ex}{x}.$$

Hence the following Rule:

Divide each term of the polynomial by the monomial, and take the sum of the partial quotients as the complete quotient.

Ex. 1. Divide $4x^3 - 52ax^3 + 24x$ by $-4x$.

$$\text{The quotient} = \frac{4x^3}{-4x} + \frac{-52ax^3}{-4x} + \frac{24x}{-4x}$$

$$= -x^2 + 13ax - 6. \text{ Ans}$$

Note. The student should carefully remember that

$$\text{Quotient} = \text{Dividend} \div \text{Divisor};$$

$$\text{Dividend} = \text{Divisor} \times \text{Quotient};$$

$$\text{Divisor} = \text{Dividend} \div \text{Quotient}.$$

EXAMPLES 23.

Divide

1. $x^2 - xy$ by x .
2. $x^2 - (3xy$ by $-x$.
3. $4x^5 - 5x^3$ by $2x^2$.
4. $9a^3 - 4ab$ by 4 .
5. $2x^3 - 3ax$ by 2 .
6. $6a^2b^3 + 9ab^3$ by $3ab$.
7. $2ax^2 - 7abx$ by $3x$.
8. $4l + 8m - 16n + 24p$ by 4 .
9. $48a^2b^5 - 36a^3b^7 + 144ab^9$ by $12ab^3$, and $\frac{1}{2}ab^5$.
10. $\frac{1}{2}x^7 - \frac{3}{4}x^6y + \frac{1}{6}x^5y^2 - \frac{2}{3}x^4y^3$ by $-\frac{2}{3}x^4$, and $\frac{1}{6}x^4$.
11. $12x^4y^2z^3 - 3x^2y^3z^3 - 6xy^2z^3$ by $-3xyz^2$, and $4y^2z^3$.
12. $-l^3m^2n^3 + \frac{1}{2}l^2m^3n^3 - \frac{2}{3}lm^2n^4$ by $-\frac{1}{3}lmn$, and $6m^2n^3$.
13. $4a^2x^3 - 9a^2x^2 + 7ax$ by $\frac{2}{3}a$, $-\frac{1}{2}x$, and $-\frac{1}{2}ax$.
14. $a^3m^7n^5 - b^3m^6n^4 + c^3m^6n^3 - d^3m^3n^3$ by $-l^3m^2n^2$, $\frac{2}{3}lmn$, $-\frac{1}{2}alm^2n^2$, and $abc^3m^3n^3$.
15. $2a^2x + 3a^2 + 4a^2$ by a^2 , $-\frac{2}{3}a$, a^{2-x} , and $-\frac{1}{2}a^{2-x}$.
16. $x^{2a}y^{3b}z^c - 4x^{2a}y^bz^c$ by $x^ay^bz^c$, $-\frac{1}{2}x^{a-1}y^{b-1}z^{c-1}$ and $-4x^{a-2}y^{b-2}z^{c-2}$.
17. If the dividend be the product of $2a^2 - 3ab$ and $x^2 - 3x$, and the divisor be $2ax$, find the quotient.
18. If the divisor be $p + 2q$, and the quotient $p - 2q$, find the dividend.
19. If the dividend be $4ax^2 - a^2x - ax$, and the quotient $-\frac{1}{2}ax$, find the divisor.
20. What must be added to $ax^3 - a^2x^2$ in order that the quotient of the resulting expression by ax may be $(x - a)^2$?

53. To divide one compound expression by another.

Rule. 1. Arrange the divisor and dividend in ascending or descending powers of some common letter

2. Divide the term on the left of the dividend by the term on the left of the divisor, and put the result in the quotient.

3. Multiply the whole divisor by this quotient, put the product under the dividend, and subtract.

4. Bring down from the dividend as many terms as may be necessary for the subsequent work.

5. Repeat the above operations till the terms of the dividend are exhausted, and a remainder, if any, is left, the first term of which is not exactly divisible by the first term of the divisor.

N.B. It will be seen that this process is akin to that of Long Division in Arithmetic.

Ex. 1. Divide $x^3 - 4x - 32$ by $x + 4$.

$$\begin{array}{r}
 x^3 \div x = x; \quad x+4 \overline{) x^3 - 4x - 32} \quad (x-8) \\
 \underline{x(x+4)} \quad \underline{-x^3 + 4x} \\
 -8x \div x = -8; \quad \underline{-8x - 32} \\
 \underline{-8(x+4)} \\
 \\

 \end{array}$$

Reqd. quotient
 $= x - 8$. Ans.

Reason for the above process :

It is easily seen that $x^3 - 4x - 32 = x^3 + 4x - 8x - 32$
 $= x(x+4) - 8(x+4)$
 $= Ax - 8A$, if A stand for $x+4$.

Therefore $(x^3 - 4x - 32) \div (x+4) = (Ax - 8A) \div A = x - 8$, Art. 52.

Ex. 2. Divide $\frac{x^4}{3} - \frac{11x^3}{12} + \frac{41x^2}{8} - \frac{23x}{4} + 6$ by $\frac{2x^2}{3} - \frac{5x}{6} + 1$.
 (P. U. 1892)

$$\begin{array}{r}
 \left(\frac{2x^2}{3} - \frac{5x}{6} + 1 \right) \overline{) \frac{x^4}{3} - \frac{11x^3}{12} + \frac{41x^2}{8} - \frac{23x}{4} + 6} \left(\frac{x^2}{2} - \frac{3x}{4} + 6 \right. \\
 \phantom{\left(\frac{2x^2}{3} - \frac{5x}{6} + 1 \right) \overline{) }} \underline{\frac{x^4}{3} - \frac{5x^3}{12} + \frac{x^2}{2}} \\
 \phantom{\left(\frac{2x^2}{3} - \frac{5x}{6} + 1 \right) \overline{) }} \phantom{\frac{x^4}{3} - \frac{5x^3}{12} + \frac{x^2}{2}} \underline{-\frac{x^3}{2} + \frac{37}{8}x^2 - \frac{23x}{4}} \\
 \phantom{\left(\frac{2x^2}{3} - \frac{5x}{6} + 1 \right) \overline{) }} \phantom{\frac{x^4}{3} - \frac{5x^3}{12} + \frac{x^2}{2}} \phantom{-\frac{x^3}{2} + \frac{37}{8}x^2 - \frac{23x}{4}} \underline{-\frac{x^3}{2} + \frac{5x^2}{8} - \frac{3x}{4}} \\
 \phantom{\left(\frac{2x^2}{3} - \frac{5x}{6} + 1 \right) \overline{) }} \phantom{\frac{x^4}{3} - \frac{5x^3}{12} + \frac{x^2}{2}} \phantom{-\frac{x^3}{2} + \frac{37}{8}x^2 - \frac{23x}{4}} \phantom{-\frac{x^3}{2} + \frac{5x^2}{8} - \frac{3x}{4}} \underline{4x^2 - 5x + 6} \\
 \phantom{\left(\frac{2x^2}{3} - \frac{5x}{6} + 1 \right) \overline{) }} \phantom{\frac{x^4}{3} - \frac{5x^3}{12} + \frac{x^2}{2}} \phantom{-\frac{x^3}{2} + \frac{37}{8}x^2 - \frac{23x}{4}} \phantom{-\frac{x^3}{2} + \frac{5x^2}{8} - \frac{3x}{4}} \underline{4x^2 - 5x + 6}
 \end{array}$$

\therefore the required quotient $= \frac{x^2}{2} - \frac{3x}{4} + 6$. Ans.

Ex. 3. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

Arrange the dividend in descending powers of a .

$$\begin{array}{r}
 a+b+c \overline{) a^3 + abc + b^3 + c^3} \quad (a^2 - ab - ac + b^2 - bc + c^2) \\
 \phantom{a+b+c \overline{) }} \underline{a^3 + a^2b + a^2c} \\
 \phantom{a+b+c \overline{) }} \underline{-a^2b - a^2c - 3abc} \\
 \phantom{a+b+c \overline{) }} \underline{-a^2b - ab^2 - abc} \\
 \phantom{a+b+c \overline{) }} \underline{-a^2c + ab^2 - 2abc} \\
 \phantom{a+b+c \overline{) }} \underline{-a^2c - ab^2 - abc}
 \end{array}$$

The required quotient

$$= a^2 - ab - ac + b^2 - bc + c^2.$$

$$\begin{array}{r}
 ab^2 - abc + ac^2 + b^3 + c^3 \\
 \underline{ab^2 + b^3 + b^2c} \\
 -ab^2c + ac^3 - b^3c + c^3 \\
 \underline{-ab^2c - b^3c - b^2c^2} \\
 +ac^3 + b^3c + c^3 \\
 \underline{ac^3 + b^3c + c^3}
 \end{array}$$

N.B. This result is usually written as $a^2 + b^2 + c^2 - bc - ca - ab$.

Note. Suppose when B is divided by A , the quotient is Q , and the remainder R .

The complete quotient in this case is $Q + \frac{R}{A}$.

For, since $B = QA + R$, $\therefore \frac{B}{A} = \frac{QA}{A} + \frac{R}{A} = Q + \frac{R}{A}$.

Ex. 4. Divide $16(x^3 - y^3) + 14x^2y - 129xy^2$ by $8x^2 + 3y^2 + 27xy$, and find the complete quotient.

Arrange in descending powers of x and ascending powers of y .

$$\begin{array}{r} 8x^3 + 27xy + 3y^3 \overline{) 16x^3 + 14x^2y - 129xy^2 - 16y^3} \\ \underline{16x^3 + 54x^2y + 6xy^3} \\ -40x^2y - 135xy^3 - 16y^3 \\ \underline{-40x^2y - 135xy^3 - 15y^3} \\ -y^3 \end{array}$$

The complete quotient

$$= 2x - 5y + \frac{-y^3}{8x^2 + 27xy + 3y^3} = 2x - 5y - \frac{y^3}{8x^2 + 27xy + 3y^3}. \text{ Ans.}$$

Ex. 5. Divide $1 - 2x$ by $2 - 3x + 4x^2$ to three terms.

$$\begin{array}{r} 1 - 2x \overline{) 1 - \frac{3}{2}x + 2x^2} \left(\frac{1}{2} - \frac{1}{4}x - \frac{1}{8}x^2 \right. \\ \underline{-\frac{1}{2}x - 2x^2} \\ -\frac{1}{2}x + \frac{3}{2}x^2 - 2x^2 \\ \underline{-\frac{1}{2}x^2 + x^2} \\ -\frac{1}{2}x^2 + x^2 \\ \underline{-\frac{1}{2}x^2 + \frac{3}{8}x^2 - \frac{1}{8}x^2} \\ -\frac{2}{8}x^2 + \frac{1}{8}x^2 \end{array}$$

\therefore quo. $= \frac{1}{2} - \frac{1}{4}x - \frac{1}{8}x^2$, and rem. $= -\frac{2}{8}x^2 + \frac{1}{8}x^2$. Ans.

EXAMPLES 24.

Divide

- $x^2 + 14x + 40$ by $x + 10$.
- $x^2 + 3x - 70$ by $x - 7$.
- $x^2 - 36$ by $x - 6$.
- $x^3 - 64y^3$ by $x - 8y$.
- $2x^3 - 13x - 24$ by $x - 8$.
- $6x^3 - 7x + 2$ by $3x - 2$.
- $2l^3 + 17l - 117$ by $2l - 9$.
- $8l^3 - 65l + 8$ by $l - 8$.
- $1 + a - 30a^2$ by $1 - 5a$.
- $1 - 2a - 48a^2$ by $6a + 1$.
- $8 - 7x - x^2$ by $x - 1$.
- $6x^3 + 19xy + 15y^3$ by $2x + 3y$.
- $ac - (bc + ad)x + bdx^2$ by $c - dx$.

14. $30x^3 + 41xy - 55y^2$ by $5y - 6x$.
15. $acx^2 + (8c + ad)x + bd$ by $ax + b$.
16. $72x^2 - 151xy + 77y^2$ by $8x - 7y$.
17. $4a^2x^2 - 7abxy - 15b^2y^2$ by $4ax + 5by$.
18. $125l^3 + 27m^3$ by $5l + 3m$, and $25l^2 - 15lm + 9m^2$.
19. $125x^3 + 225ax^2 + 130a^2x + 24a^3$ by $5x + 2a$, and $5x + 4a$.
20. $x^2 + 4xy + 4y^2 - 4x - 8y + 3$ by $x + 2y - 1$.
21. $a^{12} - b^{12}$ by $a - b$, $a^2 - b^2$, $a^3 - b^3$, and $a^4 - b^4$.
22. $2k^4 - 5k^3 + k^2 - 12$ by $2k^2 - 3k - 6$, and $k^2 - k + 2$.
23. $x^5 - x^4 + 6x^2 - 2x - 1$ by $x^3 - 2x^2 + 3x + 1$.
24. $a^4 + 2a^3 - 7a^2 - 8a + 12$ by $a^2 + a - 6$.
25. $1 - x^4 + 2x^3 + x^2$ by $1 + x^2 + x^3$.
26. $75x^6 - 28x^5 + 13x^4 - 12x$ by $-3 + x + 5x^2$.
27. $54x^3 + 12 - 24x + 81x^4 - 63x^2$ by $2 - 9x + 9x^2$.
28. $9k^4 + 13k^3 - 4k^2 + 9k^5 - 12k$ by $-3k + 1 + 9k^2$.
29. $a^6 - 4ay^2 + 4a^2y^3 - a^3 - 2a^2 - 8y^3$ by $a^3 - a - 2$.
30. $a^6 + 5b^6 + 6ab^5$ by $a^3 + 2ab + b^2$.
31. $k^6 - 2k^3 + 1$ by $1 + 2k + k^4 + 2k^3 + 3k^2$.
32. $a^3 + 8a^2b + 11ab^2 - 6b^3$ by $a^2 + 2ab - b^2$.
33. $a^4 + 4b^4$ by $a^2 + 2ab + 2b^2$ and $a^2 - 2ab + 2b^2$.
34. $2x^4 - 2y^4 + x^3y + 3xy^3 - 4x^2y^2$ by $x^2 + xy - 2y^2$.
35. $a^6 - a^4x - 2ax^4 - x^5 + 6a^2x^3$ by $x^3 + 3ax^2 - 2a^2x + a^3$.
36. $a^6 - b^6 - 3a^2b^2(a^3 - b^3)$ by $a^3 - 3ab(a - b) - b^3$.
37. $4x^4y - x^3y^3 + 4xy^5$ by $3xy^3 + 2y^2 + 2x^2y$.
38. $x^4 - (2a + 1)x^2 + 2a^2x + a^3 - a^4$ by $x^2 + a^2 - (x + a)$.
39. $5a^6 - 11a^5b + 21a^4b^2 - 13a^3b^3 + 19a^2b^4 - 12ab^5 + 9b^6$ by $a^3 - 2ab + 3b^2$.
40. $4x^5 + 12x^4 - 4x^3 + 9x^2 - 9x^2y^3 - 6x - 6xy^2 + 1 - y^4$
by $2x^2 - 3xy - y^2 + 3x - 1$.
41. $x^5 + a^4x^4 + a^5$ by $a^4 - a^2x^2 + x^4$ and $a^2 + ax + x^2$.
42. $x^6 + 2ax^5 + a^2x^4 - a^4x^2 + 2a^3x - a^5$ by $x^3 - a^2 + ax^2 + a^2x$.
43. $x^6 - 4x^3y^4 - 12x^4y^2 - y^6$ by $-x^3 + 4x^2y - 2xy^2 + y^3$.
44. $24x^6 - 46x^4y + 9x^2y^3 + 19x^3y^4 - 3xy^5$ by $3x^2 - 5x^2y + xy^2$.
45. $a^2x^2 + 2abxy + b^2y^2 + acx + bcy$ by $ax + by$.
46. $a^2x^3 - b^2y^3 + 4by - 4$ by $ax + by - 2$.
47. $3a^3 + 8ab + 4b^3 + 10a + 8b + 3$ by $a + 2b + 3$.

48. $x^4 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ by $x^2 + xy + y^2$.
 49. $a^4b^6 + a^4b^4 - a^2b^2 - 1$ by $a^2b^3 - a^2b^2 + ab - 1$.
 50. $4x^5 - 7y^2x^3 + 6x^4y + 8x^2y^3 - 36y^6 - 29xy^4$ by $x^2 - 2x^2y + 3xy^2 - y^3$.
 51. $\frac{1}{2}a^3 - \frac{1}{3}ab + \frac{1}{3}b^3$ by $a - \frac{2}{3}b$, and $\frac{2}{3}a - \frac{2}{3}b$.
 52. $5x^2 - 2\frac{1}{2}xy + y^2$ by $3x - \frac{2}{3}y$, and $\frac{1}{3}x^2 - \frac{1}{3}y$.
 53. $\frac{1}{2}x^3 - \frac{2}{3}x^2y + \frac{2}{3}xy^2 - \frac{1}{6}y^3$ by $\frac{2}{3}x - \frac{2}{3}y$, and $\frac{1}{6}x - \frac{1}{6}y$.
 54. $6x^4 - 10x^3 + 7x^2 - 5x + 2$ by $\frac{1}{2}x^3 - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{2}$.
 55. $\frac{1}{24}a^2y^4 - \frac{1}{8}a^2y^3 + \frac{1}{8}ax^2y^2 + \frac{2}{3}a^4y^3 - \frac{2}{3}a^2x^2y + \frac{1}{8}x^6$ by $x^3 - 2a^2y + \frac{1}{2}ay^2$.
 56. $9x^2 + \frac{1}{3}y^2 + 4x^2 - \frac{2}{3}ys + 3xz - 4xy$ by $\frac{2}{3}x - \frac{1}{3}y + \frac{1}{3}z$.
 57. $1 + x$ by $i + x + x^2$ to four terms.
 58. $2 - 3x$ by $1 - 2x + 3x^2$ to four terms.
 59. $\frac{1}{2} - x$ by $\frac{1}{2} - x + x^2$ to five terms.
 60. $a^6 + 7a^5 - 3a^4 + 2a^3 - 8a^2 + a - 1$ by $a^3 - 2a^2 + 3a - 2$.

CHAPTER VI.

BRACKETS.

54. The brackets in common use are (), { }, [].

These symbols are generally called brackets, but when required to distinguish them from one another, we call them parentheses, braces and crotchets respectively.

55. **Removal of brackets.** The use of brackets within brackets, and the way in which they are successively removed will be illustrated by the following example.

Ex. 1. Remove the brackets from the expression

$$a - [b - \{c - d + (e - f)\}]$$

Beginning with the innermost pair of brackets, we have

$$a - [b - \{c - d + (e - f)\}] = a - [b - \{c - d + e - f\}]$$

\therefore + before () requires no change of sign in $e - f$,

$$= a - [b - c + d - e + f]$$

\therefore - before { } requires change of signs within,

$$= a - b + c - d + e - f$$

\therefore - before [] requires change of signs.

Or we may begin with the outermost pair, and proceed thus :

$$\begin{aligned} a - [b - \{c - d + (e - f)\}] &= a - b + \{c - d + (e - f)\} \\ &= a - b + c - d + (e - f) \\ &= a - b + c - d + e - f \dots\dots\dots(1) \end{aligned}$$

N. B. In removing [], the signs of the terms within the inner brackets { } and () should not be altered ; and so in removing { }, the signs of the terms within () should not be altered.

From (1) it is evident that the brackets may be removed all at once, and the sign of any term determined by the help of the following rule :

Suppose all the other terms with their own signs suppressed ; if an odd number of minus signs now precede the term in question, its final sign is -, but otherwise +.

Thus as regards c in (1),

$$-[-\{c\}] = +c, \text{ there being 2 minuses ;}$$

As to f , $-[-\{+(-f)\}] = -f$, (3 minuses).

86. Sometimes a line is drawn over the symbols to be connected ; thus $a - \overline{b - c}$ is equivalent to $a - (b - c)$, and is therefore equal to $a - b + c$. The line is called a **vinculum**.

Ex. Simplify $12x - 5 - \{8x - (7 - 5x - 3)\}$

The expression $= 12x - 5 - \{8x - (7 - 5x + 3)\}$

$$= 12x - 5 - \{8x - (10 - 5x)\}$$

$$= 12x - 5 - \{8x - 10 + 5x\}$$

$$= 12x - 5 - \{13x - 10\}$$

$$= 12x - 5 - 13x + 10$$

$$= 5 - x. \text{ Ans.}$$

N.B.. We might remove all the brackets at once. We should then have $12x - 5 - 8x + 7 - 5x + 3$, i.e., $5 - x$.

EXAMPLES 25.

Simplify

1. $a - (b - c + d) + b - (c - a - d)$.

2. $a - \{a - (a - b)\}$.

3. $2x - \{y - (x - 2y)\}$.

4. $x - (1 - \overline{1 - x})$.

5. $10 - \{6 - \{1 - (7 - \overline{5 - 2})\}\}$.

6. $16a - [3b + \{5a + 2b - (3a - 2a - 4b)\}]$.

7. $y^2 - x^2 - [x^2 - \{x^2 - (y^2 - z^2 - x^2 - y^2 - z^2)\}]$.

8. $3a - [10a + 8b - \{7b - (2a - 3b)\}] + (a + b) - (a - b) - [6b - 5a - \{7a - (3a - 4b)\}]$
9. $\{m - n - (3x - 2y)\} - [3m + 2n - \{x - y + (m + 2n)\}]$
10. $x - [4y - \{3x - (5y - x)\}] - [3x - \{2y - (5x - 2y - x)\}]$
11. $-[-\{-(-a)\}]$ 12. $x - [-\{+(-x)\}]$
13. $16 - x - [(-x) - \{8 - (+9x) - (-6x - 3)\}]$
14. $b - \{b - (a + b) - [b - (b - a - b)] + 2a\}$
15. $x - \frac{1}{2}[x - \frac{1}{3}\{x - \frac{1}{4}(x - \frac{1}{5}x)\}]$
16. $-[15x - \{14y - (15z + 12y) - (10x - 15z)\}]$
17. $-m - 2\{-3m + 2n - 3(-5m - 8n) + 4(-3m - 2n - 3m + m)\}$
18. Given $a = -\frac{1}{2}$, $b = 2$, find the value of $\frac{1}{2}b(b - a) - 35[\frac{1}{2}(3a - 4b) - \frac{1}{10}\{3a - \frac{1}{5}(7a - 4b)\}]$
19. Given $x = -6$, $y = \frac{1}{3}$, $z = -\frac{1}{2}$, find the value of $\frac{4x}{3} - 5z\{-y - \frac{2}{3}[x - z + 2(3y - \frac{2x}{3} - \frac{y - z}{2} - \frac{x - y}{2})]\} - (x + y)$
20. What must be added to $x^3 - [x^3 - 1 - \{x - x(x^3 - 1)\}]$ to produce $x^4 - \{2x^3 - 3 - (x - 1)(x + 2) + x \cdot \overline{x - 3}\}$?

57. Insertion of brackets Part of an expression can be enclosed within brackets by attending to the following rules :

(1) If the sign + be used before the bracket, there should be no change of the signs of the terms enclosed.

(2) The sign - before the bracket requires a change of the sign of every term enclosed.

$$\begin{aligned}
 \text{Thus } a - b - c + d - f + g &= a - [b + c - d + f - g] \\
 &= a - [b + \{c - d + f - g\}] \\
 &= a - [b + \{c - (d - f + g)\}] \\
 &= a - [b + \{c - \overline{(d - f - g)}\}].
 \end{aligned}$$

EXAMPLES 26.

Enclose within brackets the terms commencing with the second .

1. $a + b - c - d$. 2. $a - b - c - d$. 3. $a - b + c + d$.
 4. $x^3 - x - 2$. 5. $x^3 + x - 2$. 6. $x^4 - x^3 + x^2 - 2x + 3$.

Enclose within parentheses the terms commencing with the 5th, within braces the terms commencing with the 4th, and within crotchets the terms commencing with the 3rd :—

7. $x - y + z - w - a + b$. 8. $a - b - c + d + e + f$.
 9. $2x - 3y - 4z + 5p - q - r - s$. 10. $3x - y - z - l - m - n$.
 11. $x^2 - y^2 - z^2 - yz - zx - xy - ax$. 12. $2x - a - 2y - 3b - 4z - c$.
 13. $ax - by - cz + by - cz - ax + 2cz$. 14. $x^3 - lx + y^3 - my + z^3 - nz + a^3$.

CHAPTER VII.

RESOLUTION INTO FACTORS.

58. **Definition.** When an algebraical expression is the product of two or more expressions, each of the latter is called a **factor** of the former, and the determination of such factors is called the **resolution** of the expression into its factors.

59. **Monomial factors.** *Rule : When each term of an expression is divisible by a common factor, enclose within brackets the quotient of each term by the common factor, and place the latter outside as a coefficient*

Ex. 1. Factorise $4x^2 - 8xy$.

Here $4x$ divides each term. Hence

$$4x^2 - 8xy = 4x \cdot x - 4x \cdot 2y = 4x(x - 2y). \quad \text{Ans.}$$

Ex. 2. Factorise $6a^2bx^4 - 24ab^2cx^3 + 30a^2b^2x^2$.

$$\begin{aligned} \text{The given expression} &= 6abx^2 \cdot ax^2 - 6abx^2 \cdot 4bcx + 6abx^2 \cdot 5ab^2 \\ &= 6abx^2(ax^2 - 4bcx + 5ab^2). \quad \text{Ans.} \end{aligned}$$

EXAMPLES 27.

Factorise

- | | | |
|---|-----------------------------------|----------------------------|
| 1. $a^2 - ab$. | 2. $a^2 + 2ab$. | 3. $2x^2 - 3xy$. |
| 4. $3yz - 4zx$. | 5. $6ab + 4bc$. | 6. $9b^2 - 6bc$. |
| 7. $abc - 3abd$. | 8. $2abc + 4ab^2$. | 9. $4xyz^2 - 8x^2y^2$. |
| 10. $8x^2yz - 8xy^2z$. | 11. $3l^2mn + 6lm^2n$. | 12. $11apqr - 22a^2p^2q$. |
| 13. $5a^2 - 5ab + 10a$. | 14. $4b^5 - 20ab^4 + 16a^2b^3d$. | |
| 15. $2a^2b^2 - 4ab^2c + 6ab^2c + 12ab^2d$. | | |
| 16. $5x^2yz - 10x^2yz^2 + 15x^2y^2z - 20x^2y^2$. | | |

17. $3x^4y^7z^8 - 9x^6y^6z^6 + 12x^7y^7z^7$.
 ✓ 18. $3a^3b^2k^3 - 6a^2b^3k^4 + 9a^4b^2k^5 - 12a^3b^3k^6$.
 ✓ 19. $x^2y^3 - x^{2+3}y^{3+2} + 2x^{3+2}y^{2+3}$.
 ✓ 20. $2ax^3y^2z^3 - 4bx^2y^3z^4 + 2cx^4y^2z^5$.
 ✓ 21. $8l^3m^2n^2p - 12al^2m^4n^4 + 16bl^3m^5n^6 - 20l^2m^3n^3$.

60. Compound factors. An expression may be resolved into factors, if it can be broken up into parts having a compound factor common to them all.

Ex. 1. Resolve into factors $x^2 - ax - bx + ab$.

$$\begin{aligned} x^2 - ax - bx + ab &= (x^2 - ax) - (bx - ab) \\ &= x(x - a) - b(x - a) \\ &= x.A - b.A, \quad A \text{ standing for } x - a, \\ &= A(x - b) \\ &= \underline{(x - a)(x - b)}, \text{ restoring the value of } A. \end{aligned}$$

Hence the following **rule**: *Arrange the terms of the given expression into groups having a common factor; enclose within brackets the quotient of each group by the common factor, and place the latter outside those brackets and within a second pair of brackets, as a compound co-efficient.*

Ex. 2. Factorise $6x^3 + 10x^2y - 9xz^3 - 15xyz$.

$$\begin{aligned} \text{The given expression} &= (6x^3 + 10x^2y) - (9xz^3 + 15xyz) \\ &= 2x^2(3x + 5y) - 3xz(3x + 5y) \\ &= x\{2x(3x + 5y) - 3z(3x + 5y)\} \\ &= \underline{x(3x + 5y)(2x - 3z)}. \end{aligned}$$

EXAMPLES 28.

Resolve into elementary factors

- | | |
|------------------------------------|---------------------------------|
| 1. $ab + ac + bd + cd$. | 2. $a^3 + ab + ac + bc$. |
| 3. $x^3 + ax^2 + a^2x + a^3$. | 4. $x^3 + ax^2 - a^2x - a^3$. |
| 5. $x^3 - ax^2 + a^2x - a^3$. | 6. $x^3 - ax^2 - a^2x + a^3$. |
| 7. $a^2x^2 + y^2 + a^2 + x^2y^2$. | 8. $2ax - 4ay - 2bx + 4by$. |
| 9. $xy - zx + y^2 - yz$. | 10. $l^2 + al - bl - ab$. |
| 11. $2p^3 + 2px - 2ap - 2ax$. | 12. $2px - pq + 2xy - qy$. |
| 13. $a^2x + abx + aby + b^2y$. | 14. $a^2x + b^2y - ab(x + y)$. |
| 15. $2pl + 3mq + 6pm + ql$. | 16. $ax^3 - bx - bx^2 + ax^2$. |

17. $2abx^2 - b^2cx + 2acx^2 - b^2x$. 18. $3x^2y - 3xy^2 + 3ax^2 - 3axy$.
 19. $4a^2b^2 - 4ab^2q - 4a^2pq + 4ab^2p$. 20. $cx + cxyx - cxx - cyx$.
 H 21. $2a(p-q) + 2b(p-q) + 2(p-q)c + 2(q-p)d$.
 H 22. $(a^2+ab)x - (a^2+ab)y - (a^2+ab)z + (a^2+ab)w$.
 H 23. $am - bm + ap - bp + aq - bq + ar - br$.
 H 24. $x^2 + ay + xy + az + ax + zx$.

61. To factorise $x^2 - a^2$.

$$\begin{aligned} x^2 - a^2 &= x^2 - ax + ax - a^2, \text{ adding and subtracting } ax, \\ &= (x^2 - ax) + (ax - a^2) \\ &= x(x - a) + a(x - a) \\ &= (x - a)(x + a). \end{aligned}$$

That is, the difference of the squares of two quantities = the product of the sum and difference of those quantities.

Ex. 1. Factorise $9a^2 - 25b^2$.

$$9a^2 - 25b^2 = (3a)^2 - (5b)^2 = (3a - 5b)(3a + 5b).$$

Ex. 2. Factorise $(2x + 3a)^2 - (x + 5a)^2$

$$\begin{aligned} \text{The expression} &= \{(2x + 3a) - (x + 5a)\} \{(2x + 3a) + (x + 5a)\} \\ &= (x - 2a)(3x + 8a), \text{ simplifying the factors.} \end{aligned}$$

Ex. 3. Factorise $a^5 - b^5$.

$$a^5 - b^5 = (a^4)^2 - (b^4)^2 = (a^4 - b^4)(a^4 + b^4) \dots \dots \dots (1)$$

$$\text{Now, } a^4 - b^4 = (a^2)^2 - (b^2)^2 = (a^2 - b^2)(a^2 + b^2)$$

$$= (a - b)(a + b)(a^2 + b^2), \text{ factorising } a^2 - b^2.$$

$$\text{Using this result in (1), } a^5 - b^5 = (a - b)(a + b)(a^2 + b^2)(a^4 + b^4).$$

Ex. 4. Find the value of $627^2 - 427^2$.

$$627^2 - 427^2 = (627 + 427)(627 - 427) = 1054 \times 200 = \underline{210800}$$

EXAMPLES 29

Resolve into elementary factors :

- | | | | |
|-------------------------|------------------------|-------------------------|-------------------------|
| 1. $1 - a^2$. | 2. $4a^2 - 9b^2$. | 3. $16a^2 - 25b^2$. | 4. $a^2b^2 - x^2y^2$. |
| 5. $ax^3 - ay^3$. | 6. $3x^3 - 27y^3$. | 7. $4ab^2 - 9ac^2$. | 8. $b^2x^2y - c^2y^2$. |
| 9. $a^2t^2 - b^2nt^2$. | 10. $abl^2 - abm^2$. | 11. $12ax^2 - 75ay^2$. | |
| 12. $625a^2 - 169b^2$. | 13. $x^4 - a^4$. | 14. $p^4q^4 - a^4b^4$. | |
| 15. $p^{16} - q^{16}$. | 16. $81a^8 - 256b^8$. | 17. $49x^4 - 1$. | |

$$H 18. 1 - 16b^4.$$

$$H 21. 49a^4 - 100b^2.$$

$$24. 16x^{18} - 25y^6.$$

$$H 27. a^2b^4 - a^6.$$

$$29. (x+a)^2 - (x-a)^2.$$

$$31. (ax+b)^2 - (bx+a)^2.$$

$$33. (a^2-1)^2 - (b^2-1)^2.$$

$$H 19. x^5 - 4a^4.$$

$$22. 768a^3 - 3.$$

$$H 25. 81c - c^3.$$

$$H 28. 2x^4y - 162y^5.$$

$$30. (4a+b)^2 - (a+4b)^2.$$

$$32. (cx-d)^2 - (dx-c)^2.$$

$$34. (a+b)^4 - (a-b)^4.$$

$$H 20. 4a^4b^4 - 81.$$

$$23. 25a^{18} - 4.$$

$$H 26. a^2b^4c^3 - x^6.$$

Find the value of

$$35. (49)^2 - (41)^2.$$

$$36. (61)^2 - (59)^2.$$

$$37. (1001)^2 - 1.$$

$$38. (88)^2 - (12)^2.$$

$$39. (295)^2 - (205)^2.$$

$$40. (9992)^2 - 64.$$

62. To factorise $x^2 + 2ax + a^2$ and $x^2 - 2ax + a^2$.

Breaking up $2ax$ into $ax + ax$, we have

$$\begin{aligned} x^2 + 2ax + a^2 &= (x^2 + ax) + (ax + a^2) \\ &= x(x+a) + a(x+a) \\ &= (x+a)(x+a) \\ &= \underline{(x+a)^2}. \end{aligned}$$

Similarly, $x^2 - 2ax + a^2 = x^2 - ax - ax + a^2$

$$\begin{aligned} &= (x^2 - ax) - (ax - a^2) \\ &= x(x-a) - a(x-a) \\ &= (x-a)(x-a) \\ &= \underline{(x-a)^2}. \end{aligned}$$

Ex. 1. Factorise $x^2 + 6x + 9$.

$$x^2 + 6x + 9 = x^2 + 2 \times x \times 3 + 3^2 = (x+3)^2. \text{ Ans.}$$

Ex. 2. Factorise $9x^2 - 12ax + 4a^2$.

$$\text{The expression} = (3x)^2 - 2(3x)(2a) + (2a)^2 = (3x-2a)^2. \text{ Ans.}$$

Ex. 3. Find the value of $(1'05)^2 + (.94)^2 + 4'24 \times .47$.

$$4'24 \times .47 = 4 \times 1'06 \times .47 = 2 \times 1'06 \times .47 \times 2 = 2 \times 1'06 \times .94;$$

$$\therefore \text{the given expression} = (1'06)^2 + (.94)^2 + 2 \times 1'06 \times .94$$

$$= (1'05 + .94)^2 = 2^2 = 4. \text{ Ans.}$$

EXAMPLES 30.

Factorise

1. $a^2 - 2a + 1.$

2. $x^2 + 4x + 4.$

3. $a^2 - 4ab + 4b^2.$

4. $4a^2 - 12ab + 9b^2.$

5. $16x^2 - 40xy + 25y^2.$

6. $25x^2 + 60xy + 36y^2.$

Factorise

7. $9x^3 - 42xy + 49y^3$. 8. $4x^3 + 44x + 121$. 9. $49a^2b^3 - 126ab + 81$.
 10. $25x^3 + 70xy + 49y^3$. 11. $36p^3 - 48p + 16$. 12. $p^2 - 26pq + 169q^2$.
 13. $x^4 + 2x^2y^2 + y^4$. 14. $4x^6 + 4x^3y^3 + y^6$. 15. $9a^3 + 60a^2b^4 + 100b^8$.
 16. $25a^2 + 40ab^3 + 16b^6$. 17. $81a^3 - 72a^2b^3 + 16b^6$.
 18. $625a^2b^3c^2 + 150abc + 9$. 19. $\frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2$.
 20. $\frac{4}{9}x^2 + \frac{1}{18}y^2 - xy$. 21. $\frac{1}{9} - \frac{2}{3}a + \frac{1}{9}a^2$.
 22. $1 - \frac{1}{8}a^2b^2c + \frac{1}{81}a^4b^4c^2$. 23. $\frac{1}{4}y^2 - \frac{1}{2}x^2y + 16x^4$.
 24. $\frac{1}{8}l^2 + \frac{1}{10}lm^2 - \frac{1}{8}lm$. 25. $2a^6 - 8a^3b + 8b^3$.
 26. $2a^2x^4 + 20a^2x^2 + 50a^2$. 27. $3a^2 - 66ab + 363b^2$.
 28. $18l^2 + 60lm + 50m^2$. 29. $45 + 20x^2y^2z^6 - 60xy^2z^3$.
 30. $4l^2m^2 + 49m^2n^2 - 28lm^2n$. 31. $100a^3 + 240a^4 + 144a^6$.
 32. $2a^2b + 64a^2b + 512ab^3$. 33. $28a^2b^3c^2 + 7b^3c^2d^2 - 28ab^3c^2d$.
 34. $x^2 + 2x(a+b) + (a+b)^2$. 35. $x^2 + (y-z)^2 + 2(xy-xz)$.
 36. $4a^2 + 25(b+c)^2 - 20a(b+c)$. 37. $a^4 + (4a^3 + 2a^2) + (2a+1)^2$.
 38. $(x+a)^2 + (x+b)^2 - 2(x+a)(x+b)$.
 39. $(a^2+3)^2 + (2a^2+6)(4a+1) + (4a+1)^2$.
 40. $2(ab+cd)^2 + 4(ab+cd)(ac+bd) + 2(ac+bd)^2$.

Find the value of

41. $(2)^2 + 4 \times 8 + (8)^2$. 42. $(2 \cdot 03)^2 + (03)^2 - 1218$.
 43. $(1985)^2 + (015)^2 + 05955$. 44. $(123)^2 + 144 - 123 \times 24$.
 45. $4x^2 - 20xy + 25y^2$, when $x=15$, and $y=-12$.
 46. $49x^2 + 126xy + 81y^2$, when $x=124$, and $y=-97$.
 47. $9a^2 + 24ab + 16b^2$, when $a=k+4$, and $b=k-3$.
 48. $25x^2 - 80xy + 64y^2$, when $x=8a-3$, and $y=5a-2$.

63. Trinomials of the form $x^2 + ax + b$.

First Method : It consists in reducing the given expression into the difference of two squares. Take $x^2 + 5x + 6$. First find out what term is wanting to make $x^2 + 5x$ a perfect square. Observe that in $x^2 + 2ax + a^2$, which is a perfect square, the third term, viz., $a^2 = (\frac{1}{2} \times \text{coefficient of } x)^2$.

∴ the term wanting is $(\frac{5}{2})^2$.

Now proceed thus : Adding and subtracting $(\frac{5}{2})^2$,

$$\begin{aligned}x^2 + 5x + 6 &= x^2 + 5x + (\frac{5}{2})^2 - (\frac{5}{2})^2 + 6 \\&= \{x^2 + 2 \times \frac{5}{2}x + (\frac{5}{2})^2\} - (\frac{25}{4} - 6) \\&= (x + \frac{5}{2})^2 - (\frac{1}{2})^2 \\&= \{(x + \frac{5}{2}) - \frac{1}{2}\} \{(x + \frac{5}{2}) + \frac{1}{2}\} \\&= \underline{(x + 2)(x + 3)}.\end{aligned}$$

Again, to factorise $x^2 - 4x - 45$, observe that the term wanting to make $x^2 - 4x$ a perfect square $= (\frac{1}{2} \text{ coef. of } x)^2 = (-2)^2 = 4$. Then proceed thus :

$$\begin{aligned}x^2 - 4x - 45 &= (x^2 - 4x + 4) - 4 - 45 \\&= (x - 2)^2 - 7^2 \\&= \underline{(x - 2 - 7)(x - 2 + 7)} \\&= \underline{(x - 9)(x + 5)}.\end{aligned}$$

Second Method : The second is the method of *inspection*, and it is the one generally used in practice.

To make this method clear, it is necessary to examine closely the product $(x + m)(x + n)$.

By actual multiplication, $(x + m)(x + n) = x^2 + (m + n)x + mn$.

We observe in the product that :

(1) *The coefficient of the second term which contains x , is the sum of the second terms, viz., m and n , of the binomial factors.*

(2) *The third term of the product is the product of the second terms, viz., m and n , of the binomial factors.*

The product may therefore be stated as

$x^2 + (\text{sum of the 2nd terms of the factors})x + \text{product of the second terms of the factors}.$

A similar rule will be readily found to apply to $(x - m)(x - n)$, $(x + m)(x - n)$ and $(x - m)(x + n)$. For example, we find that

$$(x + m)(x - n) = x^2 + (m - n)x - mn = x^2 + \{m + (-n)\}x + m \times (-n).$$

Hence we readily obtain the following practical **Rule** :

Break up the last term into two factors of which the sum or difference is equal to the coefficient of x according as the last term is positive or negative ; express that coefficient as such sum or difference ; then arrange as in the following examples.

Ex. 1. Resolve $x^2 + 5x + 6$ into factors.

Here the third term, 6, is positive, and hence we have to break

up 6 into two factors, such that their *sum* may be equal to 5, which is the coefficient of the second term.

Now, the factors of 6 are (1, 6) and (2, 3). Of these pairs 2 and 3 only have 5 as their sum. Breaking up 5 into 2+3, we have

$$\begin{aligned}x^2 + 5x + 6 &= x^2 + 2x + 3x + 6 \\&= x(x+2) + 3(x+2) \\&= \underline{(x+2)(x+3)}. \quad \text{Ans.}\end{aligned}$$

Ex. 2. Factorise $x^2 - 4x - 45$.

Since the 3rd term is negative, we have to break it up into two factors of which the *difference* shall be 4 (=coef. of 2nd term, neglecting sign).

Now, the factors of 45 are (1, 45), (3, 15) and (9, 5).

Of these pairs, $9 - 5 = 4$.

Hence, breaking up $4x$ into $9x - 5x$, we have

$$\begin{aligned}x^2 - 4x - 45 &= x^2 - 9x + 5x - 45 \\&= x(x-9) + 5(x-9) \\&= \underline{(x-9)(x+5)}. \quad \text{Ans.}\end{aligned}$$

Ex. 3. Factorise $x^2 - 8x + 15$.

The last term, 15, is positive.

The factors of 15 are (1, 15), (3, 5).

Of these pairs, $3 + 5 = 8$ (i. e., coef. of 2nd term, neglecting sign).

Hence, breaking up $8x$ into $3x + 5x$, we have

$$\begin{aligned}x^2 - 8x + 15 &= x^2 - 3x - 5x + 15 \\&= x(x-3) - 5(x-3) \\&= \underline{(x-3)(x-5)}. \quad \text{Ans.}\end{aligned}$$

Ex. 4. Factorise $x^2 - (2a+1)x + a^2 + a$.

Now, $a^2 + a$ (the 3rd term) $= a(a+1)$.

Also $2a+1$ (coef. of 2nd term, neglecting sign) $= a + (a+1)$.

Now proceed thus :

$$\begin{aligned}x^2 - (2a+1)x + a^2 + a &= x^2 - (a+a+1)x + a(a+1) \\&= x^2 - ax - (a+1)x + a(a+1) \\&= x(x-a) - (a+1)(x-a) \\&= (x-a)(x-a+1) \\&= \underline{(x-a)(x-a-1)}. \quad \text{Ans.}\end{aligned}$$

EXAMPLES 31.

Resolve into factors by two different methods :

1. $x^2 + 3x + 2$. 2. $x^2 + 4x + 3$. 3. $x^2 + 8x + 7$.
 4. $x^2 + 7x + 6$. 5. $x^2 + 9x + 8$. 6. $x^2 + 13x + 36$.
 7. $x^2 + 8x + 12$. 8. $x^2 + 10x + 16$. 9. $x^2 + 10x + 24$.
 10. $x^2 - 9x + 20$. 11. $x^2 - 11x + 24$. 12. $x^2 - 12x + 27$.
 13. $x^2 - 13x + 42$. 14. $x^2 - 15x + 56$. 15. $a^2 - 17a + 72$.
 16. $x^2 - x - 2$. 17. $x^2 + x - 2$. 18. $c^2 - c - 6$.
 19. $a^2 + 3a - 28$. 20. $x^2 + 9x - 36$. 21. $x^2 - 8x - 48$.
 22. $40 + 3x - x^2$. 23. $20 - 19x - x^2$. 24. $63 + 16a + a^2$.
 25. $80 + 2x - x^2$. 26. $60 - 7a - a^2$. 27. $108 - 21y + y^2$.
 28. $x^2 + 11xy + 30y^2$. 29. $x^2 - 21xy + 110y^2$. 30. $b^2 - 7ab - 144a^2$.
 31. $c^2 - 23cd + 120d^2$. 32. $a^2b^2 - 16ab + 60$. 33. $y^2 - 3yz - 180z^2$.
 34. $30xy + x^2 + 200y^2$. 35. $20ab - 125b^2 + a^2$. 36. $80 - 11y - y^2$.
 37. $x^2 - (2a + 3)x + a^2 + 3a$. 38. $x^2 - (2a + 3)x + (a + 1)(a + 2)$.
 39. $y^2 - (2a + 5)y + (a + 2)(a + 3)$. 40. $y^2 - y - (a + 2)(a + 3)$.
 41. $x^2 + 3x - (a - 1)(a + 2)$. 42. $x^2 - x(a - b - c) - (ac - bc)$.
 43. $x^2 + mx - (l^2 + 3lm + 2m^2)$. 44. $y^2 + (b - c)y - (a + b)(a + c)$.
 45. $y^2 - (a - b)y - (2a^2 + 5ab + 2b^2)$. ✓

64. Trinomials of the form $ax^2 + bx + c$.Ex. Factorise $8x^2 + 22x + 15$.First Method. $8x^2 + 22x + 15 = 8(x^2 + \frac{11}{4}x + \frac{15}{8})$. (A)

Now apply the method of the difference of two squares to

$$x^2 + \frac{11}{4}x + \frac{15}{8}.$$

$$x^2 + \frac{11}{4}x + \frac{15}{8} = \{x^2 + \frac{11}{4}x + (\frac{11}{8})^2\} - (\frac{11}{8})^2 + \frac{15}{8}$$

$$= (x + \frac{11}{8})^2 - \frac{11^2}{64} + \frac{15}{8}$$

$$= (x + \frac{11}{8})^2 - \frac{1}{8}$$

$$= (x + \frac{11}{8})^2 - (\frac{1}{8})^2, \text{ simplifying,}$$

$$= (x + \frac{11}{8} - \frac{1}{8})(x + \frac{11}{8} + \frac{1}{8})$$

$$= (x + \frac{5}{4})(x + \frac{3}{2}).$$

$$\therefore 8x^2 + 22x + 15 = 8(x + \frac{5}{4})(x + \frac{3}{2}), \text{ by (A),}$$

$$= 4(x + \frac{5}{4}) \times 2(x + \frac{3}{2})$$

$$= \underline{(4x + 5)(2x + 3)}, \text{ clearing of fractions.}$$

Second Method : This may be called the method of *inspection*, and is the one most largely followed. It is, however, of some difficulty on the part of the beginner, and can be mastered only after a good deal of practice.

Note the following results of multiplication :

$$(ax+b)(cx+d) = acx^2 + (bc+ad)x + bd.$$

$$(ax-b)(cx+d) = acx^2 + \{(-b)c+ad\}x + (-b) \times d.$$

$$(ax+b)(cx-d) = acx^2 + \{bc+a(-d)\}x + b \times (-d).$$

$$(ax-b)(cx-d) = acx^2 + \{(-b) \times c + a \times (-d)\}x + (-b)(-d).$$

The co-efficient of x in the product in each case is the sum of a pair of algebraical factors of the product of the co-efficient of x^2 and the third term. We thus obtain the following rules :

(1) *Multiply together the coefficient of x^2 in the given trinomial and the third term.*

(2) *Break up the product into two factors of which the sum or difference is equal to the coefficient of x according as the product is positive or negative ; express that coefficient as such sum or difference ; then arrange as in the following examples.*

Ex. 1. Factorise $8x^2 + 22x + 15$.

$8 \times 15 = 120$, which is positive, and the factors of 120 of which the sum is 22 are easily seen to be 12 and 10 ;

$$\begin{aligned} \therefore 8x^2 + 22x + 15 &= 8x^2 + 12x + 10x + 15 \\ &= (2x+3)4x + (2x+3)5 \\ &= \underline{(2x+3)(4x+5)}. \end{aligned}$$

Ex. 2. Resolve $30x^2 + 13x - 56$ into elementary factors.

$30 \times (-56) = -1680$, which is negative ; and the factors of 1680 of which the difference is 13 are 48 and 35 ;

$$\begin{aligned} \therefore 30x^2 + 13x - 56 &= 30x^2 + 48x - 35x - 56 \\ &= 6x(5x+8) - 7(5x+8) \\ &= \underline{(5x+8)(6x-7)}. \end{aligned}$$

Ex. 3. Decompose into factors $12x^2 - 17xy + 6y^2$.

$12 \times 6 = 72$, which is positive. Also the factors of 72 of which the sum is 17 are 8 and 9.

$$\begin{aligned} \therefore 12x^2 - 17xy + 6y^2 &= 12x^2 - 9xy - 8xy + 6y^2 \\ &= 3x(4x-3y) - 2y(4x-3y) \\ &= \underline{(4x-3y)(3x-2y)}. \end{aligned}$$

Ex. 4. Factorise $28 - 31a - 5a^2$.

$-5 \times 28 = -140 = -35 \times 4$; and the difference of 35 and 4 = 31.

Exhibit the work thus.

$$28 - 31a - 5a^2 = 28 + 4a - 35a - 5a^2$$

$$= 4(7+a) - 5a(7+a) = (7+a)(4-5a).$$

To apply the First Method, we should proceed thus:

$$28 - 31a - 5a^2 = -5 \left(a^2 + \frac{31}{5}a - \frac{28}{5} \right) = -5 \left\{ \left(a + \frac{31}{10} \right)^2 - \left(\frac{39}{10} \right)^2 \right\}$$

$$= -5 \left\{ \left(a + \frac{31}{10} \right) - \frac{39}{10} \right\} \left\{ \left(a + \frac{31}{10} \right) + \frac{39}{10} \right\}$$

$$= -5 \left(a - \frac{4}{5} \right) (a+7) = (4-5a)(7+a).$$

Ex. 5. Resolve $24x^4 - 26x^2 + 5$ into its simplest factors.

$24 \times 5 = 120$, which has for factors 20 and 6.

$$\therefore 24x^4 - 26x^2 + 5 = 24y^2 - 26y + 5, \text{ putting } y \text{ for } x^2,$$

$$= 24y^2 - 6y - 20y + 5$$

$$= 6y(4y-1) - 5(4y-1)$$

$$= (4y-1)(6y-5)$$

$$= (4x^2-1)(6x^2-5), \text{ restoring } x^2 \text{ for } y,$$

$$= \{(2x)^2-1^2\}(6x^2-5)$$

$$= (2x-1)(2x+1)(6x^2-5).$$

Note. We could have proceeded by putting the preceding given expression as the difference of two squares.

$$\text{Thus, } 24x^4 - 26x^2 + 5 = 24 \left(x^4 - \frac{26}{24}x^2 + \frac{5}{24} \right)$$

$$= 24 \left\{ \left(x^2 - \frac{13}{24} \right)^2 - \frac{13^2}{24^2} + \frac{5}{24} \right\}$$

$$= 24 \left\{ \left(x^2 - \frac{13}{24} \right)^2 - \left(\frac{7}{24} \right)^2 \right\}$$

$$= 24 \left(x^2 - \frac{13}{24} + \frac{7}{24} \right) \left(x^2 - \frac{13}{24} - \frac{7}{24} \right)$$

$$= 24 \left(x^2 - \frac{1}{4} \right) \left(x^2 - \frac{5}{6} \right)$$

$$= 4 \left(x^2 - \frac{1}{4} \right) 6 \left(x^2 - \frac{5}{6} \right)$$

$$= (4x^2-1)(6x^2-5), \text{ \&c.}$$

It is important to note that these methods will not succeed in all cases, as for example, in expressions like $x^4 + x^2 + 1$, &c. Such cases will be treated later on.

EXAMPLES 32.

Resolve into elementary factors :

1. $2x^2 + 3x + 1$.
2. $2x^2 + 5x + 2$.
3. $4x^2 + 5x + 1$.
4. $3a^2 + 10a + 3$.
5. $2a^2 + 7a + 6$.
6. $2b^2 + 9b + 4$.
7. $6x^2 - 5x + 1$.
8. $6x^2 - 13x + 6$.
9. $10x^2 - 23x + 12$.
10. $12y^2 - 17y + 6$.
11. $9y^2 - 31y + 12$.
12. $8y^2 - 17y + 9$.
13. $2a^2 - a - 1$.
14. $3x^2 + 11x - 20$.
15. $4x^2 + 11x - 3$.
16. $4x^2 - x - 3$.
17. $110x^2 + x - 1$.
18. $7x^2 - 123x - 54$.
19. $2 - 5x - 3x^2$.
20. $28 + 3x - x^2$.
21. $30 - 11x - 30x^2$.
22. $15 - 77x + 10x^2$.
23. $6a^2 + 55a - 50$.
24. $12 + p - 20p^2$.
25. $8x^2 - 65x + 8$.
26. $24x^2 - 29x - 4$.
27. $64x^2 + 128x + 63$.
28. $3x^2 + 23xy + 14y^2$.
29. $3x^2 - 50xy + 200y^2$.
30. $7a^2 - 145ab + 72b^2$.
31. $18 - 33ab + 5a^2b^2$.
32. $24c^2 - 37cd - 72d^2$.
33. $42x^2 - 41xy - 20y^2$.
34. $6x^2 + 38x - 28$.
35. $12x^2 + 69x + 45$.
36. $40x^2 - 190x + 175$.
37. $7ax^2 + 123ax - 54a$.
38. $7a^2x^2 + 123ax - 54$.
39. $a^2b - a^2b^2 - 2ab^2$.
40. $6x^3 - 15x^2 - 9x$.
41. $2x^4 + 7x^2 + 6$.
42. $6x^4 - 19x^2 + 14$.
43. $14x^4y^2 - 29x^2y^3 - 15x^2y^4$.
44. $72x^3y^3 - 102x^2y^2 + 15xy$.
45. $abx^2 - (a^2 + b^2)x + ab$.
46. $20a^2b^2c^2 + 11abc^3 - 3c^3$.

CHAPTER VIII.

ALGEBRAICAL LAWS AND PROCESSES.

65. In the previous chapters we have made use of some laws of ordinary Algebra without naming them. We proceed to make a collection of these laws.

(1) **Law of Association** : The operator $+$ or $-$ before any number of quantities connected by $+$ or $-$ and taken as a whole, may be applied to each, attending to the usual rule, viz., *like signs produce $+$, and unlike signs produce $-$* .

ILLUSTRATIONS : For Addition and Subtraction.

$$c + (+a - b) = c + (+a) + (-b) = c + a - b ;$$

$$c - (+a - b) = c - (+a) - (-b) = c - a + b.$$

A similar rule applies to the signs of multiplication and division.

$$\text{Thus } a \div (b \times c \div d) = a \div b \div c \times d.$$

(2) **Law of Commutation** : In any chain of addition and subtraction, or of multiplication and division, the order of operations is indifferent.

For Addition and Subtraction,

$$c - a + b = c + b - a = b + c - a = -a + b + c, \text{ \&c}$$

For Multiplication and Division,

$$c \times a \div b = c \div b \times a = a \div b \times c = a \times c \div b.$$

It is easily seen that $4 \div 2 \times 5 = 10 = 4 \times 5 \div 2$.

(3) **Law of Distribution for Multiplication** : The product of two expressions, each of which consists of a chain of additions and subtractions, is equal to the chain of additions and subtractions formed by multiplying each term of one expression by each term of the other, the partial products being subject to the usual rule of signs, *vis*, like signs give +, and unlike signs give -.

$$\begin{aligned}(a+b)(c+d) &= a \times (+c) + a \times (+d) + b \times (+c) + b \times (+d) \\ &= ac + ad + bc + bd.\end{aligned}$$

$$\begin{aligned}(a-b)(c-d) &= a \times (+c) + a \times (-d) + (-b) \times (+c) + (-b) \times (-d) \\ &= ac - ad - bc + bd.\end{aligned}$$

The law of distribution for division applies to the dividend only ; e.g., $(a-b) \div c = (a \div c) + (-b \div c) = (a \div c) - (b \div c)$

CHAPTER IX.

HARDER WORK ON THE FIRST FOUR RULES.

Ex. 1. Subtract $-(ax^2 + bx - c)$ from $(lx^2 - mx + n)$.

$\therefore a - (-b) = a + b$, we have to add $ax^2 + bx - c$ to $lx^2 - mx + n$;

\therefore the reqd. result $= (lx^2 - mx + n) + (ax^2 + bx - c)$

$$= lx^2 + ax^2 - mx + bx + n - c$$

$$= \underline{(l+a)x^2 + (b-m)x + n - c}.$$

Ex. 2. Find the sum of $(n^3 - 1)a^3 - (2n - 1)a^2 + (n^2 - n + 1)a + b$,

$(n^2 + n + 1)a^3 + n^2a^2 - mb - 2a$, and $-(n^3 + n - 2)a^3 + pb$

$$+ (n^3 + 2n + 2)a^2.$$

Arranging the terms,

$$\begin{array}{r}
 (n^3-1)a^3 - (2n-1)a^2 + (n^2-n+1)a + b \\
 (n^3+n+1)a^3 + n^2a^2 + 2a - mb \\
 -(n^3+n-2)a^3 + (n^3+2n+2)a^2 + pb \\
 \hline
 \therefore \text{sum} = (n^3+2)a^3 + (n^3+n^2+3)a^2 + (n^2-n-1)a + (1-m+pb)b.
 \end{array}$$

Note. Observe that the coefficient of a^3 in the required sum
 = the sum of the coefficients of a^3 in the summands
 = the sum of n^3-1 , n^3+n+1 and $-n^3-n+2$
 = n^3+2 .

Ex. 3. Multiply $a^4 - (m-1)a^2 + (n-m+1)a^2 - (m-1)a + 1$ by $a-1$.

$$\begin{array}{r}
 a^4 - (m-1)a^2 + (n-m+1)a^2 - (m-1)a + 1 \\
 a - 1 \\
 \hline
 a^5 - (m-1)a^4 + (n-m+1)a^3 - (m-1)a^2 + a \\
 - a^4 + (m-1)a^2 - (n-m+1)a^2 + (m-1)a - 1 \\
 \hline
 \text{Reqd. product} = a^5 - ma^4 + na^3 - na^2 + ma - 1
 \end{array}$$

Ex. 4. Find the coefficient of x^4 in the product of

$$x^4 - ax^3 + bx^2 - cx + d \text{ by } x^3 + px + q. \quad \text{C. U. 1885.}$$

The student may multiply up actually, but the following is a shorter method: We have only to find out which terms of the two factors give a product containing x^4 . Now, x^4 occurs in $x^3 \times bx^2$, $px \times (-ax^3)$ and $q \times x^4$. Collecting these terms, we have $(b - ap + q)x^4$; hence the required coefficient = $\underline{b - ap + q}$.

Ex. 5. Find the continued product of

$$a+b+c, b+c-a, c+a-b, a+b-c. \quad \text{C. U. 1866, 1867.}$$

$$\begin{aligned}
 (a+b+c)(b+c-a) &= \{(b+c)+a\}\{(b+c)-a\} \\
 &= (b+c)^2 - a^2, \quad \text{Art. 46,}
 \end{aligned}$$

$$= b^2 + 2bc + c^2 - a^2. \quad \text{Art. 46.}$$

$$\begin{aligned}
 (c+a-b)(a+b-c) &= \{a-(b-c)\}\{a+(b-c)\} \\
 &= a^2 - (b-c)^2
 \end{aligned}$$

$$= a^2 - (b^2 - 2bc + c^2)$$

$$= a^2 - b^2 + 2bc - c^2.$$

$$\begin{aligned}
 \therefore \text{the required product} &= (b^2 + 2bc + c^2 - a^2)(a^2 - b^2 + 2bc - c^2) \\
 &= \{2bc + (b^2 + c^2 - a^2)\}\{2bc - (b^2 + c^2 - a^2)\} \\
 &= 4b^2c^2 - (b^2 + c^2 - a^2)^2.
 \end{aligned}$$

Now, $(\delta^3 + c^3 - a^3)^3 = \{(\delta^3 + c^3) - a^3\}^3$
 $= (\delta^3 + c^3)^3 + (a^3)^3 - 2(a^3)(\delta^3 + c^3), \text{ Art. 46,}$
 $= (\delta^4 + 2\delta^3c^3 + c^4) + a^4 - 2a^2(\delta^3 + c^3)$
 $= \delta^4 + 2\delta^3c^3 + c^4 + a^4 - 2a^2\delta^3 - 2a^2c^3.$
 \therefore finally, the product $= 4\delta^2c^2 - (\delta^4 + 2\delta^3c^3 + c^4 + a^4 - 2a^2\delta^3 - 2a^2c^3)$
 $= 4\delta^2c^2 - \delta^4 - 2\delta^3c^3 - c^4 - a^4 + 2a^2\delta^3 + 2a^2c^3$
 $= 2\delta^2c^2 + 2a^2\delta^3 + 2a^2c^3 - a^4 - \delta^4 - c^4. \text{ Ans.}$

Ex. 6. Divide $(a-b)^2c^3 + (a-b)c^3 - (c^3 - a^3)\delta^3 + (c-a)\delta^3$
 by $(a-b)c^3 - (c-a)\delta^3. \text{ C. U. 1883.}$

Arrange the terms in the order of the powers of c .

$$\begin{array}{r} (a-b)c^2 - b^3c + ab^3 \Big) (a-b)c^3 + (a-b)^2c^3 - \delta^2c^3 + b^3c + a^2b^3 - ab^3 \Big(c + (a-b) \\ \underline{(a-b)c^3 - b^3c^2 + ab^3c} \\ (a-b)^2c^3 - (ab^3 - \delta^3)c + a^2b^3 - ab^3 \\ \underline{(a-b)^2c^3 - (ab^3 - \delta^3)c + a^2b^3 - ab^3} \end{array}$$

The required quotient $= c + (a-b), \text{ or } a-b+c. \text{ Ans.}$

EXAMPLES 33.

Add together

- $(a+b)x + (c+d)y, (a-b)x + (c-d)y.$
- $(m+n)x^2 + (n-1)x - 4, (m-h)x^2 + (n^2 - n + 1)x + 6.$
- $(a+2b-c)p + (a-b-c)q + cr + 1, (b+c-a)r + 2 + (c-b-a)p + bq,$
 $-(a-b+c)q - (b+a)r + (a+c-b)p - 3, bq + (a+c)r - cp + 2.$
- $(b+c)m - (c+a)n + (a+2b)mn, -(a+b)n - (c+a)mn + (b-c)m,$
 $-(2a+c-b)m + (a+2c-b)n, (2b-a+c)mn + (a-b-c)m.$
- $(m+n+r)ab - (2m+n+r)bc + (n+2r-m)ca,$
 $(2m+2n-r)dc - (2r+2n-m)ab + (m-n+r)ca,$
 and $(2m+n)ab - (2n-r)dc + (2r-m)ca.$
- $(x+y-2z)a - (x+z-2y)b + (y+z-2x)c,$
 $-(x-y-z)c + (x+2z-2y)a + (2y+x+2z)b,$
 $(2x+y-z)b - (x+y-z)a + (2x+2y-z)c,$
 and $(x+y)a - (x-y)b + (y-z)c.$
- $(x+2z-y)a^m - (x-y-z)a^mb^n + (y+z-x)b^n,$
 $(x+y+z)a^mb^n - (x-z+y)a^m - (y-z-x)b^n,$
 and $(x-2y+z)a^m + (2x+y+z)b^n + (2y-2z-x)a^mb^n.$

$$\begin{aligned}
 8. \quad & (l-m+2n)a^{q+r} - (l+m-n)a^{p+q} + (2l-2m+n)a^{r+p}, \\
 & (3l-2m+2n)a^{p+q} + (n+m-l)a^{r+p} + (2m-n+3l)a^{q+r}, \\
 & (l-m-n)a^{r+p} - (2m+n-l)a^{p+q} + (3m+2n-2l)a^{q+r}, \\
 & \text{and } (l+m+n)a^{p+q} + (2n-m-l)a^{r+p} - (2m-l-2n)a^{q+r}.
 \end{aligned}$$

Subtract

$$\begin{aligned}
 9. \quad & (x+y-1)pq - (y-x)qr + (x+y-z)rp \\
 & \text{from } (2x-y+3z)rp + pq + (x+z-y+1)qr.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad & (2l+m-n)x^m y^n + (l+m+n)x^m - (m+n-l)y^n \\
 & \text{from } (2m+2n-l)y^n + (m-l-n)x^m + (3l+2m-n)x^m y^n.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad & (c-b-2a)x^m + (c-b+a)x^n - (a+c-2b)x^r \\
 & \text{from } (2a+2b-c)x^m - (3c+b-2a)x^n + (2b-a-c)x^r.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad & (\frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}c^2)l^2 + (-\frac{1}{2}a^2 - \frac{1}{2}b^2 - \frac{2}{3}c^2)m^2 + (\frac{2}{3}c^2 - ab)n^2 \\
 & \text{from } (a^2 - \frac{1}{2}b^2 - \frac{2}{3}c^2)l^2 - (\frac{1}{2}a^2 + \frac{1}{2}b^2 + \frac{1}{2}c^2)m^2 + \frac{2}{3}c^2 - c^2n^2.
 \end{aligned}$$

$$\begin{aligned}
 13. \quad & (a^2+ab-b^2)x^2y^2 - (b^2+bc-c^2)y^2z^2 - (c^2-2ac+a^2)z^2x^2 \\
 & \text{from } (c^2-bc-2b^2)y^2z^2 + (2ac-c^2-2a^2)z^2x^2 - (a^2-ab+b^2)x^2y^2.
 \end{aligned}$$

Find the product of

$$14. \quad x^2 + (a-p)x + a^2 - af + q \text{ and } x - a.$$

$$15. \quad x^2 + (a-2b)x + a^2 + 3b^2 \text{ and } x - a + 2b.$$

$$16. \quad x^4 - (p-1)x^3 + (q-p+1)x^2 - (p-1)x + 1 \text{ and } x + 1.$$

$$17. \quad a^3 + (m+n)a^2 + 2mna + 1 \text{ and } (m+n)a - 1.$$

$$18. \quad x^2 - (a+b)x + ab \text{ by } (a-b)x + ab - b^2.$$

$$19. \quad (x+2z)y^2 + (x^2-2z^2)y - xz(x+z) \text{ and } y+z-x.$$

$$20. \quad a^2(b-c) + b^2(c-a) + c^2(a-b) \text{ and } a+b+c.$$

$$21. \quad x+y+z, x+y-z, y+z-x, z+x-y. \quad (\text{Continued product.})$$

$$\begin{aligned}
 22. \quad & \text{Find the coefficient of } x^3 \text{ in the product of} \\
 & 3x^4 - 3x^3 + x^2 - x + 2 \text{ by } 2x^3 - 3x^2 - x + 1.
 \end{aligned}$$

Find the coefficient of x^4 in the following products :

$$23. \quad (4x^3 - 3x^2 - 2x + 1)(2x^2 - 3x + 1).$$

$$24. \quad (7x^4 - 2x^3 - 1)(2x^3 - x^2 - 2x - 3).$$

$$25. \quad (4x^5 + 3x^4 - 2x^3 + x^2 - x + 1)(2x^5 - x^3 + x - 2).$$

$$26. \quad (ax^4 + bx^3 + cx + d)(ax^2 - bx + c).$$

$$27. \quad (ax^5 + 6x^3 + 7)(cx^3 + d).$$

Divide

$$28. \quad x^3 + (a-b-c)x^2 + (bc-ca-ab)x + abc \text{ by } x+a \text{ and } x-c.$$

$$29. \quad x^3 - (a^2-ab+b^2)x + ab(a-b) \text{ by } x-b \text{ and } x-a+b.$$

30. $a^3 + (m+1)(a^2b + ab^2) + b^3$ by $a^2 + mab + b^2$.
 31. $(a-b)(b+x) + (c-a)(c+x)$ by $a-b-c-x$.
 32. $ba(b-c) + ca(c-a) + ab(a-b)$ by $a-b$ and $b-c$.
 33. $x^2(y-z) + y^2(z-x) + z^2(x-y)$ by $x-y$ and $y-z$.
 34. $x^3(y-z) + y^3(z-x) + z^3(x-y)$ by $x-y$ and $x+y+z$.
 35. $x(y^4 - z^4) + y(z^4 - x^4) + z(x^4 - y^4)$ by $(x-y)(y-z)(z-x)$.
 36. $x(y-z)^3 + y(z-x)^3 + z(x-y)^3$ by $x(y-z) + (y^3 - z^3)$.
 37. $(x-y)(x+1)(y+1) - x(y+1)^2 + y(x+1)^2$ by $x+y(2x+1)$.
 38. $x^4 + 5ax^3 + (25a-b-29)x^2 - 5(4a+b-4)x + 4b$ by $x^2 + 5x - 4$.

CHAPTER X.

ELEMENTARY EQUATIONS.

66. An **Equation** is a statement of the equality of two expressions. Thus $2x+3=7$, $ax+b=cx+d$, $ax+by=c$, and $2x+3y=5z$, are all equations.

If we enquire what value of x makes $2x+3=7$, we shall find that it is 2; for $2 \times 2 + 3 = 7$. Here x is regarded as the **variable** or **unknown quantity**, and its determination is the **solution** of the equation.

The following axioms and principles, which will be very frequently used, should be carefully noted.

Ax. I. If equal quantities or any the same quantity be added to equal quantities, the sums will be equal.

Thus, if $a=b$, then $a+c=b+c$,

or if $a=b$, and $c=d$, then $a+c=b+d$.

Ax. II. If equal quantities or any the same quantity be subtracted from equal quantities, the remainders are equal.

Thus, if $a=b$, then $a-c=b-c$,

or if $a=b$, and $c=d$, then $a-c=b-d$.

Ax. III. If equal quantities be multiplied by equal quantities or any the same quantity, the products are equal.

Thus, if $a=b$, then $ma=mb$;

or if $a=b$, and $c=d$, then $ac=bd$.

Ax. IV. If equal quantities be divided by equal quantities or any the same quantity, the quotients are equal.

Thus, if $x=y$, then $\frac{x}{m} = \frac{y}{n}$;

or if $x=y$, and $m=n$, then $\frac{x}{m} = \frac{y}{n}$.

67. Principle of Transposition of Terms : *Any term may be transferred from one side of an equation to the other, provided its sign be changed.*

Proof : Let $a-b=c$

Add b to both sides of the equation ; then

$$a-b+b=c+b \quad \text{Ax. I.}$$

$$\text{i.e., } a=c+b.$$

Thus $-b$, which was on the left-hand side of the given equation, appears as $+b$ on the right-hand side of the deduced equation, $a=c+b$.

Again, take $a+b=c$.

Subtract b from both sides ;

then $a+b-b=c-b$; Ax. II.

$$\text{i.e., } a=c-b.$$

Thus $+b$ transferred from the left-hand side of the given equation appears as $-b$ on the right-hand side of the equation finally obtained.

68. Change of signs : *If the sign of each term of an equation be altered, the two sides of the equation will still be equal.*

For, let $a-b=c-d$.

Multiply each side by -1 ;

then $-1(a-b)=-1(c-d)$; Ax. III.

$$\text{i.e., } -a+b=-c+d.$$

69. The mode of solution of simple equations will be best illustrated by the following examples.

Ex. 1. What value of x will make $7x=35$?

Divide both sides by 7 ; then $x=35 \div 7$. Ans.

Ex. 2. What value of y will make $5y+9=3y+2$?

First reduce it to the form of Example 1. To do this, bring over by transposition to the left side the terms containing the

unknown quantity, and likewise bring over the known terms to the right side.

$$\text{Since } 5y + 9 = 3y + 9,$$

$$\text{transposing } 9 \text{ and } 3y, 5y - 3y = 2 - 9;$$

$$\therefore 2y = -7;$$

divide both sides by 2; then $y = -\frac{7}{2}$. Ans.

$$\text{Ex. 3. Solve the equation } 2x - (7 - x) = 11 - (26 - \frac{1}{2}x)$$

$$\text{Remove brackets; then } 2x - 7 + x = 11 - 26 + \frac{1}{2}x;$$

$$\text{i.e., } 2x - 7 = -15 + \frac{1}{2}x;$$

to clear this of fractions, multiply each side by 2;

$$\text{then } (x - 14) = -30 + 9x.$$

Now reduce the last equation by transposition to the form of Example 1; that is, bring the terms containing the unknown to one side of the equation, and the terms with known quantities to the other.

$$\text{Thus } 6x - 9x = 14 - 30.$$

$$\therefore -3x = -16.$$

$$\text{Change the signs; } 3x = 16.$$

$$x = \frac{16}{3} = 5\frac{1}{3}. \text{ Ans.}$$

N.B. The beginner should verify the solution. Thus in the given equation, the left side $-2 \times \frac{1}{3} - (7 - \frac{1}{3}) = 9$, and the right side also $= 11 - (26 - \frac{1}{3}) = 9$.

EXAMPLES 34.

Solve

1. $7x = 42.$
2. $11x = 14.$
3. $9x + 3 = 30.$
4. $12x + 10 = 70.$
5. $6x + 14 = 44.$
6. $2x + 9 = 7.$
7. $4x - 5 = 15.$
8. $18x - 17 = 37.$
9. $3x + 11 = 21.$
10. $7x - 5 = 40.$
11. $2x + 9 = 4$
12. $11x + 15 = 9.$
13. $7x + 4 = 7\frac{1}{2}.$
14. $\frac{3}{2}x - 1 = \frac{1}{2}.$
15. $3x - \frac{1}{2} = \frac{1}{2}.$
16. $\frac{3}{2}x - 2 = \frac{1}{2}.$
17. $\frac{1}{2}x - 1 = \frac{3}{2}.$
18. $\frac{4}{3}y - \frac{2}{3} = 1\frac{1}{3}.$
19. $3 - \frac{2}{3}y = 2.$
20. $4 - 9y = \frac{1}{2}.$
21. $2x = 3x - (2x - 1).$
22. $7x - 1 = 3x - 3.$
23. $11x - 5 = 9 - 3x.$
24. $4x - 5 = 2x + (10x - 3).$
25. $3x + 2 = (2x - 1) + 4x.$
26. $(2x - 3) + (3x - 2) = x - 4.$
27. $2(x + 1) = 3(x + 5).$
28. $3(x + 2) + 5 = 5(x + 4).$
29. $2(x - 2) + 3(x - 4) = x - 6.$
30. $x + 1 = \frac{2}{3}(x + 5).$
31. $x + 2 = \frac{1}{2}(2x + 9).$
32. $(7x - 5) - (2x - 3) = \frac{3}{2}.$
33. $7 - 3x = 11 - 2x.$

34. $9 - 4x = 7 - 5x$.
 35. $4 - 5x = 3 - (1 - 2x)$.
 36. $2x - 3 = 4 - (7 - x)$.
 37. $3 - (2 - x) = 1 - 2x$.
 38. $2x - 5 = 2x - (x - 2)$.
 39. $\frac{1}{2}x = \frac{1}{2}x - 6$.
 40. $4x - 1 = 3(x + 1) - 2$.
 41. $x + 4 = 2(x + 1) - 1$.
 42. $4(x - 3) = 5(x - 2)$.
 43. $7x + 4 = 10x + 1$.
 44. $6x + 8 = 9(x + 1) + 2$.
 45. $5(x - 3) = 7(1 - x)$.
 46. $\frac{1}{2}x + \frac{1}{3} = 5$.
 47. $\frac{1}{2}(x - 1) = \frac{1}{2}(x - 3)$.
 48. $\frac{3}{4}(x - 1) = \frac{1}{2}x - 5$.
 49. $\frac{2}{3}(x - 2) = 1 - \frac{1}{3}(2x - 7)$.
 50. For what value of x is $4 - 2x - (3x + 5) = 7 + 2x - (2 + 9x)$?
 51. For what value of x is $x + \frac{1}{2} - (2x - \frac{1}{2}) = 3x - \frac{3}{4}$?
 52. When will $m + 3 = 4 - 7m - 5(2m + 1)$?

CHAPTER XI.

TRANSLATION INTO ALGEBRAICAL LANGUAGE.

70. The following examples are intended to show how any proposed quantity may be expressed in algebraical symbols.

Ex. 1. A man having a son and a daughter, is 20 years older than the son, and twice as old as the daughter. Denoting the son's age by x , find the daughter's age.

The son's age in years = x .

Since the father is 20 years older than the son,

the father's age in years = $x + 20$.

Since the daughter is just half as old as the father,

the daughter's age required = $\frac{1}{2}(x + 20) = \frac{1}{2}x + 10$. *Ans.*

Ex. 2. A horse and its saddle together cost £ y . If the horse costs £ $(2x - a)$, what is the price of the saddle?

Cost of horse + cost of saddle = £ y ,

and cost of horse = £ $(2x - a)$;

∴ cost of saddle = $\{y - (2x - a)\}$ in £
 = £ $(y - 2x + a)$. *Ans.*

EXAMPLES 35.

1. By how much is x less than a ?
2. What must be added to a to make 8?

3. What is the quotient of 3 divided by $2x$?
4. The product of two numbers is x , and one of them is y ; what is the other?
5. In a division sum, the divisor is 16, and the remainder is 11; what is the dividend, if the quotient is x ?
6. In a division sum, the dividend is a , and the remainder is b ; if the divisor be denoted by x , what is the quotient?
7. The difference of two numbers is 8, and the greater of them is a ; what is the other?
8. How many pictures each worth 4s. can be bought for £ y ?
9. If mangoes are selling at x for an anna, what will be the cost in rupees of b mangoes?
10. What is the price in pence of x gooseberries at fourpence a score?
11. x persons pay £ $(a-5)$ in equal shares. What is the share of each in shillings?
12. The sum of $2x$ shillings and $3a$ pence is 11 b pounds; express this statement symbolically.
13. How many hours will be required to make a journey of a miles at the rate of 12 miles an hour?
14. A man travels at the rate of a miles an hour, and is out for p hours; how far has he travelled?
15. If I can walk x miles in m days, what is my rate per hour?
16. Find three consecutive numbers of which the middle one is $2n$.
17. Write down the product of three consecutive odd numbers of which the middle one is $2x+1$, and find the product of the even numbers just before and just after $2x+1$.
18. A man is x years old; what will be his age a years hence, and what was his age a years back?
19. How old is a man who in x years hence will be thrice as old as his son aged 5 years now?
- * 20. In 7 years hence a boy will be x years old; what is the present age of his father, who is twice as old as the son?
21. If A can do a work w in x hours, how much work does he do per hour? If B can do the same work in y hours, what is the rate of work per hour of A and B together?
22. How long will it take a man to walk m miles, if he walks at the rate of n miles in b hours?

23. A and B began to play a game, having respectively $\text{£}x$ and $\text{£}y$. A loses 3s., but afterwards gains half of what B has at the beginning of the second chance. Find the money of each at last.

24. A bag contains $\text{£}x$ in sovereigns, shillings and pence. If the numbers of the sovereigns and shillings be respectively y and z , find the number of the pence.

CHAPTER XII.

SIMPLE PROBLEMS.

- 71. Solution of equational problems involving one unknown quantity. The method of procedure is as follows :

Denote the unknown quantity by a symbol x , and express in algebraical language the conditions of the question ; solve the resulting simple equation.

Ex. 1. Four times a number, diminished by 25, equals 63 ; find it.

Let x denote the number required.

Then, four times of it $= 4x$.

\therefore four times the number, diminished by 25 $= 4x - 25$;
by the question, the last result $= 63$.

$\therefore 4x - 25 = 63$. Now solve this equation.

Transposing, $4x = 63 + 25 = 88$;

dividing both sides by 4, $x = 22$, required number. *Ans.*

N.B. Verify the solution thus : $4 \times 22 - 25 = 63$.

Ex. 2. A and B begin to play a game with equal sums of money. After A has won Rs. 4 from B, he finds that he has then $\frac{2}{3}$ of what B then has. What sum had each at first ?

Let x denote the required sum in rupees.

A wins Rs. 4, and B of course loses the same amount ; therefore after the play A's money in rupees $= x + 4$,

and B's " " " $= x - 4$.

Since after the play A's money $= \frac{2}{3}$ of B's,

by the question, $x + 4 = \frac{2}{3}(x - 4)$.

To solve this, clear it of fractions by multiplying each side by 7.

Thus, $7x + 28 = 9(x - 4) = 9x - 36$;

transposing, $7x - 9x = -36 - 28$;

i.e., $-2x = -64$;

changing the sign, $2x = 64$;

dividing both sides by 2, $x = 32$.

\therefore the sum required = Rs. 32. *Ans.*

N.B. After the play A's money = Rs. 32 + Rs. 4 = Rs. 36, and B's money = Rs. 32 - Rs. 4 = Rs. 28, and 36 = $\frac{3}{2}$ of 28.

EXAMPLES 36.

- Four times a number increased by 7 is 51 ; find it
- What number multiplied by 12, and then diminished by 14, gives 82 as remainder ?
- If 5 be added to a number, and the sum be multiplied by 16, the product is 144 ; find it
- What number is greater than two-thirds of itself by 60 ?
- What number is less than 35 by two-fifths of itself ?
- When a number is divided by 5, and the quotient is increased by 7, the result is 23 ; find the number.
- The quotient of a number by 12, being diminished by 11, gives 9 as remainder ; find it
- What number is as much above 17 as under 25 ?
- If a certain number be increased by 5, the sum multiplied by 5, and the product diminished by 5, the result is 70 ; find the number.
- Find a number such that if it be diminished by 5, the remainder divided by 6, and the quotient multiplied by 7, the result is 84.
- What number should be subtracted from 17 in order that the remainder may be less than two-thirds of the same by 3 ?
- My annual income is Rs. 40 ; by what sum should it increase in order that the new income may be six times the increase ?
- A person spends $\frac{1}{3}$ of his income, and has Rs. 100 left. what is his income ?
- Half of an income goes to pay the butcher's bill, and one-third to meet the other expenses ; if the remainder be Rs. 80, what is the whole income ?

15. A man bequeathed two-thirds of his property to his wife, and one-fourth to his son ; if he has still property worth £120, what is the value of the whole property ?

16. After cutting off one-third of a log of wood, and then one-fourth of the remainder, I find that the log still measures 2 yards ; determine its original length

17. I give away two-fifths of a stock of rice, and then two-thirds of the remainder ; if 20 maunds are now left, what was the whole stock ?

18. A man spends £2, and then borrows as much as he has left ; he again spends £2, and again borrows as much as he has left ; he finally spends £2, and has nothing left. How much had he at first ?

19. A and B begin to play a game with equal sums of money. After A has won Rs. 10 from B he finds he has five-fourths of what B then has ; what sum had each at first ?

20. A and B have equal sums of money ; B gives A £5, and then takes one-third of A's money ; after these exchanges, they have again equal sums ; what sum had each at first ?

21. How much water should be added to 32 seers of milk at 3 *as.* 9*d.* per seer, in order to reduce the price of the mixture to 3 *as.* per seer ?

22. Assam tea is worth Re. 1. 8 *as.* per lb., and Ceylon tea Rs. 2 per lb. ; what quantity of Assam tea should be mixed with 12 lbs. of Ceylon tea, in order that I may neither gain nor lose by selling off the mixture at Re. 1. 14 *as.* per lb. ?

23. A man buys a certain number of oxen at Rs. 45 per head, and immediately two of them die ; he then finds that he should charge Rs. 5 more than he paid per head, in order to clear his outlay. How many oxen did he buy ?

24. A has Rs. 150, and B Rs. 210 ; what sum should A give away to B, in order that A may then have just half as much as B ?

25. A spendthrift ran through one-third of his fortune in 3 months, one-fourth in two months more, and five-sixths of what was left in 7 months more ; at the close of the year he had only £75 left. What was the original amount of his fortune ?

26. I lay out on business a certain sum of money, and gain successively half and one third of it. If I have at last £110, what was the outlay ?

27. I bought a certain number of mangoes at 3 pice apiece, and then half as many at 4 pice apiece, and gained 5 pice by selling the whole at 3½ pice apiece. How many did I buy at first ?

28. A zemindar lets out two-fifths of his estate, and has in his own hands 80 bighas more than a half ; how large is the estate ?

72. When more than one unknown quantity is involved, denote one of the unknowns by x , and then express the rest in terms of it.

Ex. 1. Find two numbers whose sum is 800, and difference 60.

Let x denote the smaller number.

Since the difference of the numbers is 60, the larger number $= x + 60$.

Since the sum of the two numbers is 800, $x + (x + 60) = 800$,

$$\text{i.e., } 2x + 60 = 800.$$

Now solve the last equation.

$$\text{Transposing, } 2x = 800 - 60 = 740 ;$$

dividing both sides by 2, $x = 370$;

$$\therefore x + 60 = 370 + 60 = 430.$$

\therefore the required numbers are 370 and 430. (Verify).

Ex. 2. Divide 80 into two parts, so that 10 times the smaller may exceed 200 by as much as 7 times the greater falls short of 450.

Let x be the smaller part.

Since the two parts together make up 80, the larger part $= 80 - x$.

By the problem, 10 times the smaller part $- 200$

$$= 450 - 7 \text{ times the greater.}$$

$$\therefore 10x - 200 = 450 - 7(80 - x) = 450 - 560 + 7x ;$$

transposing, $10x - 7x = 450 - 560 + 200 = 90$;

$$\text{i.e., } 3x = 90 ;$$

dividing by 3, $x = 30$.

$$\therefore 80 - x = 50$$

\therefore the required parts are 30 and 50. (Verify).

Ex. 3. Divide £89 between A, B and C, so that C may have £10 more than A, and £12 less than B

Let C's share be £ x .

Since A has £10 less than C, A's share $= £(x - 10)$.

Since B has £12 more than C, B's share $= £(x + 12)$.

$$\therefore \text{the total sum} = £x + £(x - 10) + £(x + 12) = £(3x + 2).$$

Now, by the question, this $= £89$.

$$\therefore 3x + 2 = 89 ;$$

transposing, $3x = 89 - 2 = 87$;

dividing by 3, $x = 29$.

$$\therefore x - 10 = 19, \text{ and } x + 12 = 41.$$

\therefore the shares of A, B and C are respectively £19, £41 and £29.

N.B. The student is advised to work out the problem by representing A's money by x .

Ex. 4. Tea sells at 2s. 6d. per lb., and coffee at 3s. 6d. per lb.; I spend £4. 2s. on them, and get 28 lbs. in all; how much of each do I buy?

Let x denote the quantity of tea in lbs.

Then, since the total quantity bought is 28 lbs., the quantity of coffee in lbs. = $28 - x$.

The cost of x lbs. of tea at 2s. 6d. per lb. = $\frac{3}{2}x$ shillings.

The cost of $(28 - x)$ lbs. of coffee at 3s. 6d. per lb. = $\frac{7}{2}(28 - x)$ shillings.

$$\begin{aligned}\therefore \text{the total cost in shillings} &= \frac{3}{2}x + \frac{7}{2}(28 - x) \\ &= \frac{3}{2}x + 98 - \frac{7}{2}x \\ &= 98 - x.\end{aligned}$$

By the question, the total cost = £4. 2s. = 82s.

$$\therefore 98 - x = 82;$$

$$\text{transposing, } -x = 82 - 98 = -16;$$

$$\text{changing the sign, } x = 16.$$

$$\therefore 28 - x = 12.$$

\therefore the quantity of tea required = 16 lbs.

" " " coffee " = 12 lbs.

EXAMPLES 37.

1. Divide 60 into two parts such that their difference will be equal to thrice the less.

2. Divide 88 into two parts such that one part will exceed the other by 12.

3. Find two numbers such that their sum is 50, and difference 20.

4. Find two consecutive numbers such that their sum is 91.

5. The difference of two numbers is 20, and five times the greater equal nine times the less; find the numbers.

6. The sum of two numbers is 98, and the greater exceeds twice the less by 14; find the numbers.

7. Divide 32 into two parts such that twice the greater added to thrice the less may be 76.

8. Divide 45 into two parts such that a fourth of one part may be equal to a fifth of the other.

9. A third of one number is a fifth of another, and their difference is 40; find them.

10. Divide 93 into two parts such that 14 times one part may be equal to 17 times the other.

11. Divide Rs 200 between A and B so that B may have Rs. 30 more than A.

12. Divide £40 among A, B and C, so that A may have twice as much as B, and B thrice as much as C.

13. A has £6 more than B, B has £8 more than C, and they have £58 between them. How much has each?

14. A has Rs. 10 more than B. B has Rs. 6 less than C, and they have Rs 178 between them. How much has each?

15. A, B and C have a certain sum among them. A has one-half of the whole, B one-third of the whole, and C £125. How much has each?

16. A has £20 less than B, C has as much as A and B together, and they have £164 between them. How much has each?

17. Divide £140 among 4 men and 6 children so that each man may have twice as much as each child.

18. Divide £166 among 8 men and 10 children so that each man may have 10s less than thrice as much as each child.

19. Divide £1200 among 60 boys and 80 girls so that a boy's share may be thrice as much again as a girl's

20. Divide a purse worth Rs 271 among 10 men, 12 women and 20 girls, so that each man may get twice as much as a woman, and each woman Rs. 4 more than each girl

21. Divide Rs. 952 among 30 men, 40 women and 60 girls so that a woman may get Rs 8 less than a man, and Rs. 6 more than a girl.

22. A subscription of £175 was raised by A, B and C. A subscribed as much as C and £20 more, and B subscribed as much as A and £30 more. Find the contribution of each

23. Three brothers divide a legacy of £360 in such a way that the middle brother gets £40 more than the eldest, and £40 less than the youngest; find the share of each

24. A man gave away Rs. 39 to a batch of old men and women, 24 in all; if each man received Re. 1. 4 as., and each woman Re. 1. 2 as., how many of each sex were there?

25. A sum of money is divided between A, B and C in such a way that A and B have Rs. 80 between them, B and C Rs 100 between them, and C and A Rs 90 between them; find the share of each.

26. A and B have Rs. 100 between them. A gives B as much as the latter had originally, and then finds that B's purse is now four times his own. How much had each at first?

27. A father's age is thrice that of his son; in 8 years the father will be twice as old as the son; how old are they?

28. A father is older than his son by 20 years ; in 10 years hence the father will be twice as old as the son ; how old are they now ?

29. The sum of the ages of a father and his son is 42 years, and 9 years hence the father's age will be thrice as great as that of the son ; find their present ages.

30. The sum of the ages of a father and his son is 46 years, and 7 years back the father's age was thrice as great as that of the son ; find their respective ages.

31. I spend a certain sum on oranges at 2 a penny ; had they been offered at 3 a penny, I would have got 8 more for the same sum. How many oranges do I buy, and what sum do I spend ?

32. I buy two kinds of tea for £35, one at 1s. 3d. per lb, and the other at 1s. 6d. per lb ; if the total quantity bought be 520 lbs, how much of each kind was there ?

33. I buy some hens and geese, 65 in all, the hens at 1s. 6d. per head, and the geese at 2s. per head ; if the total price paid be £5. 9s, how many hens are bought ?

34. I sell 45 lbs. of coffee, some at 1s. 3d. per lb, and the rest at 1s. 9d. per lb, and realise £3 8s 9d by the sale ; what quantities have been sold at the different rates ?

35. I bought 25 yards of cloth for Rs. 223 8as ; for a part I paid Rs. 8 8as. a yard, and for the rest Rs. 9. 8as. a yard ; how many yards were there at each price ?

36. A can copy 40 pages of a manuscript per day, and B 60 pages per day ; how long should each work in order to finish 320 pages in 7 days ?

37. A milkman buys a certain quantity of milk at 3 as per seer, and then adds water so as to make up a mixture of 20 seers ; he next sells the mixture at 2 as. 9d. per seer, gaining 7as. altogether ; what quantity of water was added ?

38. A contractor agreed to finish a work in a fixed time, during which he was to be paid 8as per day ; the work was, however, finished in 40 days, and the man was fined 4 as. for each day by which he exceeded the appointed time. If he received only Rs. 8, in what time was he to finish the work, and by how much did he exceed it ?

39. Two sums of money are together equal to £54. 12s., and there are as many pounds in the one as shillings in the other ; find the sums ?

40. A sum of Rs 63. 4 as. was paid in rupees and two-anna pieces ; the total number of coins being 100 ; how many of each kind were used ?

MISCELLANEOUS EXAMPLES I.

1. Find the value of $a^3 + b^3 + c^3 + 3abc$, when $a = 12$, $b = 13$, $c = 25$.
2. Add together $2x^4 - 3x^3 + 7x^2 - 3x + 4$, $9x^3 + 7x^2 - 6x - 5$, $12x^5 - x^3 - 1$, $7 - x^3 - 1 - 9x^2 - 6x^4$, $13 - 5x^4 - 9x^3 - 11x^2 - 15x$, and $3 - x + 3x^4$.
3. Simplify $a - 2[b - 3\{c - 4(d - e)\}]$.
4. Subtract the expression $13x^6 - 11x^5 + 9x^4 - 7x^3 + 6x^2 - 5x - 9$ from the expression $6x^6 - 7x^4 - x^2 + 12$.
5. Multiply $x^3 - 2x^2 + 3x - 4$ by $2x^3 - 5x^2 + 4x - 6$.
6. Reduce to the simplest form $(2a^3 - a^2 + 3a - 4)(2a^3 + a^2 + 3a + 4) + (2a^3 + a^2 - 3a + 4)(2a^3 + a^2 + 3a - 4)$.
7. Divide $2a^5 - 11a^4 + 16a^3 - 23a^2 + 32a - 16$ by $a^2 - 5a + 4$.
8. For what value of l is $2l + 3 = 3(l - 1) - 5$?
9. Solve $4x + 3(x - 1) = 9(x - 2) - 5(x - 3)$.
10. A certain number being multiplied by 3, the product increased by 5, and the sum divided by 7, the result is 5; find the number.
- ✓ 11. Find the value of $\frac{x^3 - y^3}{x^3 + y^3} + \frac{y^3 - zw}{2y^3 + zw} + \frac{z^3 - y^3}{3xyz}$, being given that $x = 1$, $y = -1$, $z = 2$, $w = 0$.
12. Add together $(n - 1)x^4 + mx^3 + (r - n)x^2 - nx - 2$, $mx^3 - (r - m)x^2 + (m + n)x + 1$, $(m - n + 1)x^4 - (2m - n)x^3 + (m + n)x^2 + (m - 2n)x + 1$.
13. Simplify $2x - y - 3\{x - 2y - 4(2x + y) + 5(x + y)\}$.
14. Subtract $(a^2 - b^2)(x^2 + y^2)$ from $(a^2 + c^2)x^2 - (b^2 - c^2)y^2$.
15. Find the expression whose quotient by $1 - 2a + 3a^2 - 4a^3$ is the expression $1 + a + 2a^2 + 3a^3$.
16. Find the continued product of $x - a$, $x - b$ and $x - c$, and thence deduce $(x - 2)^3$.
17. Divide $x^3 - 16a^3$ by $x^2 - 2ax + 2a^2$.
18. Resolve into factors $p^2 - pq - 72q^2$, and $12a^2 + ab - 20b^2$.
19. For what value of y will $2y + 1 + 2(2y + 3) = 4(2y + 5)$?
- ✓ 20. A man buys 35 seers of milk at 3as. per seer; how much water should he mix with it in order to reduce the price to 2as. 6p. per seer?
- ✓ 21. Find the value of $2\sqrt{\frac{a^2 + b^2}{d - c + b - a}} + \frac{1}{d - c + b - a}\sqrt{(a^2 + b^2 - c^2 + 6a^2)}$, being given that $a = 3$, $b = 4$, $c = 5$, and $d = 6$.
22. Add together $a^3 - 2ab + b^3$, $(n - 1)a^3 - (n - 2)ab + (n - 1)b^3$, $(n^2 - n + 1)a^3 + (n^2 + n + 2)ab + (n^2 - n + 1)b^3$.

23. Take $(a-b)x^2 + (b-c)xy + (c-a)y^2$ from $(a-b)(x^2 + xy + y^2)$.
24. Simplify $3x - 4\{2x - \{3y - (2x - 3x) - 4(x - y)\}\}$.
25. Simplify $(a+1)(a+2)(a+3) - (a-1)(a-2)(a-3)$.
26. Multiply $x^2 + y^2$ by $x - y$, and divide the product by $x + y$.
27. Find the coefficient of x in the product of $x^2 + x(a-b) - ab$ and $x^2 + x(a+b) + ab$.
28. Resolve into elementary factors $9(a+b)^2 - 16$, $x^2 - 34x + 288$, $a^6 - a^2$, and $5y^2 - 38y + 21$.
29. Solve $1 - 2\{x - 3(x-4)\} = 0$, and $4(1-2x) + 6(1+x) = 13$.
30. There are two men, one of 35, and the other of 25; when was the first twice as old as the second?
31. Evaluate the expression $\frac{\sqrt{(x^2 + 2yz)}}{z} + \frac{\sqrt{(y^2 + xz)}}{y} + \frac{\sqrt{(z^2 + xy)}}{x}$, when $x = 4$, $y = 3$, $z = -2$.
32. Add together $2x^2 - \frac{1}{2}x^2y + \frac{1}{3}xy^2 - \frac{2}{3}y^3$, $\frac{1}{3}y^3 + xy^2 - x^2y + \frac{2}{3}x^2$, $\frac{1}{2}xy^2 + \frac{1}{3}y^3 + \frac{1}{2}x^2$, $\frac{1}{2}x^2 + \frac{2}{3}xy^2 - y^3$.
33. Simplify $(a+b)^2 - (a+b)(a-b) - \{a(2b-a) - b(2a-b)\}$.
34. Find the product of $x^2 + y^2 + 1 - 2xy - x - y$ and $x + y + 1$.
35. The product of two expressions is $a^6 + a^5b + a^4b^2 - a^3b^3 + b^6$, and one of them is $a^3 + ab + b^3$; find the other.
36. What must be subtracted from $(a-b-c)^2$ that the remainder may be $(a+b+c)^2$?
37. If $x^2 + 9x + a$ be exactly divisible by $x + 3$, find a .
38. Resolve into elementary factors
(1) $(a^2 + 2b^2)^2 - (2a^2 + b^2)^2$; (2) $75ab^3 - 130a^2b^4 - 9a^3b^5$.
39. Shew that $a^3 = a(a-1)(a-2) + 3a(a-1) + a$.
40. A man paid a bill of £50 in sovereigns and crowns, using in all 125 coins. How many coins of each sort did he use?
41. Evaluate $\frac{\sqrt{(17-4m-m^2)}}{m^2 - \sqrt{(1-m^2)}}$, when $m + z = 0$.
42. From $(a+b-2c)x^2 + (2a+b-c)x^2 + (b+2c-a)x + b+c - 2a$ take $(a+b+c)x^2 + (a+b-c)x^2 + (b+c-a)x + 2(b+c-a)$.
43. On dividing an expression by $3a^2 - 4ab + b^2$, the quotient is $5a^2 - 6ab + 7b^2$, and the remainder $b^4 - 17ab^3$, find the expression.
44. Shew that $2x^3 - (a+6b+4c)x^2 + (12bc+2ca+3ab)x - 6abc = 0$, being given that $2x = a$.
45. Write down at once the product $(5xy - 4ab)(6xy - 5ab)$.
46. Divide $1 - 3a(1-a+a^3-a^{10}) - a^{12} - 9a^4(1-a+a^5-a^8) + 2a^6(1-a^6)$ by $(1-a)(1+a^2)$.

47. Factorise (1) $50y^2 - 5y - 28$, (2) $125a^3 - 20a - 12$,

(3) $(3x^2 - 4)^2 - (2x^3 - 3)^2$.

48. Solve (1) $3(x+5) - 12x = 2x - 18$, (2) $195 - 3(x-5) = 5(x+26)$.

49. What number is that from which 'if 16 be subtracted, three-sevenths of the remainder shall be 33?

50. A cod's tail weighed 20 maunds, its head as much as its tail and one-third of its body, and its body as much as the head and tail together. Find the weight of the fish.

51. Find the value of $\{x - (y - z)\}^2 + \{y - (z - x)\}^2 + \{z - (x - y)\}^2$, when $x = -2$, $y = -3$, $z = -4$.

52. Simplify $2a - 3\{b - c - 4\{a - c - 5(a - b)\}\}$

53. Subtract the sum of $(m+n)x^3 + (m-1)x^2 + (n-1)x + 1$ and $(n+1)x^3 + (m+1)x^2 + (n+1)x + n$ from that of $(m+n-1)x^3 + mx^2 + nx + 1$ and $nx^3 + x^2 + x + n - 1$

54. If $a = 2x - 5y + 6z$, $b = 2x + 5y + 6z$, and $c = 2x + y + 6z$, find $2a + 3b - 5c$ in terms of x, y, z .

55. Multiply $x^2 + y^2 + 1 - xy - x - y$ by $x + y - 1$.

56. Find the expression which, when multiplied by $x - b + 2a$, gives $x^3 + a(4b - a)x - (b - 2a)(3a^2 + b^2)$.

57. Shew that

$$(x+5)(y+5) - 5(x+1)(y+1) + 5(x-1)(y-1) - (x-5)(y-5) = 0.$$

58. Find the expression which, when divided by $x^3 - 2x^2 + 3x - 5$, gives a quotient $x^3 + 2x^2 - 3x - 5$, and a remainder $7x^2 - 6x + 13$.

59. Resolve into simple factors

(1) $21x^2 - 17x - 50$, (2) $3x^3 - 27x$, (3) $4x^4 + 4x^6 - 224x^7$.

60. If x is the cost in pence of y lbs. of tea, how many shillings will be required to buy z ounces?

61. When $x = \frac{16}{25}$, find the value of $\frac{\sqrt{1-x}}{(1+a)\sqrt{x} + \sqrt{1-x}}$

62. Simplify $2a - [6b + (4b - 3c) - 15c + \{2a - (6b - 5c - 4b)\}]$.

63. Find the sum of $ax^4 - bx^3 + cx^2 - dx + e$, $bx^4 - ax^3 + bx^2 + dx - e$ and $cx^4 - cx^3 + ax^2 + dx + e$, and subtract the resulting sum from $(a+b+c)(x^4 - x^3 + x^2 + x + 1)$.

64. Multiply $4a^2 + 2ax + x^2 - 8x + 4n + 4$ by $2(a-1) - x$.

65. Divide $13ax^2 + \frac{1}{2}a^4 + 12x^4 - \frac{5}{2}a^2x^2 - \frac{1}{2}a^3x$ by $\frac{1}{2}ax - \frac{1}{2}a^2 + 6x^2$.

66. Find the coefficient of x^3 in the product of $1 - 2x - 3x^2 + x^3 - x^4$ and $1 - 3x + 4x^2 - 5x^3 + 6x^4 - x^5$.

67. Simplify $\{2(a-b)\}^2 + (a+b)^2 - (2a-b)^2 - (a-2b)^2$.

68. Divide $(a+b)(a^3 - b^3) + 3ab(a-b) - 54a + 81$ by $3a - b$.

69. Shew that $x^3 = x(x-1)(x-2) + 3x(x-1) + x$.

70. How old is a man who a years ago was twice as old as his son, now aged b years?

71. When $x=64$, and $y=12$, find the value of the expression

$$\frac{x-64}{\sqrt{x-49}} - (\sqrt{x-7})(\sqrt{x-2}).$$

72. Add together $(x-1)(x+3)$, $2(\sqrt{2+x})(\sqrt{2-x})$, $(x+1)(x-2)$.

73. To what must the sum of $2a^2 - 3bc + 2ac$, $3b^2 - 4ab + 7ac$, and $3c^2 - 2ab - 4ac$ be added in order that the total may be equal to $a^2 + b^2 + c^2 + bc + ca + ab$?

74. Find the continued product of

$$\frac{a}{2} - \frac{b}{3}, \quad \frac{a}{2} + \frac{b}{3}, \quad \frac{a^2}{4} + \frac{b^2}{9}, \quad \frac{a^4}{16} + \frac{b^4}{81}.$$

75. Divide $x^7 - 64x$ by $x^2 - 2x + 4$

76. Shew that

$$(a+2x)(b+2x) - 4(a+x)(b+x) + 6ab + (a-2x)(b-2x) = 4(a-x)(b-x).$$

77. Determine the coefficient of x^3 in the product of $x^2 - x + 1$ and $x^3 + (m-n)x^2 - (m-r+n)x - m + n$.

78. Resolve into their elementary factors (1) $21x + 23x^2 - 20x^3$, (2) $(13a^2 - 5b^2)^2 - (12a^2 + 4b^2)^2$, (3) $(nl - mr)^2 - (lr - mn)^2$.

79. What value would you give to x so as to make the expression $5x - [4x - \{3x + 2 - (x - 1)\}] - \{6x - 3(2x - 1)\}$ equal to 0?

80. How much sugar at 4d per lb. must be mixed with 20 lbs. at 5½d. per lb., so that the mixture may be worth 5d. per lb.?

81. If $x=4$, $y=3$, and $z=2$, find the value of the expression

$$\frac{x^2 + y^2 - z^2}{yz + zx - xy} - \frac{\sqrt{(x+y+z)}}{2(y+z-x)}.$$

82. Add together $3x^3 - 5x^2y + 2y^3$, $8x^2y - 3y^3 + 2xy^2$, $5xy^2 - 4x^3 - 3x^2y$, $2x^3 - 6xy^2 + 4y^3$, and subtract the sum from $x^3 + xy^2$.

83. Simplify the expression

$$(a^2b^2 - abxy + x^2y^2)(ab + xy) + (a^2b^2 + abxy + x^2y^2)(ab - xy).$$

84. Multiply $m^6 + m^5n - m^4n^2 + mn^5 + n^6$ by $m^2 - mn + n^2$.

85. Divide the expression

$$x^2(y+z+2) + 2x(y+z)(y+z+2) - y-z-2 \text{ by } x^2 + 2xy + 2xz - 1.$$

86. Simplify completely the following expression :

$$(3x-y)^2 - (2x^2-y^2)(3x+y) - 3x\{(y-3x)^2 - (2x^2-y^2) - 2(2x-y)y\},$$

87. Assuming $A = a^2 - bc$, $B = b^2 - ca$, and $C = c^2 - ab$, prove that $bA + cB + aC = 0 = cA + aB + bC$.

88. Resolve into factors (1) $x^2 - x - 306$, (2) $25x^2 - 105x + 68$, (3) $3 + 28x - 20x^2$, (4) $3(l+m)^2 - 2(l^2 - m^2) - l(l+m)$.

99. Find an algebraical expression which exceeds $ax^2 - ax^2y^2 + xxy^4$ by as much as it falls short of $bx^4y - dx^2y^3 + yy^3$.

100. I ride from Howrah to Hugli in x hours at the rate of a miles per hour, and after stopping at the latter place for y hours, ride back at the rate of b miles per hour past Howrah to a place distant c miles from it; if the whole time spent be z hours, find the relation between x , y and z .

101. If $l=3$, $m=4$, $n=5$, $r=6$, find the value of

$$\frac{\sqrt{l^2+m^2} + \sqrt{l^2+m^2+n^2}}{m-l+r-n}, \text{ and } \frac{(l+m)(n+r) - (m+n)(l+r)}{lm-mn+nr-lr}.$$

102. Simplify completely the expression $(a^2d - 3abc + 2b^2)x^3 + 3(abd + b^2c - 2ac^2)x^2y + 3(2b^2d - acd - bc^2)xy^2 + (3bcd - ad^2 - 2c^3)y^3$, when $z = -b$, and $y = a$.

103. Multiply $ax^3 + bx^2 + cx + d$ by $lx^2 + mx^2 + nx + r$, and thence deduce the product of $3x^3 + 10x^2 + 7x - 2$ and $3x^3 + 13x^2 - 17x + 6$.

104. One of two expressions is $c - \frac{3b}{4} + \frac{1ca}{3}$, and their product is $50a^2 - \frac{275ab}{12} + \frac{145}{9}ac + \frac{21}{8}b^2 - \frac{15}{4}bc + \frac{c^2}{3}$, find the other

105. Find the divisor when $(4x^2 + 7xy + 5y^2)^2$ is the dividend, $8(x+2y)^2$ the quotient, and $y^2(9x+11y)^2$ the remainder

106. Simplify $(1-2x+x^2)(1+2x+x^2)(1+2x^2+x^4)(1+2x^4+x^8)$ and $(a^4b^4 - a^2b^2 + 1)(a^2b^2 + 1) - (a^4b^4 + a^2b^2 + 1)(ab - 1)(ab + 1)$.

107. Simplify completely $(a+b+c)(p+q+r) + (b+c-a)(q+r-p) + (c+a-b)(r+p-q) + (a+b-c)(p+q-r)$.

108. Find the factors of (1) $49a^2 - 154ab + 121b^2$, (2) $x^2 + \frac{2}{3}x + \frac{1}{9}$, (3) $a^3 + \frac{3}{2}a + \frac{1}{2}$, (4) $2a^2b - a^3 - ab^2$.

109. If a be greater than $3b$, find an expression which exceeds $a-2b$ by as much as $a-2b$ exceeds b .

100. The number of months in the age of a man on his birthday in 1850 was exactly half the number denoting the year in which he was born. In what year was he born?

101. Divide $(x^2 - xy + \frac{1}{4}y^2)(x^2 + y^2)$ by $x - \frac{1}{2}y$.

102. Find the factors of $3x^3 + 6x^2 - 129x$

103. Divide 1 by $(1+x)^2$ to four terms.

104. Multiply $x^2 + (a+2)x + 3$ by $x^2 - 3x + a + 2$

105. Show that

$$(ax+b)^2 + (cx+d)^2 + (bx-a)^2 + (dx-c)^2 = (a^2+b^2+c^2+d^2)(x^2+1).$$

106. If $2x + 3y = 7$, and $xy = 2$, show that $(2x - 3y)^2 = 1$.

107. For what value of a is $a^2 + 9 = (a+1)^2$?

108. Solve $x - (2x-1) = (x+1)^2 - (x-1)^2$.

109. The side of a square court-yard is a ft. ; if the length were increased by 1 ft, and the breadth diminished by 2 ft, find the change in its area

110. A man buys some apples for 12 as , and gains 2 as , by selling at 8 p . apiece all but 3, which become rotten : how many did he buy ?

CHAPTER XIII.

IMPORTANT FORMULÆ.

73. The student should prove by actual multiplication, and then commit to memory the following very useful identities.

$$(a+b)^2 = a^2 + 2ab + b^2. \quad \dots \quad \dots \quad \text{I.}$$

$$(a-b)^2 = a^2 - 2ab + b^2. \quad \dots \quad \dots \quad \text{II.}$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \quad \dots \quad \dots \quad \text{III.}$$

$$a^2 - b^2 = (a+b)(a-b) \quad \dots \quad \dots \quad \text{IV.}$$

$$x+a)(x+b) = x^2 + (a+b)x + ab \quad \dots \quad \dots \quad \text{V.}$$

$$(x+a)(x-b) = x^2 + (a-b)x - ab \quad \dots \quad \dots \quad \text{VI.}$$

$$x-a)(x+b) = x^2 + (-a+b)x - ab. \quad \dots \quad \dots \quad \text{VII.}$$

$$x-a)(x-b) = x^2 - (a+b)x + ab \quad \dots \quad \dots \quad \text{VIII.}$$

$$(x+a)(x+b)(x+c) \\ = x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc \quad \dots \quad \text{IX.}$$

$$a+b)^3 = a^3 + 3ab(a+b) + b^3 = a^3 + 3a^2b + 3ab^2 + b^3. \quad \dots \quad \text{X.}$$

$$a-b)^3 = a^3 - 3ab(a-b) - b^3 = a^3 - 3a^2b + 3ab^2 - b^3. \quad \dots \quad \text{XI.}$$

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a+b)(b+c)(c+a) \quad \dots \quad \text{XII.}$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) \quad \dots \quad \dots \quad \text{XIII.}$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2). \quad \dots \quad \dots \quad \text{XIV.}$$

$$a^2 + b^2 + c^2 - bc - ca - ab = \frac{1}{2}\{(a-b)^2 + (b-c)^2 + (c-a)^2\} \quad \text{XV.}$$

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - bc - ca - ab) \\ = \frac{1}{2}(a+b+c)\{(a-b)^2 + (b-c)^2 + (c-a)^2\}. \quad \text{XVI.}$$

$$(a+b)(b+c)(c+a) = a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc \\ = a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) + 2abc \\ = bc(b+c) + ca(c+a) + ab(a+b) + 2abc \quad \text{XVII.}$$

$$(b-c) + (c-a) + (a-b) = 0. \quad \dots \quad \dots \quad \text{XVIII.}$$

$$a(b-c) + b(c-a) + c(a-b) = 0. \quad \dots \quad \dots \quad \text{XIX.}$$

$$a^2(b-c) + b^2(c-a) + c^2(a-b) = -(a-b)(b-c)(c-a) \\ = bc(b-c) + ca(c-a) + ab(a-b). \quad \text{XX.}$$

$$a^4 + a^2b^2 + b^4 = (a^2 - ab + b^2)(a^2 + ab + b^2). \quad \dots \quad \text{XXI.}$$

74. The formulæ I, II and III, can be brought under a general rule :

The square of the sum of any number of quantities = the sum of their squares + twice the sum of the products of every two of the quantities (with their proper signs).

Thus, since $a-b = a + (-b)$, i.e., sum of a and $-b$,

$$(a-b)^2 = a^2 + b^2 - 2ab.$$

Note that this result can be deduced from formula I by changing $+b$ into $-b$.

N.B. By 'sum' in the above formula, we mean 'algebraic sum'; thus the sum of a and $-b$ is $a-b$.

The following forms should also be noted :

$$\left. \begin{aligned} (a+b)^2 &= (a-b)^2 + 4ab, \\ \text{and } (a-b)^2 &= (a+b)^2 - 4ab \end{aligned} \right\} \begin{array}{l} \text{These can be easily proved by} \\ \text{expansion.} \end{array}$$

Formula III can be deduced from I.

Let $a+b = X$; then $a+b+c = X+c$.

$$\begin{aligned} \therefore (a+b+c)^2 &= (X+c)^2 = X^2 + 2cX + c^2 && \dots && \dots && \text{by I,} \\ &= (a+b)^2 + 2c(a+b) + c^2, && \text{putting } a+b \text{ for } X, \\ &= (a^2 + 2ab + b^2) + 2(ac + bc) + c^2 && \dots && \dots && \text{by I,} \\ &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca, && \text{re-arranging terms.} \end{aligned}$$

N.B. In practice we go on thus: $\{(a+b)+c\}^2 = (a+b)^2 + 2c(a+b) + c^2$.

Similarly, $(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 +$ twice the sum of the products of every two of a, b, c, d ,

$$\begin{aligned} &= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd, \\ (a+b+c+d)^2 &= a^2 + b^2 + (-c)^2 + (-d)^2 + 2ab + 2a(-c) + 2a(-d) \\ &\quad + 2b(-c) + 2b(-d) + 2(-c)(-d) \\ &= a^2 + b^2 + c^2 + d^2 + 2ab - 2ac - 2ad - 2bc - 2bd + 2cd. \end{aligned}$$

N.B. Note carefully the mode of formation of the terms, which is indicated more conveniently in the Chapter on Involution.

Ex. 1. Find $(1639)^2$

$$\begin{aligned} 1639^2 &= (1600 + 40 - 1)^2 \\ &= 1600^2 + 40^2 + (-1)^2 + 2 \times 1600 \times 40 + 2 \times 1600(-1) + 2 \times 40(-1) \\ &= 2560000 + 1600 + 1 + 128000 - 3200 - 80 = \underline{2686321}. \end{aligned}$$

Ex. 2. Expand $(2x - 3y + 4z)^2$.

$$\begin{aligned}(2x - 3y + 4z)^2 &= (2x)^2 + (-3y)^2 + (4z)^2 + 2(2x)(-3y) + 2(2x)(4z) \\ &\quad + 2(-3y)(4z) \\ &= 4x^2 + 9y^2 + 16z^2 - 12xy + 16xz - 24yz.\end{aligned}$$

Ex. 3. If $2s = a + b + c$, simplify $s^2 + (s - a)^2 + (s - b)^2 + (s - c)^2$

Expanding, the given expression

$$\begin{aligned}&= s^2 + (s^2 - 2as + a^2) + (s^2 - 2bs + b^2) + (s^2 - 2cs + c^2) \\ &= 4s^2 - 2as - 2bs - 2cs + a^2 + b^2 + c^2, \text{ re-arranging terms,} \\ &= 4s^2 - 2s(a + b + c) + a^2 + b^2 + c^2 \\ &= 4s^2 - 2s \times 2s + a^2 + b^2 + c^2, \text{ putting } 2s \text{ for } a + b + c, \\ &= 4s^2 - 4s^2 + a^2 + b^2 + c^2 \\ &= a^2 + b^2 + c^2.\end{aligned}$$

EXAMPLES 38.

Write down the squares of

1. $4x + 5y$; $7x + 8y$.
2. $3x - 7y$; $6a - 8b$.
3. $x - y + z$; $x - y - z$.
4. $a - b + c - d$; $a - b - c - d$.
5. $a + b - c - d + e$.
6. $a - b - c + d - e$.
7. $2x - 3y + 5z + w$.
8. $2x - 4y - 5z - w$.
9. $a^2 + b^2 + a^2b + ab^2$.
10. $a^3 - b^3 + a^2b - ab^2$.
11. $x^2 + y^2 + 2ax + 2by$.
12. $x^2 + y^2 - 2ax - 2by$.
13. $x^2 + 2y^2 + 3a^2 + 4ax$.
14. $x^2 - 2y^2 - 3a^2 - 4ax$.
15. $\frac{2}{3}x - \frac{1}{2}y + \frac{1}{3}z$.
16. $\frac{1}{2}a^2x^2 + \frac{2}{3}b^2y^2 - 3abx^2y$.
17. $a^2 + b^2 + c^2 - ab - bc - ca$.
18. $a^3 - b^3 + c^3 - ab + bc - ca$.
19. $x^4 + 2x^2y^2 + y^4 - 2a^2x^2 - 2a^2y^2 + a^4$.

Simplify

20. $(a + b + c + d)^2 + (a + b - c - d)^2 + (a - b - c + d)^2 + (a - b + c - d)^2$.
21. $(x + y + z)^2 - x(y + z - x) - y(z + x - y) - z(x + y - z)$.
22. $(2a + b)^2 + (a + 2b)^2 - 5(a + b)^2 + 2ab$.
23. $(a^2b^2 + b^2c^2 + c^2a^2)^2 - a^4(b^2 + c^2)^2 - b^4(c^2 + a^2)^2 - c^4(a^2 + b^2)^2$.

Shew that

24. $(x - y)^2 + (y - z)(z - x) = (y - z)^2 + (x - y)(x - z)$
 $= (z - x)^2 + (y - z)(y - x)$.
25. $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2 = (ad + bc)^2 + (ac - bd)^2$.

$$26. (ax+by+cz)^2 + (bx-cy)^2 + (ay-bx)^2 + (cx-as)^2 \\ = (a^2+b^2+c^2)(x^2+y^2+z^2).$$

If $s=a+b+c$, prove that

$$27. (s-a)^2 + (s-b)^2 + (s-c)^2 - s^2 = a^2 + b^2 + c^2.$$

$$28. (s+c)^2 - 4(s-a)(s-b) = (a-b)^2.$$

$$29. \frac{1}{2}(s-2a-b)^2 + \frac{1}{2}(s-2b-c)^2 + \frac{1}{2}(s-2c-a)^2 \\ = a^2 + b^2 + c^2 - bc - ca - ab$$

If $2s=a+b+c$, prove that

$$30. (s-a)^2 + (s-b)^2 + (s-a)(s-b) = c^2 - (s-a)(s-b).$$

$$31. (s-a)^2 + (s-b)(s-c) + as = a^2 + bc.$$

$$32. (s-b)^2 + (s-c)^2 + 2s(s-a) = s^2 + (s-a)^2 - 2(s-b)(s-c).$$

$$75. \text{ Formula IV. } a^3 - b^3 = (a+b)(a-b).$$

In words, *the difference of the squares of two quantities = the product of the sum and difference of those quantities.*

Ex. 1. Factorize $a^3 - b^3 - 2a + 1$.

$$\begin{aligned} a^3 - b^3 - 2a + 1 &= (a^3 - 2a + 1) - b^3, \text{ re-arranging terms,} \\ &= (a-1)^2 - b^3 \\ &= \{(a-1)+b\}\{(a-1)-b\} \quad \dots \text{ Formula IV.} \\ &= \underline{(a+b-1)(a-b-1)}, \text{ re-arranging terms.} \end{aligned}$$

Ex. 2. Prove that $(x+y)^2 - (x-y)^2 = 4xy$.

$$\begin{aligned} (x+y)^2 - (x-y)^2 &= \{(x+y) + (x-y)\}\{(x+y) - (x-y)\} \\ &= 2x \times 2y \\ &= 4xy. \end{aligned}$$

Note this result.

Ex. 3. Prove the following identity :

$$(a+b)^2(c-d)^2 + 4cd(a-b)^2 = (a-b)^2(c+d)^2 + 4ab(c-d)^2.$$

$$\therefore 4cd = (c+d)^2 - (c-d)^2, \text{ Ex. 2,}$$

we easily get $(a+b)^2(c-d)^2 + 4cd(a-b)^2$

$$\begin{aligned} &= (a+b)^2(c-d)^2 + \{(c+d)^2 - (c-d)^2\}(a-b)^2 \\ &= (a+b)^2(c-d)^2 + (a-b)^2(c+d)^2 - (a-b)^2(c-d)^2 \\ &= (a-b)^2(c+d)^2 + \{(a+b)^2 - (a-b)^2\}(c-d)^2 \\ &= (a-b)^2(c+d)^2 + 4ab(c-d)^2. \quad \text{Ex. 2.} \end{aligned}$$

EX. 4. Factorize $x^4 + a^2x^2 + a^4$, $x^4 + 4a^4$ and $x^8 - 16a^8$.

$$x^4 + a^2x^2 + a^4 = (x^4 + 2a^2x^2 + a^4) - a^2x^2, \text{ adding and subtracting } a^2x^2,$$

$$= (x^2 + a^2)^2 - (ax)^2 \quad \dots \quad \text{Formula I.}$$

$$= \{(x^2 + a^2) + ax\} \{(x^2 + a^2) - ax\} \quad \text{Formula IV.}$$

$$= \underline{(x^2 + ax + a^2)(x^2 - ax + a^2)}, \text{ re-arranging terms.}$$

$$x^4 + 4a^4 = (x^4 + 4a^2x^2 + 4a^4) - 4a^2x^2, \text{ adding and taking } 4a^2x^2,$$

$$= (x^2 + 2a^2)^2 - (2ax)^2$$

$$= \{(x^2 + 2a^2) + 2ax\} \{(x^2 + 2a^2) - 2ax\} \quad \text{Formula IV.}$$

$$= \underline{(x^2 + 2ax + 2a^2)(x^2 - 2ax + 2a^2)}, \text{ re-arranging.}$$

$$x^8 - 16a^8 = (x^4)^2 - (4a^4)^2$$

$$= (x^4 - 4a^4)(x^4 + 4a^4) \quad \dots \quad \text{Formula IV.}$$

• Now break up $x^4 - 4a^4$ and $x^4 + 4a^4$ separately into factors

$$x^4 - 4a^4 = (x^2)^2 - (2a^2)^2 = (x^2 - 2a^2)(x^2 + 2a^2). \quad \text{Formula IV.}$$

$$x^4 + 4a^4 = (x^2 + 2ax + 2a^2)(x^2 - 2ax + 2a^2), \text{ already found.}$$

$$\therefore x^8 - 16a^8 = \underline{(x^2 - 2a^2)(x^2 + 2a^2)(x^2 + 2ax + 2a^2)(x^2 - 2ax + 2a^2)}.$$

N.B. Note that an expression of the form $4a^4 + b^4$ can be factorized.

EX. 5. Find the value of 1688×1712 .

$$1688 = 1700 - 12, \text{ and } 1712 = 1700 + 12.$$

$$\therefore 1688 \times 1712 = (1700 - 12)(1700 + 12)$$

$$= 1700^2 - 12^2$$

$$= 2890000 - 144$$

$$= \underline{2889856}. \quad \text{Ans.}$$

EXAMPLES 39.

Factorize

1. $a^4 - b^4$; $a^8 - b^8$.

2. $a^8x^8 - y^8$; $a^8b^8 - c^8d^8$.

3. $a^4 - 16$; $x^8 - 81$.

4. $256 - a^4$; $1 - a^{16}$.

5. $16x^4 - 81a^4$; $2a^4 - 32b^4$

6. $a^5b^4 - 625ac^4$; $50a^3b^3 - 8b^3c^3$.

7. $a^2 + b^2 + 2ab - c^2$.

8. $a^2 + b^2 - 2ab - 1$.

9. $(a + b)^2 - (b + c)^2$.

10. $a^4 - (b - c)^4$.

11. $36 - (5p + 6q)^2$.

12. $25 - (1 + x^2)^2$.

13. $4a^2 - 9(b - 4c)^2$.

14. $25(x^2 + y^2)^2 - 16(y^2 + z^2)^2$.

15. $(1 + x^2)^2 - 4x^2$.

16. $(a^3 + b^3)^2 - 4a^2b^3$.

17. $a^2 + 4b^2 - 1 - 4ab$.

18. $a^3 - b^3 + ac - bc$.

19. $(5a-4b)^2 - (4a-5b)^2$ 20. $(6a^2-5b^2)^2 - (5a^2-6b^2)^2$.
 21. $(3ax+2by)^2 - (2ax+3by)^2$. 22. $(ax-by)^2 - (ay-bx)^2$.
 23. $(ab+pq)^2 - (ap+bq)^2$. 24. $1-2y+y^2-x^2$.
 25. $4l^2-4lm+m^2-n^2$. 26. $1-b^2+2ab+a^2b^2$.
 27. $35-y^2-z^2+2yz$. 28. $2ab-a^2b^2+a^2c^2-1$.
 29. $4a^2+9b^2-4-12ab$. 30. $p^2+9q^2-16r^2-6pq$.
 31. $25(p^2-r^2)-36q^2-60qr$. 32. $a^2+2ab+b^2-c^2-2cd-d^2$.
 33. $a^2+4x^2-9y^2-b^2-4ax-6by$. 34. $x^2-2xy+y^2-2x+2y+1-x^2$.
 35. a^2-2a-b^2-2b . 36. $a^2-2ab-c^2-2bc$.
 37. $2(ab+c)-a^2-b^2+c^2+1$. 38. $(a^2x^2-1)^2-(a^2-x^2)^2$.
 39. $a^4+4b^4; x^3+64; 48a^3-3$. 40. $x^4+324; 1+a^3+a^4$.

Find the value of

41. 169×171 . 42. 511×489 .
 43. 1675×1645 . 44. 1769×1751 .

76. Formulæ, V-IX, relate to the product of binomial factors having the same first term. They can be brought under a general rule, which we break up here into 3 parts:

(1) *The highest power of x is the product of the first terms; diminishing its index continuously by unity, obtain the successive lower powers of x until the index is reduced to unity.*

(2) *The co-efficients of the different powers of x in the several terms in order, beginning with the highest power, are:*

- (a) *Unity;*
 (b) *sum of the 2nd terms of the binomials;*
 (c) *sum of the products of every two of the 2nd terms;*
 (d) *sum of the products of every three of the 2nd terms;*
and so on.

(3) *The last term is the product of all the 2nd terms.*

N.B. Here sum = algebraic sum.

Let us find $(x-a)(x+b)$.

By part (1) of the rule, the highest power of $x = x \times x = x^2$; the next lower power is x , and no more.

By part (2) of the rule, the coef. of $x^2 = 1$,

and that of $x = -a+b$.

By part (3), the last term $= (-a) \times b = -ab$.

Hence the result $= x^2 + (-a+b)x - ab$, i.e., $x^2 + (a-b)x - ab$.

Next let us find $(x+a)(x-b)(x-c)(x+d)$.

The powers of x are . The co-efficients are

$$\begin{array}{llll} x^4, & \dots & & 1, \\ x^3, & \dots & & a-b-c+d, \\ x^2, & \dots & a(-b)+a(-c)+ad+(-b)(-c)+(-b)d+(-c)d, \\ \text{and } x & \dots & a(-b)(-c)+a(-b)d+a(-c)d+(-b)(-c)d. \end{array}$$

The last term $= a(-b)(-c)d = abcd$.

Hence, simplifying the co-efficients, we have the result

$$\begin{aligned} x^4 + (a-b-c+d)x^3 - (ab+ac-ad-bc+bd+cd)x^2 \\ + (abc-abd-acd+bcd)x + abcd. \quad \text{Ans.} \end{aligned}$$

Ex. 1. Find by formula $(x+3)(x+2)(x-5)$.

$$\begin{aligned} \text{The expression} &= x^3 + (3+2-5)x^2 + \{3 \times 2 + 3(-5) + 2(-5)\}x \\ &\quad + 3 \times 2(-5) \\ &= x^3 - 2x^2 - 30x - 30, \text{ simplifying.} \end{aligned}$$

Ex. 2. Find by formula $(2x+7)(2x-3)(2x-1)$.

Put y for $2x$; then the given expression

$$\begin{aligned} &= (y+7)(y-3)(y-1) \\ &= y^3 + (7-3-1)y^2 + \{7(-3) + 7(-1) + (-3)(-1)\}y + 7(-3)(-1) \\ &= y^3 + 3y^2 - 25y + 21, \text{ simplifying,} \\ &= (2x)^3 + 3(2x)^2 - 25(2x) + 21, \text{ replacing } y \text{ by } 2x, \\ &= 8x^3 + 12x^2 - 50x + 21. \end{aligned}$$

Ex. 3. Divide $x(1+y^2)(1+z^2) + y(1+z^2)(1+x^2) + z(1+x^2)(1+y^2) + 4xyz$ by $1+xy+xz+yz$. C. U. 1878.

$$(1+y^2)(1+z^2) = 1^2 + 1(y^2+z^2) + y^2z^2, \text{ \&c. Formula V.}$$

$$\therefore \text{dividend} = x(1+y^2+z^2+y^2z^2) + y(1+x^2+z^2+x^2z^2)$$

$$+ z(1+x^2+y^2+x^2y^2) + 4xyz$$

$$= x^2(y+yz^2+z+y^2z) + x(1+y^2+z^2+y^2z^2+4yz)$$

$$+ y+z+y^2z+yz^2, \text{ arranging according to the powers of } x,$$

$$= x^2\{y+z+yz(y+z)\} + \&c.$$

$$= x^2(y+z)(1+yz) + x(1+y^2+z^2+y^2z^2+4yz) + (y+z)(1+yz).$$

The divisor arranged $= x(y+z) + 1+yz$. Now divide :

$$\begin{array}{r} x(y+z) \overline{) x^2(y+z)(1+yz) + x(1+y^2+z^2+y^2z^2+4yz) + \&c.} \\ \underline{x^2(y+z)(1+yz) + x(1+2yz+y^2z^2)} \\ x(y^2+z^2+2yz) + (y+z)(1+yz) \\ \underline{x(y+z)^2 + (y+z)(1+yz)} \end{array}$$

The required quotient $= x+y+z+xyz$.

EXAMPLES 40.

Write down at once the following products :

1. $(x-2)(x-3)$.
2. $(x-4)(x+11)$.
3. $(x+6)(x+7)$.
4. $(x-11)(x+12)$.
5. $(a+4b)(a-10b)$.
6. $(a+7b)(a-13b)$.
7. $(2x+5)(2x+9)$.
8. $(2y+11)(2y-9)$.
9. $(2z+5)(2z-10)$.
10. $(3x+4y)(3x-5y)$.
11. $(11x+9y)(11x-13y)$.
12. $(7a+15b)(7a+16b)$.
13. $(x+1)(x+2)(x+3)$.
14. $(x-4)(x+5)(x+6)$.
15. $(y+11)(y-2)(y-7)$.
16. $(x-a)(x+2a)(x-4a)$.
17. $(x-b)^2(x+2b)$.
18. $(x^2-9)(2x+5)(2x+4)$.
19. $(4x-1)(4x-7)(x+2)$.
20. $(3x-2)(3x+5)(x-1)$.
21. Simplify $(1+a)(1+b)-(1-a)(1-2b)+3(2-a)(2-b)+ab$.
22. Divide $(a^2+3b^2)(a^2+5b^2)(a^2+7b^2)-(a^2+b^2)(a^2+2b^2)(a^2+3b^2)$
by a^2+11b^2 .
23. Divide the expression $(1+a^2)(1+b^2)(1+c^2)-(1-a^2)(1-b^2)(1-c^2)$
 $+ (1-a^2)(1-b^2)(1+c^2)-(1+a^2)(1+b^2)(1-c^2)$ by $1+a^2b^2$.
24. If $2s=a+b+c$, show that $s(s-a)(s-b)+s(s-b)(s-c)$
 $+s(s-a)(s-c)-(s-a)(s-b)(s-c)=abc$.

77. Formulæ X and XI can be put under a single rule :

Cube of the sum of two quantities [Sum=algebraic sum]

=sum of their cubes + 3 times their product \times their sum

Thus $(a-b)^3 = \{a+(-b)\}^3$

$$= a^3 + (-b)^3 + 3a(-b)\{a+(-b)\}$$

$$= a^3 - b^3 - 3ab(a-b)$$

$$= a^3 - 3ab(a-b) - b^3, \text{ re-arranging terms,}$$

$$= a^3 - 3a^2b + 3ab^2 - b^3.$$

Formula XII can be deduced from Formula X thus :

$$\begin{aligned}
 (a+b+c)^3 &= (a+y)^3 \text{ putting } y \text{ for } b+c, \\
 &= a^3 + 3ay(a+y) + y^3, \quad \text{Formula X} \\
 &= a^3 + 3a(b+c)(a+b+c) + (b+c)^3, \text{ restoring } b+c, \\
 &= a^3 + 3a(b+c)(a+b+c) + \{b^3 + 3b(b+c) + c^3\} \\
 &= a^3 + b^3 + c^3 + 3a(b+c)(a+b+c) + 3bc(b+c), \\
 &\quad \text{re-arranging terms,} \\
 &= a^3 + b^3 + c^3 + 3(b+c)\{a(a+b+c) + bc\} \\
 &= a^3 + b^3 + c^3 + 3(b+c)\{a(a+b) + ac + bc\} \\
 &= a^3 + b^3 + c^3 + 3(b+c)\{a(a+b) + c(a+b)\} \\
 &= a^3 + b^3 + c^3 + 3(b+c)(c+a)(a+b), \text{ as usually put,} \\
 &= a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(c+a) + 3c^2(a+b) \\
 &\quad + 6abc, \text{ as will be seen from Formula XVII.}
 \end{aligned}$$

Notice that $(a+b-c)^3$ can be obtained from the above result by changing c into $-c$; thus

$$\begin{aligned}
 (a+b-c)^3 &= a^3 + b^3 + (-c)^3 + 3(a+b)(b-c)(-c+a) \\
 &= a^3 + b^3 - c^3 + 3(a+b)(b-c)(a-c).
 \end{aligned}$$

Ex. 1. Simplify the expression

$$(3a-b)^3 - 3(3a-b)^2(a-b) + 3(3a-b)(a^3 - 2ab + b^2) - (a-b)^3.$$

$$\begin{aligned}
 \text{The given expn.} &= (3a-b)^3 - 3(3a-b)^2(a-b) + 3(3a-b)(a^3 - 2ab + b^2) - (a-b)^3 \\
 &= x^3 - 3x^2y + 3xy^2 - y^3, \text{ putting } x \text{ for } 3a-b, y \text{ for } a-b, \\
 &= x^3 - 3xy(x-y) - y^3 \\
 &= (x-y)^3 \\
 &= \{(3a-b) - (a-b)\}^3, \text{ restoring the values of } x \text{ and } y, \\
 &= (2a)^3, \text{ simplifying,} \\
 &= 8a^3 \quad \text{Ans}
 \end{aligned}$$

Ex. 2. Expand $(2x - 3y + 4z)^3$

$$\begin{aligned}
 (2x - 3y + 4z)^3 &= (2x - 3y)^3 + 3(2x - 3y)4z\{(2x - 3y) + 4z\} + (4z)^3 \\
 &= (2x - 3y)^3 + 12z(2x - 3y)^2 + 48z^2(2x - 3y) + 64z^3 \\
 &= \{(2x)^3 - 3(2x)(3y)(2x - 3y) - (3y)^3\} \\
 &\quad + 12z\{(2x)^2 - 2 \cdot 2x \cdot 3y + (3y)^2\} + 48z^2(2x - 3y) + 64z^3 \\
 &= (8x^3 - 36x^2y + 54xy^2 - 27y^3) + 12z(4x^2 - 12xy + 9y^2) \\
 &\quad + 48z^2(2x - 3y) + 64z^3 \\
 &= 8x^3 - 36x^2y + 48x^2z + 54xy^2 - 144xyz + 96xz^2 \\
 &\quad - 27y^3 + 108y^2z - 144yz^2 + 64z^3. \quad \text{Ans.}
 \end{aligned}$$

EXAMPLES 41

Expand

1. $(2x+3y)^2$.
2. $(4a-b)^2$.
3. $(3a+2b)^2$.
4. $(6a-5b)^2$.
5. $(5ab-2)^2$.
6. $(2ab+c)^2$.
7. $(3ab+2)^2$.
8. $(7ab-4cd)^2$.
9. $(a+b+2c)^2$.
10. $(2a-b-c)^2$.
11. $(3x-y+z)^2$.
12. $(ax-by+cz)^2$.

Simplify

13. $(2a-b)^2+9a(2a-b)(a+b)+(a+b)^2$.
14. $(a-b)^2+6a(a^2-b^2)+(a+b)^2$.
15. $(2a-b)^2-3(2a-b)(a+b)(a-2b)-(a-2b)^2$.
16. $(a+b+c)^2+(a-b+c)^2+(a+b-c)^2+(b+c-a)^2$.
17. Divide $(x+y+z)^2-x^2-y^2-z^2$ by $x+y$.
18. Divide $(2x+y-z)^2-8x^2-y^2+z^2$ by $y-z$.
19. If $x+y=p$, and $xy=q$, then will $x^2+y^2=p^2-3pq$.
20. If $2a+3b=7$, and $ab=2$, find the value of $8a^2+27b^2$.

78. Note that Formula XIV can be deduced from XIII by replacing b by $-b$. Also observe that a^2-ab+b^2 and a^2+ab+b^2 cannot be factorized. [Apply the usual method.]

We can deduce XIII from X thus:

$$\begin{aligned}
 a^3+3ab(a+b)+b^3 &= (a+b)^3; & \dots & \text{Formula X} \\
 \text{transposing,} \quad a^3+b^3 &= (a+b)^3-3ab(a+b) \\
 &= (a+b)\{(a+b)^2-3ab\} \\
 &= (a+b)\{a^2+2ab+b^2-3ab\} \\
 &= \underline{(a+b)(a^2-ab+b^2)}, \text{ simplifying.}
 \end{aligned}$$

Similarly deduce XIV from XI.

Ex. 1. Factorize $64x^3+125y^3$.

The given expression $= (4x)^3 + (5y)^3$

$$\begin{aligned}
 &= (4x+5y)\{(4x)^2-4x \times 5y+(5y)^2\} \dots \text{XIII.} \\
 &= \underline{(4x+5y)(16x^2-20xy+25y^2)}.
 \end{aligned}$$

Ex. 2. Resolve into elementary factors $8x^3 + (2x + y)^3 + y^3$.

The given expression $= 8x^3 + y^3 + (2x + y)^3$, re-arranging terms,

$$\begin{aligned} &= \{(2x)^3 + y^3\} + (2x + y)^3 \\ &= (2x + y)\{(2x)^2 - (2x)y + y^2\} + (2x + y)^3 \\ &= (2x + y)\{4x^2 - 2xy + y^2\} + (2x + y)^3 \\ &= (2x + y)\{4x^2 - 2xy + y^2 + 4x^2 + 4xy + y^2\} \\ &= (2x + y)(8x^2 + 2xy + 2y^2) \\ &= (2x + y) \cdot 2(4x^2 + xy + y^2) \\ &= \underline{2(2x + y)(4x^2 + xy + y^2)}. \end{aligned}$$

It can be easily seen that $4x^2 + xy + y^2$ cannot be resolved, as it cannot be expressed as the *difference* of two squares, for

$$4x^2 + xy + y^2 = 4x^2 + xy + \frac{1}{4}y^2 + (y^2 - \frac{1}{4}y^2) = (2x + \frac{1}{2}y)^2 + \frac{3}{4}y^2.$$

Ex. 3. Factorize $(ax + by)^3 - (ax + by - 2)^3 - 8$.

We have the given expression

$$\begin{aligned} &= \{(ax + by)^3 - 2^3\} - (ax + by - 2)^3 \\ &= (ax + by - 2)\{(ax + by)^2 + 2(ax + by) + 2^2\} - (ax + by - 2)^3 \\ &= (ax + by - 2)\{(ax + by)^2 + 2(ax + by) + 4 - (ax + by - 2)^2\}. \end{aligned}$$

Now simplify the quantity within the brackets { } ; this

$$\begin{aligned} &= (ax + by)^2 + 2(ax + by) + 4 - (ax + by - 2)^2 \\ &= (ax + by)^2 + 2(ax + by) + 4 - \{(ax + by)^2 - 2 \times 2(ax + by) + 4\} \\ &= 6(ax + by), \text{ removing the brackets and simplifying.} \end{aligned}$$

\therefore the given expression $= (ax + by - 2)6(ax + by)$

$$= \underline{6(ax + by - 2)(ax + by)}.$$

Otherwise thus : Expand $(ax + by - 2)^3$; the given expression

$$\begin{aligned} &= (ax + by)^3 - \{(ax + by)^3 - 3 \times 2(ax + by)(ax + by - 2) - 2^3\} - 8 \\ &= 6(ax + by)(ax + by - 2), \text{ removing } \{ \} \text{ and simplifying} \end{aligned}$$

Ex. 4. Shew that $(ax + by + cz)^2 + (bx + cy + az)^2$ is exactly divisible by $(a + b)x + (b + c)y + (c + a)z$.

Denoting $ax + by + cz$ by A, and $bx + cy + az$ by B, we have $(ax + by + cz)^2 + (bx + cy + az)^2 = A^2 + B^2 = (A + B)(A^2 - AB + B^2)$;

evidently the right side is divisible by A + B, and

$$A + B = (ax + by + cz) + (bx + cy + az) = (a + b)x + (b + c)y + (c + a)z.$$

Hence $(ax + by + cz)^2 + (bx + cy + az)^2$ is divisible by

$$(a + b)x + (b + c)y + (c + a)z.$$

EXAMPLES 42.

Resolve into elementary factors

1. $64a^3 + 125$.
2. $8a^3 - 343b^3$.
3. $a^3x^3 + 216b^3y^3$.
4. $x^3y^6 + 8x^9$.
5. $(4x - 3y)^3 - 8x^3$.
6. $64 - (7x - 4y)^3$.
7. $(x^2 + 2)^3 - 27x^2$.
8. $(x^3 + 3)^3 + 64x^3$.
9. $x^3(x + 5)^3 + 216$.
10. $(2x^2 + 3)^3 - 343x^3$.
11. $216(x^2 - 1)^3 - 125x^3$.
12. $(x^3 + 3ax^2 + 3a^2x + a^3) - b^3$.
13. $x^3 + 3ax^2 + 3a^2x - 7a^3$.
14. $x^3 + 3x^2 + 3x - 26$.
15. $x^3 - 6x^2 + 12x - 16$.
16. $8x^3 + 1 + (2x + 1)$.
17. $(x^3 + y^3)^3 - 8x^3y^3$.
18. $a^3 + 512b^3$.
19. $a^6 - 9a^3 + 8$.
20. $a^9 - a^6 - 64a^3 + 64$.

Write down the quotient of

21. $(a + b)^3 - 27b^3$ by $a - 2b$.
22. $27a^6 + b^3$ by $3a^2 + b$.
23. $(x - y)^3 + 8y^3$ by $x + y$.
24. $a^6b^6 - 64c^6$ by $ab - 2c$.
25. $a^3 + b^3$ by $a^3 - ab + b^2$.
26. $a^6 + 2a^3b^3 + b^6$ by $(a + b)^2$.
27. $(x^3 - y^3)(x + y)$ by $x^2 - y^2$.
28. $(8a^3 - b^3)(a + 2b)$ by $2a^2 + 3c$.
29. The product of two quantities is $(4x + 5y)^3 + (5x + 4y)^3$, and one of them is $x + y$; find the other.
30. The product of two quantities is $(5x + 6y)^3 - (x - 8y)^3$, and one of them is $2x + 4y + 3z$; find the other.
31. Show that $(6x^3 + 5x + 4)^3 + (2x^3 + 9x + 1)^3$ is divisible by $4x + 5$, and by $2x + 1$.
32. Show that $a^3b^3 + a^3 + b^3 + 1$ is divisible by $ab + a + b + 1$, and find the quotient.

79. Formulæ XV and XVI.

$$\begin{aligned}
 &\text{By expansion } \frac{1}{2}\{(a-b)^3 + (b-c)^3 + (c-a)^3\} \\
 &= \frac{1}{2}\{(a^3 - 2ab^2 + b^3) + (b^3 - 2bc^2 + c^3) + (c^3 - 2ca^2 + a^3)\}, \text{ Form. II.} \\
 &= \frac{1}{2}\{2a^3 + 2b^3 + 2c^3 - 2bc - 2ca - 2ab\}, \text{ collecting like terms,} \\
 &= \frac{1}{2} \times 2 \times (a^3 + b^3 + c^3 - bc - ca - ab) \\
 &= \underline{a^3 + b^3 + c^3 - bc - ca - ab}.
 \end{aligned}$$

Now, $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$, on multiplying out the right side;

$$\begin{aligned}
 \therefore \text{ by substitution for } a^3 + b^3 + c^2 - bc - ca - ab, \text{ we have also} \\
 a^3 + b^3 + c^3 - 3abc = (a + b + c) \frac{1}{2}\{(a-b)^3 + (b-c)^3 + (c-a)^3\} \\
 = \frac{1}{2}(a + b + c)\{(a-b)^3 + (b-c)^3 + (c-a)^3\}.
 \end{aligned}$$

We can factorize $a^3 + b^3 + c^3 - 3abc$ directly thus :

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b). \quad \text{Formula X.}$$

$$\therefore a^3 + b^3 = (a+b)^3 - 3ab(a+b), \text{ by transposition.}$$

Adding $c^3 - 3abc$ to both sides, we have

$$\begin{aligned} a^3 + b^3 + c^3 - 3abc &= \{(a+b)^3 + c^3\} - 3ab(a+b) - 3abc \\ &= (a+b+c)\{(a+b)^2 - c(a+b) + c^2\} - 3ab(a+b+c) \\ &= (a+b+c)\{(a+b)^2 - c(a+b) + c^2 - 3ab\} \\ &= (a+b+c)(a^2 + b^2 + c^2 - bc - ca - ab), \text{ reducing.} \end{aligned}$$

Ex. 1. Factorize $a^3 - b^3 + c^3 + 3abc$.

$$\begin{aligned} \text{Given expn.} &= a^3 + (-b)^3 + c^3 - 3a(-b)c \\ &= \{a + (-b) + c\}\{a^2 + (-b)^2 + c^2 - (-b)c - ca - a(-b)\} \\ &= \underline{(a-b+c)(a^2 + b^2 + c^2 + bc - ca + ab)}. \end{aligned}$$

N.B. Note that this result is obtained by changing b into $-b$ in the formula for $a^3 + b^3 + c^3 - 3abc$.

Ex. 2. Factorize $x^3 + y^3 + 3xy - 1$.

$$\begin{aligned} \text{The expn.} &= x^3 + y^3 + (-1)^3 - 3xy(-1) \\ &= \{x + y + (-1)\}\{x^2 + y^2 + (-1)^2 - xy - x(-1) - y(-1)\} \\ &= \underline{(x+y-1)(x^2 - xy + y^2 + x + y + 1)}. \end{aligned}$$

Ex. 3. Factorize $8x^3 - 18xy - 27y^3 - 1$.

$$\begin{aligned} \text{The given expn.} &= 8x^3 - 27y^3 - 1 - 18xy, \text{ re-arranging,} \\ &= (2x)^3 + (-3y)^3 + (-1)^3 - 3 \times 2x(-3y)(-1) \\ &= (2x-3y-1)\{(2x)^2 + (-3y)^2 + (-1)^2 \\ &\quad - 2x(-3y) - (-3y)(-1) - (-1)2x\} \\ &= (2x-3y-1)(4x^2 + 9y^2 + 1 + 6xy - 3y + 2x) \\ &= \underline{(2x-3y-1)(4x^2 + 6xy + 9y^2 + 2x - 3y + 1)}, \text{ arranging} \end{aligned}$$

Ex. 4. Simplify $(.452)^3 + (.548)^3 + 3 \times .452 \times .548$.

Representing $.452$ by x and $.548$ by y , we have the given expression

$$\begin{aligned} &= x^3 + y^3 + 3xy \\ &= x^3 + y^3 + 3xy - 1 + 1 \\ &= \{x^3 + y^3 + (-1)^3 - 3xy(-1)\} + 1 \\ &= \underline{(x+y-1)\{x^2 + y^2 + (-1)^2 - xy - x(-1) - y(-1)\} + 1}. \end{aligned}$$

Now, $x+y-1 = .452 + .548 - 1 = 1 - 1 = 0$.

\therefore given expn. $= 0 \times (x^2 - xy + y^2 + x + y + 1) + 1 = 1$. *Ans.*

Ex 5. If $s = a + b + c$, shew that $(2s + 3a)^2 + (2s + 3b)^2 + (2s + 3c)^2 - 3(2s + 3a)(2s + 3b)(2s + 3c) = 81(a^2 + b^2 + c^2 - 3abc)$.

Denoting $2s + 3a$ by x , $2s + 3b$ by y , and $2s + 3c$ by z , we have the 1st. member $= x^2 + y^2 + z^2 - 3xyz$

$$\begin{aligned} &= \frac{1}{2}(x + y + z)\{(x - y)^2 + (y - z)^2 + (z - x)^2\} \\ \text{Now, } x + y + z &= (2s + 3a) + (2s + 3b) + (2s + 3c) \\ &= 6s + 3(a + b + c) \\ &= 6(a + b + c) + 3(a + b + c), [\because s = a + b + c] \\ &= 9(a + b + c). \end{aligned}$$

$$\text{Also } (x - y)^2 = \{(2s + 3a) - (2s + 3b)\}^2 = \{3(a - b)\}^2 = 9(a - b)^2,$$

$$(y - z)^2 = 9(b - c)^2, \text{ similarly,}$$

$$\text{and } (z - x)^2 = 9(c - a)^2.$$

$$\begin{aligned} \therefore x^2 + y^2 + z^2 - 3xyz &= \frac{1}{2} \times 9(a + b + c) \times 9\{(b - c)^2 + (c - a)^2 + (a - b)^2\} \\ &= 81 \times \frac{1}{2}(a + b + c)\{(b - c)^2 + (c - a)^2 + (a - b)^2\} \\ &= 81(a^2 + b^2 + c^2 - 3abc). \text{ Ans} \end{aligned}$$

EXAMPLES 43

Factorize

1. $a^3 + b^3 - c^3 + 3abc$.
2. $a^3 - b^3 - c^2 - 3abc$.
3. $x^3 + y^3 - 3xy + 1$.
4. $x^3 - y^3 + 3xy + 1$.
5. $x^3 - y^3 - 3xy - 1$.
6. $x^3 + 8y^3 + 27z^3 - 6xyz$.
7. $8x^3 - 27y^3 - 64z^3 - 72xyz$.
8. $1 - a^3 + 8b^3 + 6ab$.
9. $a^6 + b^6 - c^6 + 3a^2b^2c^2$.
10. $a^2(b^3 + 1) - (3a^2b - 1)$.
11. $a^3b^3 + b^3c^3 + c^3a^3 - 3a^2b^2c^2$.
12. $ab^2(a^2b + 3c) + 1 - b^3c^3$.
13. $a^3 - 8 - 9b(3b^2 + 2a)$.
14. $a^6 + 4a^3 - 1$.
15. $x^6 + 32x^3 - 64$.
16. $a^3(1 - 8b^3) + 4a^2b^2(2bc^2 + 3)$.
17. Shew that $(a + b)^3 + (b + c)^3 + (c + a)^3 - 3(a + b)(b + c)(c + a) = 2(a^3 + b^3 + c^3 - 3abc)$.
18. Shew that $a^3 + b^3 + c^3 - 3abc = (a - b)^3 + (b - c)^3 + (c - a)^3$, being given that $a + b + c = 2$.
19. Shew that $(s + a)^3 + (s + b)^3 + (s + c)^3 - 3(s + a)(s + b)(s + c) = 4(a^3 + b^3 + c^3 - 3abc)$, if $s = a + b + c$.
20. Shew that $(x + a)^3 + (x + b)^3 + (x + c)^3 - 3(x + a)(x + b)(x + c)$ is exactly divisible by $a^2 + b^2 + c^2 - bc - ca - ab$.
21. Find the quotient of $a^6 + 2a^2b^3 - 9a^2b^2c^2 + 6ab^3c^4 + b^6 - c^6$ by $(a + b)^3 - c^3$.

22. If $a+b=2c$, show that $a^3+b^3+6abc=8c^3$.
 23. Find the value of $(218)^3+(782)^3+3 \times 782 \times 218$.
 24. Find the value of $(29)^3+(16)^3-(41)^3+750 \times 16 \times 41$.

80. Formula XVII is easily obtained by multiplying out the factors on the left. It can also be established by manipulating the right-hand expression, thus :

$$\begin{aligned} & a^3(b+c) + b^3(c+a) + c^3(a+b) + 2abc \\ &= a^3(b+c) + b^3c + ab^3 + ac^3 + bc^3 + 2abc, \text{ multiplying out partly.} \\ &= a^3(b+c) + (b^3c + bc^3) + ab^3 + ac^3 + 2abc, \text{ re-arranging,} \\ &= a^3(b+c) + bc(b+c) + a(b+c)^2 \\ &= (b+c)\{a^3+bc+a(b+c)\} \\ &= (b+c)\{a^3+ac+bc+ab\}, \text{ re-arranging within the crotchets,} \\ &= (b+c)\{a(a+c)+b(c+a)\} \\ &= (b+c)(c+a)(a+b) \end{aligned}$$

EXAMPLES 44

1. Show that the expression $(a+b+c)^3 - a^3 - b^3 - c^3$
 $= 3\{bc(b+c) + ca(c+a) + ab(a+b)\} + 6abc$.
 2. Show that $a(b+c)^2 + b(c+a)^2 + c(a+b)^2 - 3abc$ is divisible by $a+b$, $b+c$ and $c+a$.
 3. Show that $b^3c^3(b^3+c^3) + c^3a^3(c^3+a^3) + a^3b^3(a^3+b^3) + 2a^3b^3c^3$ can be expressed as the product of six factors.
 4. Show that $(bc+ca+ab)(a+b+c) = (a+b)(b+c)(c+a) + abc$.
 Factorize
 5. $a(b^2+c^3) + b(c^3+a^3) - c(a^3+b^3) - 2abc$
 6. $a(b^2+c^3) - b(c^3+a^3) - c(a^3+b^3) + 2abc$.
 7. $b^4(a^2-c^3) + c^4(a^3-b^3) - a^4(b^3+c^3) + 2a^2b^2c^2$.
 8. $x^3(2y+z) + 4y^2(z+x) + z^2(x+2y) + 4xyz$.
 9. $x^2(y+3z) + y^2(3z+x) + 9z^2(x+y) + 6xyz$.
 10. $2xy(x+2y) + 6yz(2y+3z) + 3zx(3z+x) + 12xyz$.

81. Formula XVIII is at once apparent by addition.

As to XIX, $a(b-c) + b(c-a) + c(a-b)$

$$= ab - ac + bc - ab + ac - bc, \text{ multiplying out.}$$

$$= ab - ab - ac + ac + bc - bc, \text{ re-arranging,}$$

$$= 0.$$

Ex. 1. Shew that $(x+a)(b-c) + (x+b)(c-a) + (x+c)(a-b) = 0$.

By multiplication, the left-hand side

$$\begin{aligned} &= \{x(b-c) + a(b-c)\} + \{x(c-a) + b(c-a)\} + \{x(a-b) + c(a-b)\} \\ &= x\{(b-c) + (c-a) + (a-b)\} + \{a(b-c) + b(c-a) + c(a-b)\} \\ &= x \times 0 + 0 = 0. \quad \text{Ans} \end{aligned}$$

EXAMPLES 45.

Shew that

1. $(a+1)(b-c) + (b+1)(c-a) + (c+1)(a-b) = 0$.
2. $(b+c)(b-c) + (c+a)(c-a) + (a+b)(a-b) = 0$.
3. $(mx+na)(b-c) + (mx+nb)(c-a) + (mx+nc)(a-b) = 0$.
4. $(x^2-a^2)(b^2-c^2) + (x^2-b^2)(c^2-a^2) + (x^2-c^2)(a^2-b^2) = 0$
5. $a(1+b)(b-c) + b(1+c)(c-a) + c(1+a)(a-b) = 0$
6. $(x+b+c)(b-c) + (x+c+a)(c-a) + (x+a+b)(a-b) = 0$.
7. $(ax+p)(by-cx) + (by+p)(cx-ax) + (cx+p)(ax-by) = 0$
8. $(a+b-c)(b+c) + (b+c-a)(c+a) + (c+a-b)(a+b)$
 $= 2(ab+bc+ca)$

82 Formula XX is very important. It can be easily obtained by multiplying out the factors $a-b$, $b-c$ and $c-a$.

Otherwise thus : Since $a-b = -(b-a) = -(b-c+c-a)$, the expression $a^2(b-c) + b^2(c-a) + c^2(a-b)$

$$\begin{aligned} &= a^2(b-c) + b^2(c-a) - c^2\{(b-c) + (c-a)\} \\ &= a^2(b-c) + b^2(c-a) - c^2(b-c) - c^2(c-a) \\ &= a^2(b-c) - c^2(b-c) + b^2(c-a) - c^2(c-a), \text{ re-arranging,} \\ &= -(b-c)(c^2-a^2) + (c-a)(b^2-c^2), \text{ grouping suitably,} \\ &= -(b-c)(c-a)(c+a) + (c-a)(b-c)(b+c) \\ &= -(b-c)(c-a)\{c+a-(b+c)\} \\ &= \underline{-(b-c)(c-a)(a-b)}. \end{aligned}$$

Also, $bc(b-c) + ca(c-a) + ab(a-b)$

$$\begin{aligned} &= bc(b-c) + ca(c-a) - ab\{(b-c) + (c-a)\} \\ &= bc(b-c) + ca(c-a) - ab(b-c) - ab(c-a) \\ &= bc(b-c) - ab(b-c) + ca(c-a) - ab(c-a), \text{ re-arranging,} \\ &= b(b-c)(c-a) + a(c-a)(c-b), \text{ grouping suitably,} \\ &= b(b-c)(c-a) - a(c-a)(b-c), \quad \because c-b = -(b-c), \\ &= (b-c)(c-a)(b-a) \\ &= \underline{-(b-c)(c-a)(a-b)}, \quad \because b-a = -(a-b). \end{aligned}$$

The same formula can be otherwise put thus :

$$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2) = (b - c)(c - a)(a - b).$$

N.B. From the above method of work the student will learn how to factorize expressions such as $a^3(b - c) + b^3(c - a) + c^3(a - b)$.

EX. 1. Factorize the following expression :

$$(x^2 + ax + a^2)(b - c) + (x^2 + bx + b^2)(c - a) + (x^2 + cx + c^2)(a - b).$$

$$(x^2 + ax + a^2)(b - c) = (b - c)x^2 + a(b - c)x + a^2(b - c), \text{ and so on ;}$$

hence collecting into respective groups terms containing x^2 and x ,

$$\text{the 1st group} = x^2\{(b - c) + (c - a) + (a - b)\} = 0;$$

$$\text{the 2nd group} = x\{a(b - c) + b(c - a) + c(a - b)\} = 0;$$

$$\therefore \text{the given expn.} = a^2(b - c) + b^2(c - a) + c^2(a - b)$$

$$= -\frac{(b - c)(c - a)(a - b)}{1}, \text{ by formula.}$$

EX. 2. Shew that the expression $a^3(b - c) + b^3(c - a) + c^3(a - b)$

$$= -(b - c)(c - a)(a - b)(a + b + c).$$

$$\text{Since } a - b = -(b - a) = -(b - c + c - a),$$

$$\therefore \text{we have } a^3(b - c) + b^3(c - a) + c^3(a - b)$$

$$= a^3(b - c) + b^3(c - a) - c^3(b - c + c - a)$$

$$= a^3(b - c) + b^3(c - a) - c^3(b - c) - c^3(c - a)$$

$$= (b - c)(a^3 - c^3) + (c - a)(b^3 - c^3)$$

$$= -(b - c)(c^3 - a^3) + (c - a)(b^3 - c^3), \because a^3 - c^3 = -(c^3 - a^3),$$

$$= -(b - c)(c - a)(c^2 + ca + a^2) + (c - a)(b - c)(b^2 + bc + c^2)$$

$$= -(b - c)(c - a)\{(c^2 + ca + a^2) - (b^2 + bc + c^2)\}$$

$$= -(b - c)(c - a)(ca + a^2 - b^2 - bc), \text{ simplifying,}$$

$$= -(b - c)(c - a)\{a^2 - b^2 + (ac - bc)\}$$

$$= -(b - c)(c - a)\{(a - b)(a + b) + c(a - b)\}$$

$$= -(b - c)(c - a)(a - b)(a + b + c). \quad \text{Ans.}$$

EXAMPLES 48.

Factorize

1. $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2).$

2. $a^4(b^2 - c^2) + b^4(c^2 - a^2) + c^4(a^2 - b^2).$

3. $a^2(b^4 - c^4) + b^2(c^4 - a^4) + c^2(a^4 - b^4).$

4. $b^2c^2(b^2 - c^2) + c^2a^2(c^2 - a^2) + a^2b^2(a^2 - b^2).$

5. $bca(b^2 - c^2) + cab(c^2 - a^2) + abc(a^2 - b^2).$

6. $a(b^3 - c^3) + b(c^3 - a^3) + c(a^3 - b^3).$

7. $(a^2+1)(b-c) + (b^2+1)(c-a) + (c^2+1)(a-b)$.
 8. $(x^2-a^2)(b-c) + (x^2-b^2)(c-a) + (x^2-c^2)(a-b)$.
 9. $a(a-1)(b-c) + b(b-1)(c-a) + c(c-1)(a-b)$.
 10. $(mx+na)(b^2-c^2) + (mx+nb)(c^2-a^2) + (mx+nc)(a^2-b^2)$.
 11. $(a+1)^2(b-c) + (b+1)^2(c-a) + (c+1)^2(a-b)$.
 12. $(b-1)(c-1)(b-c) + (c-1)(a-1)(c-a) + (a-1)(b-1)(a-b)$.
 13. $(a-1)(a^2+a+1)(b-c) + (b-1)(b^2+b+1)(c-a) + (c-1)(c^2+c+1)(a-b)$

Shew that

14. $a^4(b-c) + b^4(c-a) + c^4(a-b) = -\{a(b^4-c^4) + b(c^4-a^4) + c(a^4-b^4)\}$
 $= bc(b^3-c^3) + ca(c^3-a^3) + ab(a^3-b^3)$
 $= -(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$
 15. $a^2(b^3-c^3) + b^2(c^3-a^3) + c^2(a^3-b^3) = -\{a^3(b^3-c^3) + b^3(c^3-a^3) + c^3(a^3-b^3)\}$
 $= -(b-c)(c-a)(a-b)(bc+ca+ab)$
 16. $a(b-c)^3 + b(c-a)^3 + c(a-b)^3 = (b-c)(c-a)(a-b)(a+b+c)$.
 17. $a^2(b-c)^3 + b^2(c-a)^3 + c^2(a-b)^3 = (b-c)(c-a)(a-b)(bc+ca+ab)$.
 18. $x^2(y-1)^3 - y^2(x-1)^3 + (x-y)^3 = (x-1)(y-1)(y-x)(x+y+xy)$.
 19. $mnb(cmb-nc) + nlc(a-nl-a) + lma(b-la-mb)$
 $= -(mb-nc)(nc-la)(la-mb)$.
 20. $6bc(2b-3c) + 3ca(3c-a) + 2ab(a-2b) = -(2b-3c)(3c-a)(a-2b)$.

83. Formula XXI.

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2a^2b^2 - a^2b^2 + b^4 \\ &= (a^4 + 2a^2b^2 + b^4) - a^2b^2, \text{ re-grouping,} \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + b^2 - ab)(a^2 + b^2 + ab) \\ &= (a^2 - ab + b^2)(a^2 + ab + b^2), \text{ re-arranging terms.} \end{aligned}$$

Ex. 1. Factorize $a^4 + 4a^2 + 16$.

$$\begin{aligned} a^4 + 4a^2 + 16 &= a^4 + 8a^2 - 4a^2 + 16 \\ &= (a^4 + 8a^2 + 16) - 4a^2, \text{ re-grouping,} \\ &= (a^2 + 4)^2 - (2a)^2 \\ &= (a^2 + 4 - 2a)(a^2 + 4 + 2a) \\ &= (a^2 - 2a + 4)(a^2 + 2a + 4) \quad \text{Ans.} \end{aligned}$$

Ex. 2 Factorize immediately from formula $16x^4 + 36x^2 + 81$.

The expn. $= (2x)^4 + (2x)^2 \cdot 3^2 + 3^4$, i.e., of the form $a^4 + a^2b^2 + b^4$.

$$\begin{aligned} \therefore 16x^4 + 36x^2 + 81 &= \{(2x)^2 - 2x \cdot 3 + 3^2\} \{(2x)^2 + 2x \cdot 3 + 3^2\} \\ &= (4x^2 - 6x + 9)(4x^2 + 6x + 9). \quad \text{Ans.} \end{aligned}$$

Ex. 3. Resolve into factors $x^8 + x^4y^4 + y^8$.

$$\begin{aligned} x^8 + x^4y^4 + y^8 &= (x^8 + 2x^4y^4 + y^8) - x^4y^4 \\ &= (x^4 + y^4)^2 - (x^2y^2)^2 \\ &= (x^4 + y^4 + x^2y^2)(x^4 + y^4 - x^2y^2) \quad \dots (A) \end{aligned}$$

Similarly, $x^4 + y^4 + x^2y^2 = (x^2 - xy + y^2)(x^2 + xy + y^2)$ Art. 83.

\therefore by (A), the given expn. $= (x^2 - xy + y^2)(x^2 + xy + y^2)(x^4 - x^2y^2 + y^4)$.

EXAMPLES 47.

Factorize .

1. $a^4 + a^2 + 1$. 2. $a^4 + 4a^2b^2 + 16b^4$. 3. $16x^4 + 4x^2y^2 + y^4$.
4. $a^4 + 9a^2 + 81$. 5. $a^4 + 9a^2b^2 + 81b^4$. 6. $16x^4 + 36x^2y^2 + 81y^4$.
7. $x^4 + 25x^2y^2 + 625y^4$. 8. $16x^4 + 100x^2y^2 + 625y^4$.
9. $(x+y)^4 + (x+y)^2 + 1$. 10. $256(a-b)^4 + 16(a-b)^2 + 1$.
11. $a^8 + a^4b^4 + b^8$. 12. $a^8 + a^4 + 1$.
13. $256a^8 + 16a^4 + 1$. 14. $x^8 + 81x^4y^4 + 6561y^8$.

Divide (without actual operation)

15. $(a-b)^4 + (a-b)^2c^2 + c^4$ by $a^2 + b^2 + c^2 - 2ab - ac + bc$.
16. $(a-1)^4 + (a-1)^2 + 1$ by $a^2 - 3a + 3$.
17. $(x-y)^4 + (x-y)^2y^2 + y^4$ by $x^2 - xy + y^2$.
18. $x^{12} + 4x^6y^6 + 16y^{12}$ by $x^6 - 2x^3y^3 + 4y^6$.
19. Find the value of $x^4 + x^2 + 5$, when $(x+1)^2 = x$.
20. Shew that $x^4 + x^2y^2 + y^4 = 0$, when $(x+y)^2 = xy$ or $3xy$.

84. Useful hint. Judicious *breaking up* of terms and their subsequent *re-grouping* are the artifices that are of the greatest service in attempting algebraic factorization. The following examples will illustrate their importance.

Ex. 1. Resolve into factors $x^3 - 3x + 2$.

$$\begin{aligned} x^3 - 3x + 2 &= x^3 - x - 2x + 2 \\ &= x(x^2 - 1) - 2(x - 1) \\ &= x(x+1)(x-1) - 2(x-1) \\ &= (x-1)\{x(x+1) - 2\} \\ &= (x-1)(x^2 + x - 2) \\ &= (x-1)(x^2 - x + 2x - 2) \\ &= (x-1)\{x(x-1) + 2(x-1)\} \\ &= (x-1)(x-1)(x+2) \\ &= (x-1)^2(x+2). \quad \text{Ans.} \end{aligned}$$

Otherwise thus :

$x^3 - 3x + 2 = x^3 + 1^3 + 1^3 - 3x \cdot 1 \cdot 1$, which is of the form $x^3 + y^3 + z^3 - 3xyz$;
 \therefore the given expn. $= (x+1+1)(x^2+1^2+1^2-x-1-x)$... Formula XVI
 $= (x+2)(x^2-2x+1)$, simplifying,
 $= (x+2)(x-1)^2$.

Ex. 2. Factorize $x^4 + 5x^3 + 2x^2 - 20x - 24$.

The given expn. $= x^4 + 5x^3 + 6x^2 - 4x^2 - 20x - 24$
 $= x^3(x^2 + 5x + 6) - 4(x^2 + 5x + 6)$
 $= (x^2 - 4)(x^2 + 5x + 6)$
 $= (x-2)(x+2)(x^2 + 2x + 3x + 6)$
 $= (x-2)(x+2)\{x(x+2) + 3(x+2)\}$
 $= (x-2)(x+2)\{(x+2)(x+3)\}$
 $= (x-2)(x+2)^2(x+3)$. *Ans.*

Ex. 3. Factorize $(a^2 - b^2)^2 - 2c^2(a^2 + b^2) + c^4$.

It is easy to see that $(a^2 - b^2)^2 = (a^2 + b^2)^2 - 4a^2b^2$.

\therefore the given expn. $= \{(a^2 + b^2)^2 - 2c^2(a^2 + b^2) + c^4\} - 4a^2b^2$
 $= (a^2 + b^2 - c^2)^2 - (2ab)^2$
 $= (a^2 + b^2 - c^2 + 2ab)(a^2 + b^2 - c^2 - 2ab)$
 $= \{(a+b)^2 - c^2\}\{(a-b)^2 - c^2\}$
 $= (a+b+c)(a+b-c)(a-b+c)(a-b-c)$. *Ans.*

EXAMPLES 48.

Factorize

1. $x^3 - 7x + 6$.
2. $x^3 - 7x - 6$.
3. $x^3 - 12x + 16$.
4. $x^3 - 13x + 12$.
5. $x^3 - 19x + 30$.
6. $x^3 - (a^2 + b^2)x + ab^2$.
7. $2x^5 - 3x^3 + 1$.
8. $3x^5 - 7x^3 + 4$.
9. $x^3 - 3x^2 + 4$.
10. $x^3 + x^2 + 4x + 4$.
11. $x^3 - 2x^2 + 1 - 2$.
12. $2x^3 + x^2 + 2x + 3$.
13. $y^3 + 3y^2 - 6y - 8$.
14. $x^4 + x^3 - x - 1$.
15. $x^4 - x^3 - 10x^2 + 4x + 24$.
16. $x^4 - 3x^2 - a^2x^2 + 3a^2$.
17. $x^5 - 4x^3 + 5x^2 - 20$.
18. $x^5 - x^3 - 8x^2 + 8$.
19. $x^5 - x^4 - 9x + 9$.
20. $a^4 + a^3(a-b)^2 - ab(a-b)^2$.
21. $(1-x^2)^2 - 2y^2(1+x^2) + y^4$.
22. $(x+y)^4 - 2(x^2+y^2)(x+y)^2 + 2(x^2-y^2)^2$.
23. $(a+1)^4 - 2(a+1)^2(a^2+1) + 2(a^2-1)^2$.
24. $(a+b)^4 - 2(a+b)^2(a^2b^2+1) + (1-a^2b^2)^2$.
25. $(ac+bd)^4 - 2(ac+bd)^2(a^2b^2+c^2d^2) + (a^2b^2-c^2d^2)^2$.
26. $(a+3)^3 + (a+2)^3 - 1$, and $(x-1)^4 + (x-2)^3 - 1$.
27. $(x^2+ax+2a^2)^2 + a^2(x+2a)^2 - 5a^4 + 2a^3(x+a)$.
28. $(x+1)^2(y-1)^2 + 4y(x-1)^2 - 4x(y-1)^2$.

CHAPTER XIV.

HARDER WORK WITH FORMULÆ.

[To be omitted at the first reading.]

85. Artifices. We now propose to illustrate some artifices by examples. No hard and fast rule can be given for the selection of the proper formulæ and artifice in any particular example. Constant practice will lead to the right choice. It is, however, of the utmost importance to examine at the outset the form of the given expression or identity, which generally goes a great way towards removing the difficulty in the selection of the proper method.

Ex. 1. Prove the following identity :

$$\begin{aligned}(x+2y-z)^2 + (2x-y+z)^2 - 2(3x-y+z)(2y-z) \\ = (3x-y+z)^2 + (2y-z)^2 - 2(x+2y-z)(2x-y+z).\end{aligned}$$

It is readily seen that

$$(x+2y-z) + (2x-y+z) = 3x+y = (3x-y+z) + (2y-z);$$

$$\begin{aligned}\text{squaring, } (x+2y-z)^2 + (2x-y+z)^2 + 2(x+2y-z)(2x-y+z) \\ = (3x-y+z)^2 + (2y-z)^2 + 2(3x-y+z)(2y-z); \end{aligned}$$

$$\begin{aligned}\text{by transposition, } (x+2y-z)^2 + (2x-y+z)^2 - 2(3x-y+z)(2y-z) \\ = (3x-y+z)^2 + (2y-z)^2 - 2(x+2y-z)(2x-y+z).\end{aligned}$$

N.B. If the ordinary method of expansion and multiplication be attempted, the work becomes very tedious.

Ex. 2. If $x+y+z=0$, prove that $x^2+y^2+z^2 = -2(xy+yz+zx)$.

$$\text{Given } x+y+z=0;$$

$$\text{squaring, } x^2+y^2+z^2+2xy+2yz+2zx=0;$$

$$\text{transposing, } x^2+y^2+z^2 = -2(xy+yz+zx).$$

N.B. This result deserves attention, as it leads to other important ones.

Ex. 3. Prove that $2(a-b)(a-c) + 2(b-c)(b-a) + 2(c-a)(c-b)$ is the sum of three squares. B. U. 1884.

Since $(a-b) + (b-c) + (c-a) = 0$, we can, in Ex. 2, replace x by $a-b$, y by $b-c$, and z by $c-a$. We shall then have

$$\begin{aligned}(a-b)^2 + (b-c)^2 + (c-a)^2 \\ = -2(a-b)(b-c) - 2(b-c)(c-a) - 2(c-a)(a-b) \\ = 2\{-(a-b)\}(b-c) + 2\{-(b-c)\}(c-a) + 2\{-(c-a)\}(a-b) \\ = 2(b-a)(b-c) + 2(c-b)(c-a) + 2(a-c)(a-b); \end{aligned}$$

that is, $2(a-b)(a-c) + 2(b-c)(b-a) + 2(c-a)(c-b)$ is equal to the sum of the squares, $(a-b)^2$, $(b-c)^2$ and $(c-a)^2$. *Ans.*

Ex. 4. Shew that $x(y^2 + z^2 - x^2) + y(z^2 + x^2 - y^2) + z(x^2 + y^2 - z^2) + 6xyz$ is divisible by $x + y + z$.

Breaking up $6xyz$ into $2xyz + 2xyz + 2xyz$, the first given expn.
 $= \{x(y^2 + z^2 - x^2) + 2xyz\} + \{y(z^2 + x^2 - y^2) + 2xyz\}$
 $+ \{z(x^2 + y^2 - z^2) + 2xyz\}$ (A)

The quantity within the first pair of braces

$$= x(y^2 + z^2 - x^2 + 2yz) = x\{(y+z)^2 - x^2\} = x(y+z-x)(y+z+x)$$

Similar expressions for the other quantities in (A) may be written down by comparison; since $x + y + z$ is a common factor of these, the whole expn. $= (x+y+z)\{x(y+z-x) + y(z+x-y) + z(x+y-z)\}$
 $= (x+y+z)(2xy + 2yz + 2zx - x^2 - y^2 - z^2)$, simplifying.

$\therefore x + y + z$ divides $x(y^2 + z^2 - x^2) + \&c$ exactly,
 and the quotient $= 2xy + 2yz + 2zx - x^2 - y^2 - z^2$. *Ans.*

Ex. 5. Shew that $(x^3 - 3x)^4 - 8(x^6 - 6x^4 + 9x^2 - 2)$ is an exact square, and resolve the whole expression into factors M. U. 1878

$$\begin{aligned} \text{The given expn.} &= (x^3 - 3x)^4 - 8(x^6 - 6x^4 + 9x^2 - 2) + 16 \\ &= (x^3 - 3x)^4 - 8(x^3 - 3x)^2 + 16 \\ &= y^4 - 8y^2 + 4^2, \text{ putting } y \text{ for } x^3 - 3x, \\ &= (y^2 - 4)^2 \\ &= \{(y+2)(y-2)\}^2 \\ &= (y+2)^2(y-2)^2 \\ &= (x^3 - 3x + 2)^2(x^3 - 3x - 2)^2, \text{ restoring } x^3 - 3x \end{aligned}$$

Now, $x^3 - 3x + 2 = (x-1)^2(x+2)$, by Ex. 1, Art. 84

and similarly $x^3 - 3x - 2 = (x+1)^2(x-2)$. [Work out]

\therefore the given expression $= \{(x-1)^2(x+2)\}^2 \{(x+1)^2(x-2)\}^2$
 $= (x-1)^4(x+2)^2(x+1)^4(x-2)^2$. *Ans.*

Ex. 6. Resolve $x^3 - 2xy - 3y^2 + x + 5y - 2$ into factors,

First method: To express the given expression as the difference of two squares, begin by arranging it in descending powers of some letter.

$$\begin{aligned} \text{The given expression } x^3 - 2xy - 3y^2 + x + 5y - 2, \text{ as arranged,} \\ &= [x^3 - x(2y-1) + \{\frac{1}{2}(2y-1)\}^2] - [\{\frac{1}{2}(2y-1)\}^2 + 3y^2 - 5y + 2], \text{ Art. 62,} \\ &= [x - \{\frac{1}{2}(2y-1)\}]^2 - [(y - \frac{1}{2})^2 + 3y^2 - 5y + 2] \\ &= \{x - (y - \frac{1}{2})\}^2 - (4y^2 - 6y + \frac{9}{4}), \text{ simplifying,} \\ &= (x - y + \frac{1}{2})^2 - (2y - \frac{3}{2})^2 \\ &= \{x - y + \frac{1}{2} + (2y - \frac{3}{2})\} \{x - y + \frac{1}{2} - (2y - \frac{3}{2})\}, \text{ Formula IV, Art. 75,} \\ &= (x + y - 1)(x - 3y + 2), \text{ simplifying. } \textit{Ans.} \end{aligned}$$

Second method : We propose a method of *inspection*. We first resolve the part of the second degree in x and y of the given expression.

$$x^2 - 2xy - 3y^2 = (x+y)(x-3y), \text{ by the usual method.}$$

$$\therefore \text{ the given expression} = (x+y)(x-3y) + (x+5y) - 2 \quad (A)$$

We now adopt a method similar to the one given in Art. 64, page 56. Taking $(x+y)(x-3y)$ as the 1st term, $x+5y$ as the 2nd term, and -2 as the last term of the given expression, and observing that the product, 1st term \times last term, is negative, we have to break up $2(x+y)(x-3y)$ into factors, such that their difference = the 2nd term, viz., $x+5y$.

$$\text{It is readily seen that } 2(x+y) - (x-3y) = x+5y.$$

Hence proceed thus : By (A), we have

$$\begin{aligned} \text{the given expn.} &= (x+y)(x-3y) + 2(x+y) - (x-3y) - 2 \\ &= (x+y)(x-3y+2) - 1(x-3y+2) \\ &= (x+y-1)(x-3y+2). \end{aligned}$$

N.B. We might have conveniently broken up $x+5y$ by examining (A) twice in the following manner. Putting $y=0$ in (A), the expn. $= x+2$, to factorize which x must be put as $2x-x$, see Rule, Art. 64. Putting $x=0$, the expn. $= -3y^2+5y-2$, to factorize which $5y$ must be put as $2y+3y$. Hence in our actual factorization we must evidently put $x+5y = 2x-x+2y+3y = 2(x+y) - (x-3y)$, as we have done above.

Ex. 7. Factorize $x^2 - xy - 2y^2 - z^2 + 3yz$.

$$\begin{aligned} \text{The given expn.} &= (x^2 - xy - 2y^2) + 3yz - z^2, \text{ re-arranging,} \\ &= (x-2y)(x+y) + 3yz - z^2. \end{aligned} \quad (A)$$

We now regard $(x-2y)(x+y)$ as the 1st term, $3yz$ as the 2nd, and $-z^2$ as the last term of the given expression. We have to break up $z^2(x-2y)(x+y)$ into two factors, such that their difference may be $3yz$.

$$\text{Now, } z(x+y) - z(x-2y) = 3yz, \text{ by subtraction ;}$$

$$\begin{aligned} \therefore \text{ by (A), the given expn.} &= (x-2y)(x+y) + z(x+y) - z(x-2y) - z^2 \\ &= (x+y)(x-2y+z) - z(x-2y+z) \\ &= (x+y-z)(x-2y+z). \quad \text{Ans.} \end{aligned}$$

N.B. The student is advised to attempt the above example otherwise by the method of the difference of two squares.

The breaking up of $3yz$, upon which depends the success of the method we have used, should be carefully noted. The answer may be guessed out very readily from beginning. Putting $z=0$ in (A), the given expression $= (x-2y)(x+y)$; putting $x=0$ in the same expression, it $= -2y^2+3yz-z^2 = (-2y+z)(y-z)$. Therefore we readily infer* that the entire expression $= (x-2y+z)(x+y-z)$. Now verify the last result by mental multiplication.

The method of inspection here explained will be found to be very convenient in all examples of the above type where factorization is at all possible.

Ex. 8. Factorize the expression, $6x^3 + 17x^2 + 11x + 2$, being given that it is exactly divisible by $3x + 1$, and thence break up 7812 into its prime factors.

Divide $6x^3 + 17x^2 + 11x + 2$ by $3x + 1$; or proceed thus :

$$\begin{aligned} 6x^3 + 17x^2 + 11x + 2 &= (3x + 1)2x^2 + (3x + 1)5x + 2(3x + 1) \\ &= (3x + 1)(2x^2 + 5x + 2). \end{aligned}$$

Now, $2x^2 + 5x + 2 = (2x + 1)(x + 2)$, by the usual method.

$$\therefore 6x^3 + 17x^2 + 11x + 2 = (3x + 1)(2x + 1)(x + 2).$$

Putting $x = 10$ in the last result, we have.

$$\begin{aligned} \text{the left side} &= 6 \times 10^3 + 17 \times 10^2 + 11 \times 10 + 2 \\ &= 6000 + 1700 + 110 + 2 \\ &= 7812; \end{aligned}$$

$$\begin{aligned} \text{and the right side} &= (3 \times 10 + 1)(2 \times 10 + 1)(10 + 2) \\ &= 31 \times 21 \times 12 \\ &= 31 \times 7 \times 3 \times 3 \times 2^2 \\ &= 2^2 \times 3^2 \times 7 \times 31. \end{aligned}$$

$$\therefore 7812 = 2^2 \times 3^2 \times 7 \times 31. \quad \text{Ans.}$$

Ex. 9. Show that $x^3 + 6(y+z)x^2 + 12(y+z)^2x + 8(y+z)^3$
 $= 4(2y+3x+6z)y^2 + (x+6y+2z)(x+2z)^2$. M. U. 1881.

$$\begin{aligned} x^3 + 6(y+z)x^2 + 12(y+z)^2x + 8(y+z)^3 &= x^3 + 3x^2 \cdot 2(y+z) + 3\{2(y+z)\}^2x + \{2(y+z)\}^3 \\ &= x^3 + 3ax^2 + 3a^2x + a^3, \text{ if } a = 2(y+z), \\ &= (x+a)^3 \dots \dots \dots \text{Formula X, Art. 73,} \\ &= \{x + 2(y+z)\}^3, \because a = 2(y+z), \\ &= \{2y + (x+2z)\}^3, \text{ re-arranging,} \\ &= (2y)^3 + 3 \cdot (2y)^2(x+2z) + 3 \cdot 2y(x+2z)^2 + (x+2z)^3 \\ &= \{8y^3 + 12y^2(x+2z)\} + \{6y(x+2z)^2 + (x+2z)^3\} \\ &= 4y^2\{2y + 3(x+2z)\} + (x+2z)^2(6y + x+2z) \\ &= 4(2y+3x+6z)y^2 + (x+6y+2z)(x+2z)^2, \text{ re-arranging.} \end{aligned}$$

Ex. 10. Resolve into factors $(9x^2 + 2)^3 + 12$.

By expansion, the given expression

$$\begin{aligned} &= 81x^6 + 36x^2 + 16 \\ &= (81x^4 + 72x^2 + 16) - 36x^2 \\ &= (9x^2 + 4)^2 - (6x)^2 \\ &= (9x^2 + 6x + 4)(9x^2 - 6x + 4). \quad \text{Art. 75. Ans.} \end{aligned}$$

N.B. It will be seen that $81x^4 + 36x^2 + 16 = (3x)^4 + 2^3 \cdot (3x)^2 + 2^4$, i.e., is of the form $a^4 + a^2b^2 + b^4$, which suggests our method.

The following examples show that the *method of substitution* is a powerful algebraical method of simplification. To use it successfully, however, requires careful examination of the form of the given expression and of the relation between its terms.

Ex. 11. Shew that

$$(2a+b+c)^3 - 3(a-b)(2a+b+c)(a+2b+c)^2 - (a+2b+c)^3 = (a-b)^3.$$

Put x for $2a+b+c$, and y for $a+2b+c$;

then $x-y=2a+b+c-(a+2b+c)=a-b$.

∴ by substitution, the left member of the proposed identity

$$= x^3 - 3(x-y)xy^2 - y^3$$

$$= (x-y)^3 \dots \dots \dots \text{Formula XI, Art. 73.}$$

$$= (a-b)^3, \because x-y=a-b.$$

Ex. 12. Find the value of

$$(a+c-b)^3 + (a+b-c)^3 + (b+c-a)^3 + 24abc. \quad \text{B. U. 1859.}$$

By formula, $(x+y+z)^3 = x^3 + y^3 + z^3 + 3(x+y)(y+z)(z+x)$. (A).

$$\left. \begin{array}{l} \text{If } x=a+c-b, \\ y=a+b-c, \\ z=b+c-a, \end{array} \right\} \text{ then } \left. \begin{array}{l} x+y+z=a+b+c, \\ x+y=2a, \\ y+z=2b, \\ z+x=2c. \end{array} \right\} \text{ By addition.}$$

$$\therefore 3(x+y)(y+z)(z+x) = 3 \times 8abc = 24abc.$$

∴ by substitution, the right side of (A) = the given expression,
and the left side = $(a+b+c)^3$

∴ the required value = $(a+b+c)^3$. *Ans*

Ex. 13. Shew that

$$(x^2-yz)^3 + (y^2-zx)^3 + (z^2-xy)^3 - 3(x^2-yz)(y^2-zx)(z^2-xy) = (x^3+y^3+z^3-3xyz)^2.$$

Put $x^2-yz=u$, $y^2-zx=v$, and $z^2-xy=w$. Then

$$\begin{aligned} (x^2-yz)^3 + (y^2-zx)^3 + (z^2-xy)^3 - 3(x^2-yz)(y^2-zx)(z^2-xy) \\ = u^3 + v^3 + w^3 - 3uvw \\ = \frac{1}{2}(u+v+w)\{(u-v)^2 + (v-w)^2 + (w-u)^2\}. \quad \text{(A).} \end{aligned}$$

$$\begin{aligned} \text{Now, } u-v &= (x^2-yz) - (y^2-zx) = x^2 - y^2 + z(x-y) \\ &= (x-y)(x+y+z); \end{aligned}$$

similarly, $v-w = (y-z)(x+y+z)$, $w-u = (z-x)(x+y+z)$.

$$\begin{aligned} \therefore \frac{1}{2}\{(u-v)^2 + (v-w)^2 + (w-u)^2\} \\ = \frac{1}{2}(x+y+z)^2\{(x-y)^2 + (y-z)^2 + (z-x)^2\} \\ = (x+y+z)^2(x^2+y^2+z^2-yz-zx-xy). \end{aligned}$$

Also $a+b+c = x^3+y^3+z^3-yz-zx-xy$.

∴ the right side of (A) = $\{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)\}^2$
= $(x^3+y^3+z^3-3xyz)^2$. Hence the identity.

Otherwise thus : By expansion, we have

$$\begin{aligned}(x^2 - yz)^2 - (y^2 - zx)(z^2 - xy) \\&= (x^4 - 2x^2yz + y^2z^2) - (y^3z^2 - x^2yz + x^2yz) \\&= x^4 + xy^3 + xz^3 - 3x^2yz \\&= x(x^3 + y^3 + z^3 - 3xyz).\end{aligned}$$

Multiplying both sides by $x^2 - yz$, we have

$$(x^2 - yz)^3 - (x^2 - yz)(y^2 - zx)(z^2 - xy) = (x^3 - xyz)(x^3 + y^3 + z^3 - 3xyz);$$

similarly we obtain,

$$(y^2 - zx)^3 - (x^2 - yz)(y^2 - zx)(z^2 - xy) = (y^3 - xyz)(x^3 + y^3 + z^3 - 3xyz);$$

$$(z^2 - xy)^3 - (x^2 - yz)(y^2 - zx)(z^2 - xy) = (z^3 - xyz)(x^3 + y^3 + z^3 - 3xyz).$$

Adding up the last three identities, we obtain the proposed identity.

Ex. 14. Prove that $a^3 + b^3 + c^3 = 3abc$, if $a + b + c = 0$.

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - bc - &c), \text{ by formula}$$

$$= 0 \times (a^2 + b^2 + c^2 - bc - &c) = 0, \text{ if } a + b + c = 0;$$

transposing, $a^3 + b^3 + c^3 = 3abc$.

Otherwise thus :

$$\text{Given } a + b + c = 0;$$

$$\text{transposing, } a + b = -c;$$

$$\text{cubing, } a^3 + b^3 + 3ab(a + b) = -c^3;$$

$$\text{transposing, } a^3 + b^3 + c^3 = -3ab(a + b)$$

$$= -3ab(-c), \therefore a + b = -c,$$

$$= 3abc.$$

N.B. The above example should be carefully studied and remembered. We add below some of its consequences.

Ex. 15. Resolve into factors $(x - y)^3 + (y - z)^3 + (z - x)^3$. C. U. 1866.

$$\text{Put } x - y = a, y - z = b, z - x = c.$$

$$\text{Then } a + b + c = x - y + y - z + z - x = 0$$

\therefore proceeding as in the last Ex., we get

$$a^3 + b^3 + c^3 = 3abc;$$

$$\text{i.e., } (x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x). \text{ Ans.}$$

N.B. Many identities can be immediately deduced from Ex. 14. For instance, $(b + c - 2a) + (c + a - 2b) + (a + b - 2c) = 0$, identically ;

$$\therefore (b + c - 2a)^3 + (c + a - 2b)^3 + (a + b - 2c)^3 \\= 3(b + c - 2a)(c + a - 2b)(a + b - 2c).$$

Again, $a(b - c) + b(c - a) + c(a - b) = 0$, Formula XIX, Art. 73 ;

$$\therefore a^2(b - c)^3 + b^2(c - a)^3 + c^2(a - b)^3 = 3abc(b - c)(c - a)(a - b).$$

Ex. 16. Factorise $a^4(b^2+c^2)-b^4(c^2+a^2)+c^4(a^2-b^2)$.

It is readily seen that the given expression results from the substitution of a^2, b^2 and c^2 for a, b and c respectively in the right-hand side of Formula XX, Art. 73. We therefore employ the method of Art. 82. Substituting $b^2+c^2+a^2-b^2$ for c^2+a^2 ,

$$\begin{aligned} \text{the given expn.} &= a^4(b^2+c^2)-b^4\{(b^2+c^2)+(a^2-b^2)\}+c^4(a^2-b^2) \\ &= \{a^4(b^2+c^2)-b^4(b^2+c^2)\}-\{b^4(a^2-b^2)-c^4(a^2-b^2)\} \\ &= (a^4-b^4)(b^2+c^2)-(a^2-b^2)(b^4-c^4) \\ &= (a^2-b^2)(b^2+c^2)\{(a^2+b^2)-(b^2-c^2)\} \\ &= (a^2-b^2)(b^2+c^2)(c^2+a^2), \text{ simplifying,} \\ &= (a-b)(a+b)(b^2+c^2)(c^2+a^2). \quad \text{Ans.} \end{aligned}$$

Ex. 17. Show that

$$\begin{aligned} (x-a)(x-b)(a-b) &+ (x-b)(x-c)(b-c) + (x-c)(x-a)(c-a) \\ &= -(a-b)(b-c)(c-a). \end{aligned}$$

Put $x-a=A, x-b=B, x-c=C$;

then by subtraction, $B-A=a-b, C-B=b-c, A-C=c-a$.

∴ by substitution, the left side of the proposed identity

$$\begin{aligned} &= AB(B-A) + BC(C-B) + CA(A-C) \\ &= -(B-A)(C-B)(A-C), \dots\dots\dots \text{Formula XX, Art. 73,} \\ &= -(a-b)(b-c)(c-a). \end{aligned}$$

Otherwise thus : Multiplying out,

$$\begin{aligned} (x-a)(x-b)(a-b) &= \{x^2-x(a+b)+ab\}(a-b) \\ &= x^2(a-b)-x(a^2-b^2)+ab(a-b). \end{aligned}$$

We now at once obtain the following results :

$$\begin{aligned} (x-a)(x-b)(a-b) &= x^2(a-b)-x(a^2-b^2)+ab(a-b), \\ (x-b)(x-c)(b-c) &= x^2(b-c)-x(b^2-c^2)+bc(b-c), \\ (x-c)(x-a)(c-a) &= x^2(c-a)-x(c^2-a^2)+ca(c-a). \end{aligned}$$

Adding up these identities, and observing that on the right side the co-efficients of x^2 taken together $= a-b+b-c+c-a=0$, and similarly the sum of the co-efficients of $x=0$, we get ultimately,

$$\begin{aligned} (x-a)(x-b)(a-b) + \&c. = ab(a-b) + bc(b-c) + ca(c-a) \\ &= -(a-b)(b-c)(c-a), \text{ by formula.} \end{aligned}$$

N.B. Compare the two processes given above. But what led us to the first process ? Let us begin with $ab(a-b)+bc(b-c)+ca(c-a)$. Here each term is the continued product of two quantities and their difference ; e.g., the first term is the product of a, b and $(a-b)$. Now let us examine $(x-a)(x-b)(a-b) + \&c$. Observe that $a-b=x-b-(x-a)$. Hence here also each term is the continued product of two quantities and their difference, and this fact has suggested the transformation in the first method.

Ex. 18. If $2s = a + b + c$, prove that $2(s-a)(s-b)(s-c) + a(s-b)(s-c) + b(s-c)(s-a) + c(s-a)(s-b) = abc$.

If we put $s-a = x$, $s-b = y$, and $s-c = z$, then
 $x + y = s - a + s - b = 2s - (a + b) = c$, $\therefore 2s = a + b + c$.

Similarly $y + z = a$, and $z + x = b$.

\therefore by substitution, $2(s-a)(s-b)(s-c) + a(s-b)(s-c) + \&c.$
 $= 2xyz + (y+z)yz + (z+x)zx + (x+y)xy$
 $= (y+z)(z+x)(x+y) \dots \dots \dots \text{Formula XVII, Art. 73,}$
 $= abc$, by substitution.

Otherwise thus : Applying Formulæ VIII and IX, we get

$$\begin{aligned} 2(s-a)(s-b)(s-c) &= 2s^3 - 2(a+b+c)s^2 + 2(ab+bc+ca)s - 2abc, \\ a(s-b)(s-c) &= as^2 - (ab+ca)s + abc, \\ b(s-c)(s-a) &= bs^2 - (bc+ab)s + abc, \\ c(s-a)(s-b) &= cs^2 - (ca+bc)s + abc; \end{aligned}$$

adding up, $2(s-a)(s-b)(s-c) + a(s-b)(s-c) + \&c.$
 $= 2s^3 - (a+b+c)s^2 + abc$
 $= 2s^3 - 2s \times s^2 + abc$, putting $2s$ for $a+b+c$,
 $= abc$.

86. Difference of two squares. We propose here to show how an expression given as the product of some others may be transformed into the difference of two squares.

Evidently, $4xy = (x+y)^2 - (x-y)^2$. See Ex. 2, Page 84.

Dividing both sides by 4, $xy = \frac{(x+y)^2}{4} - \frac{(x-y)^2}{4}$
 $= \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$.

Hence the following mnemonic rule :

Product of two quantities = (half-sum)² - (half-difference)².

Ex. 1. Express $(2x^2 + 4x - 3)(2x^2 - 6x^2 + 1)$ as the difference of two squares.

We know that $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$, identically. (A)

Put $a = 2x^2 + 4x - 3$, and $b = 2x^2 - 6x^2 + 1$, then will

$$\frac{a+b}{2} = \frac{1}{2}\{(2x^2 + 4x - 3) + (2x^2 - 6x^2 + 1)\} = 2x^2 - 3x^2 + 2x - 1,$$

$$\frac{a-b}{2} = \frac{1}{2}\{(2x^2 + 4x - 3) - (2x^2 - 6x^2 + 1)\} = 3x^2 + 2x - 2.$$

\therefore by substitution, (A) becomes

$$(2x^2 + 4x - 3)(2x^2 - 6x^2 + 1) = \underline{(2x^2 - 3x^2 + 2x - 1)^2 - (3x^2 + 2x - 2)^2}.$$

Ex. 2. Express $(x+3a)(x+5a)(x+7a)(x+9a)$ as the difference of two squares. C. U. 1887.

We shall first multiply up two of the factors as well as the other two.

First Solution : The given expression

$$\begin{aligned}
 &= \{(x+3a)(x+5a)\} \{(x+7a)(x+9a)\} \\
 &= (x^2+8ax+15a^2)(x^2+16ax+63a^2) \\
 &= AB, [A=x^2+8ax+15a^2, B=x^2+16ax+63a^2] \\
 &= \left(\frac{A+B}{2}\right)^2 - \left(\frac{A-B}{2}\right)^2, \text{ Art. 86,} \\
 &= (x^2+12ax+39a^2)^2 - (-4ax-24a^2)^2 \\
 &= (x^2+12ax+39a^2)^2 - (4ax+24a^2)^2. \quad \text{Ans.}
 \end{aligned}$$

Second Solution : Re-arranging factors,

$$\begin{aligned}
 \text{the given expression} &= \{(x+3a)(x+9a)\} \{(x+5a)(x+7a)\} \\
 &= (x^2+12ax+27a^2)(x^2+12ax+35a^2) \\
 &= AB, \text{ suppose,} \\
 &= \left(\frac{A+B}{2}\right)^2 - \left(\frac{A-B}{2}\right)^2 \\
 &= (x^2+12ax+31a^2)^2 - (4a^2)^2 \\
 &= (x^2+12ax+31a^2)^2 - (4a^2)^2. \quad \text{Ans. [Note this.]}
 \end{aligned}$$

N. B. It is evident that by combining the factors in other ways, we shall get other solutions. For instance, we can put the given expression $= \{(x+3a)(x+7a)\} \{(x+5a)(x+9a)\}$, or $= (x+3a)\{(x+5a)(x+7a)(x+9a)\}$, &c. It will be seen, however, that the second solution given above is the simplest, and is the one that is usually adopted. In this case we generally begin by putting the given expression as

(greatest \times least factor) \times (product of others).

EXAMPLES 49.

Prove that

- $(2x+y)^2 + (y+2z)^2 = 4(x+y+z)^2 - (4x+2y)(y+2z).$
- $(2x+a)^2 + (x+2b)(5x+2a+2b) = (a+b)^2 + (3x+b)(3x+2a+3b).$
- $(a-b)^2 + (b-c)^2 + (c-a)^2 = 3(a^2+b^2+c^2), \text{ if } a+b+c=0.$
- $2bc(a-b)(a-c) + 2ca(b-c)(b-a) + 2ab(c-a)(c-b)$
= the sum of three squares.
- $a^4+b^4+c^4-2a^2b^2+2b^2c^2+2c^2a^2 = \frac{1}{2}(a^2+b^2+c^2)^2, \text{ if } a+b+c=0.$
- $(x-y)^4 + (y-z)^4 + (z-x)^4 = \frac{1}{2}\{(x-y)^2 + (y-z)^2 + (z-x)^2\}^2.$

7. $x^3(y^4 + z^4 - x^4) + y^3(z^4 + x^4 - y^4) + z^3(x^4 + y^4 - z^4) + 6x^2y^2z^2$
 $= (x^2 + y^2 + z^2)(x + y + z)(y + z - x)(z + x - y)(x + y - z).$
8. $(x^3 - 7x)^3 - 12(x^3 - 7x - 2) =$ the product of six factors.
9. $(4x^3 - 3x)^4 - (32x^6 - 48x^4 + 18x^2 - 1)$ is a perfect square. Resolve the same expression into factors.

Resolve into factors

10. $(x^3 + 3x)^2 - 8(x^2 + 3x) - 20.$ 11. $x^4 + 4x^3 + 8x^2 + 8x + 3.$
12. $x^4 + 2x^3 - 7x^2 - 8x + 12.$ 13. $x^3 + xy - 6x + y - 7.$
14. $x^2 - 4y^3 + 5x + 2y + 6.$ 15. $2xy + 3y^2 + 4(x + y - 1)$
16. $x^2 + 4xy + 3y^2 + 6x + 10y + 8.$ 17. $x^2 + 2xy + x(c - 1) - 2y - c.$
18. $a^2 - ab - 2b^2 + 4ca + bc + 3c^2.$ 19. $2a^2 + ab - b^2 + ac + bc.$
20. $x^2 + xy(1 - a) - ay^2 + by(1 + a) - b^2.$
21. $x^2 + (a + c)xy + (b + d)x + acy^2 + (bc + ad)y + bd.$
22. $24x^3 + 58x^2 + 37x + 6$, being given that $2x + 3$ is a factor, and thence factorize 30176.
23. Resolve into their simplest factors.
- (1) $a^2(b + c) - b^2(c + a) + c^2(a - b).$
 - (2) $a(b^2 - c^2) - b(c^2 - a^2) - c(a^2 - b^2).$
 - (3) $a^2b^2(a^2 + b^2) + b^2c^2(b^2 - c^2) - c^2a^2(c^2 + a^2).$
 - (4) $a^3b^3(b^3 - a^3) + b^3c^3(b^3 + c^3) - c^3a^3(c^3 + a^3).$
24. Factorize as far as possible :
- (1) $b^3(a - c) + c^3(a - b) - a^3(b + c) + 2abc.$
 - (2) $a^3b^3(b^3 - a^3) + b^3c^3(b^3 - c^3) + c^3a^3(c^3 + a^3) - 2a^3b^3c^3.$
 - (3) $(a^2 - b^2 + c^2)(c^2a^2 - a^2b^2 - b^2c^2) + a^2b^2c^2.$
25. Prove that $(x + y - z)^3 + (x - y + z)^3 + 6x^2 - 6x(y - z)^2$
 $= (2a - c)^3 + (2x - 2a + c)^2 + 6x(2a - c)(2x - 2a + c).$
26. Show that $(b^2 - c^2)^2 + (c^2 - a^2)^2 + (a^2 - b^2)^2$ is divisible by the sum and difference of any two of the quantities a , b and c .
27. Show that $(a - b)^3(c + 1)^3 + (b - c)^3(a + 1)^3 + (c - a)^3(b + 1)^3$ can be expressed as the product of six algebraical factors.
28. Resolve into factors $a^3(bx - cy)^3 + b^3(cx - az)^3 + c^3(ay - bx)^3.$
29. Factorize $(x^2 + 2)^3 + 12$ and $(a^3 + 2a + 3)^2 + 12.$
30. Factorize $(x + y)^3 - (x - y)^3 - 6(x^2 - y^2) - 8.$
31. If $x + 2y - 3z = a$, $z + 2x - 3y = b$, $y + 2z - 3x = c$, find the value of $a^3 + b^3 + c^3 - 3abc.$
32. If $a = (y - z)(x - w)$, $b = (z - x)(y - w)$, $c = (x - y)(z - w)$, find the value of $a^3 + b^3 + c^3 - 3abc.$

33. Factorize $27x^3 - (x+y+z)^3 - (x+y-2z)^3 - (x-2y+z)^3$.

34. Prove that

$$(x+2y)^3 = x^3 + (y+z)^3 + (y-z)^3 + 6x(x+y+z)(x+y-z)$$

35. Show that $26x^3 + (x+y)^3 - (x-y)^3 = 6x(4x^2 - y^2)$.

36. Factorize $7(a+c)^3 - (a-b)^3 - (b+c)^3$.

37. Simplify $(x-y)^3(x+y-2z)^2 + (y-z)^3(y+z-2x)^2$
 $+ (z-x)^3(z+x-2y)^2$.

38. Prove that $(x+a)^3(b-c) + (x+b)^3(c-a) + (x+c)^3(a-b)$
 $= -(a-b)(b-c)(c-a)(a+b+c+3x)$.

39. Resolve into factors $(x-2a)^3(b-c) + (x-2b)^3(c-a)$
 $+ (x-2c)^3(a-b)$.

40. Simplify $(b+c)^3(b-c) + (c+a)^3(c-a) + (a+b)^3(a-b)$.

41. If $a+b+c=0$, show that $a^3(b+c) + b^3(c+a) + c^3(a+b) + 3abc = 0$.

42. If $s = a+b+c$, show that

$$(s-a)^3 + (s-b)^3 + (s-c)^3 = 8s^3 - 3(s+a)(s+b)(s+c).$$

43. If $s = a+b+c$, show that

$$s^3 - (s-2a)^3 - (s-2b)^3 - (s-2c)^3 = 24abc.$$

44. Given that $s = a+b+c$, prove that

$$(s-3a)^3 + (s-3b)^3 + (s-3c)^3 = 3\{(a-b)^3 + (b-c)^3 + (c-a)^3\}$$

45. If $s = \frac{1}{2}(a+b+c)$, prove that $s(s-a)(s-b) + s(s-b)(s-c)$
 $+ s(s-c)(s-a) = (s-a)(s-b)(s-c) + abc$

46. If $2s = a+b+c$, prove that

$$(s-a)^3(s-b) + (s-b)^3(s-c) + (s-c)^3(s-a) + a^3b + b^3c + c^3a = s^3.$$

47. Show that $x(y+z)(y^2+z^2-x^2) + y(z+x)(z^2+x^2-y^2)$
 $+ z(x+y)(x^2+y^2-z^2) = 2xyz(x+y+z)$.

48. Show that $2\{(a+b)^3 - (b+c)^3 + (c+a)^3 - (d+a)^3\}$ is exactly
divisible by $3\{(a+b)^2 - (b+c)^2 + (c+d)^2 - (d+a)^2\}$.

Express as the difference of two squares :

49. $(x+1)(x+2)(x+3)(x+4)$. 50. $(x^2-a^2)(x+3a)(x+11a)$

51. $(x+1)(x+10)(x-1)(x+6)$. 52. $x(x^2-a^2)(x+2a) - 24a^4$.

Show that

53. $x(x-a)(x+2a)(x-3a) - 16a^4 = (x^2-ax+2a^2)(x^2-ax-8a^2)$.

54. $(x+1)(x+2)(x+3)(x+8) - 3(4x+11)^2$
 $= (x^2-x-9)(x^2+15x+35)$.

55. $(x-1)(x-7)(x+5)(x+11) + 720 = (x^2+4x-17)(x^2+4x-65)$.

56. $(x+1)(x+3)(x+6)(x+8) + (2x+9)^2 = (x^2+9x+15)^2$.

57. $(x+2)(x-4)(x+8)(x-10)+144(x-1)^2=(x^2-2x+28)^2$.
58. Express as the sum of two squares:
 (i) $(a^2+b^2)(x^2+y^2)$; (ii) $2(a^2+b^2)^2+2a^2b^2$.
59. Show that twice the sum of the squares of three quantities diminished by twice the product of every two is equal to the sum of three squares.
60. Reduce to the sum of three squares: $2(x^2-xy+y^2-x-y+1)$.

CHAPTER XV.

HIGHEST COMMON FACTOR.

87. Definitions. Each of the letters occurring in a term is called a **dimension** of the term, and the total number of letters involved is called its **degree** and denotes the **number of its dimensions**. Thus the product $abcd$ is said to be of *four dimensions*, or of the *fourth degree*; and x^4y^3 is of *eight dimensions*, or of the *eighth degree* ($x^4=x \times x \times x \times x$, and $y^3=y \times y \times y$). Thus *the degree of a term is the sum of the indices of the powers of the several letters in it*.

A numerical coefficient is not counted. Thus the degree of $9x^4y^3z$ or $9x^4y^3z^1=4+2+1=7$.

We sometimes speak of the dimensions of a term with respect to any one of the letters it involves. Thus $9x^4y^3z$, which is of seven dimensions, is said to be of *four dimensions in x* , of *two dimensions in y* , and of *one dimension in z* .

The **degree of an expression** with respect to any particular letter involved is the index of the highest power of that letter occurring in it. Thus the expression $ax^5+bx^3+cx^2+dx+e$ is of the fifth degree in x .

A compound expression is said to be **homogeneous** when all its terms are of the same dimensions. Thus $4x^5-3x^4y+7x^3y^2-11xy^4+y^6$ is a *homogeneous expression of five dimensions*.

The **highest common factor** of two or more expressions is the *expression of highest dimensions* which divides each of them without a remainder. It is written in brief as H. C. F. It is also called the **highest common divisor** (briefly H. C. D.), and the **greatest common measure** (briefly G. C. M.).

88. H. C. F. of simple Expressions. The H. C. F. of simple expressions is easily found by *inspection*.

Ex. 1. Find the H. C. F. of x^3 and x^5 .

Evidently x^3 is the *highest power of x* which divides each of x^3 and x^5 without a remainder. Hence x^3 is the required H. C. F. *Ans.*

Ex. 2. Find the H. C. F. of ax^2y^2 , bx^4y^6 and $a^2x^3y^4$.

x^2 is the highest power of x which is common to all the given expressions, and therefore divides them exactly. Similarly y^2 is the highest power of y which is a common factor. Hence x^2y^2 is evidently the H. C. F. sought. Note here that a is common to ax^2y^2 and $a^2x^3y^4$, but not to bx^4y^6 as well.

Ex. 3. Find the H. C. F. of $36x^2yz^3$, $42x^3y^2z^2$ and $30xy^3z^2$.

The G. C. M. of 36, 42 and 30 = 6.

• The H. C. F. of x^2yz^3 , $x^3y^2z^2$ and $xy^3z^2 = xyz^2$.

Hence evidently the required H. C. F. is $6xyz^2$. *Ans.*

89. Rule. The rule for finding the H. C. F. of two or more simple expressions can thus be generally put :

Take the product of

(1) *The G. C. M. of the numerical coefficients, if any ;*

(2) *The highest power of each common letter which divides exactly each of the given expressions.*

EXAMPLES 50.

Find the H. C. F. of

1. x^4 and x^2y ; x^2y^3 and xy^3 ; x^4y^2 and x^3y^5 .
2. $a^5b^3c^4$ and $a^3b^5c^3$; $a^4b^3c^6$ and $a^6b^4c^3$; $a^3b^4c^7$ and $a^2b^3c^2$.
3. $2a^2x^3y^3$ and $4a^3x^2y^3$; $3a^4b^3c^4$ and $8a^2c^2d$; $6l^4m^3$ and $8l^2m^7n^3$.
4. $8a^3x^4$ and $-12a^2x^2y^4$; $-9x^3y^2$ and $6x^2z^3$; $10x^4y^4z^4$ and $15a^5x^5y^5$.
5. $24a^6b^7c^8d^4$ and $36a^7b^5c^4d^6$; $144a^4b^6x^5y^3$ and $156a^5c^6x^4y^3z^3$.
6. a^3b^2 , abc^2 and $a^2b^3c^3$; $a^4b^4c^4$, $a^3b^5c^6$ and $a^2b^3c^2d^2$.
7. $x^3y^2z^3$, $x^4y^4z^4$ and a^2xyz ; l^4m^5 , $l^3m^3n^2$ and $l^2m^2n^3$.
8. $6a^2b^2x^2y^3$, $4a^6x^6$ and $8a^4b^5x^5$; $2a^7b^3$, $3a^4b^4x^3$ and $4a^5b^3x^2y^3$.
9. $7x^5y^3z^5$, $5x^4y^4z^6$ and $x^4y^2z^6$; $9x^5y^2z^3$, $12x^3y^4z^4$ and $6x^6y^6z^6$.
10. $12a^3b^3c^3p^3q^3r^3$, $35a^3b^3p^4q^4$, $48a^3b^3p^3q^2r^3$ and $60a^3b^3c^4p^3q^4r^3$.
11. $24a^3b^4c^6x^4y^4$, $40a^5b^3x^5z^3$, $32a^3b^3x^4$ and $72a^4b^4c^3x^3y^3z^3$.
12. $x^3y^2z^3$, $x^4y^3z^3$, $x^3y^3z^3$, $2x^5y^4z^6$ and $6ax^6y^6z^6$.
13. $4a^3b^4l^3m^2$, $6a^2b^3l^3m^3n^3$, $8a^4b^3l^4m^3x^3$, $10a^3b^3l^3m^2z^3$ and $12a^3b^3c^3l^4m^4$.

14. $14a^2b^2c^2d^2$, $21a^2b^2c^2d^2$, $28a^2b^2c^2d^2$, $35a^2b^2c^2d^2$ and $49a^2b^2c^2d^2$.

15. $6l^2m^2n^2$, $8a^2l^2mn$, $24a^2b^2l^2m^2$, $42abclm^2$ and $20a^2x^2l^2m$.

90. H. C. F. of compound expressions. The H. C. F. of such compound expressions as can be readily resolved into factors is found in a manner analogous to the one shown in the case of simple expressions.

Ex. 1. Find the H. C. F. of $4(a^2 - b^2)$ and $2a^2 - 2ab$.

$$4(a^2 - b^2) = 2^2(a - b)(a + b),$$

$$\text{and } 2a^2 - 2ab = 2(a - b)a;$$

$$\therefore \text{H. C. F.} = 2(a - b). \text{ Ans.}$$

Ex. 2. Find the H. C. F. of $4x^3 - 9x^2 + 2x$, $3x^3 - 4x^2 - 4x$ and $2x^4 - 3x^3 - 2x^2$.

$$\text{We easily find } 4x^3 - 9x^2 + 2x = x(4x^2 - 9x + 2) = x(x - 2)(4x - 1),$$

$$3x^3 - 4x^2 - 4x = x(3x^2 - 4x - 4) = x(x - 2)(3x + 2),$$

$$\text{and } 2x^4 - 3x^3 - 2x^2 = x^2(2x^2 - 3x - 2) = x^2(x - 2)(2x + 1);$$

$$\therefore \text{H. C. F.} = x(x - 2). \text{ Ans.}$$

Ex. 3. Find the H. C. F. of $8(x^3 + y^3)(x - y)^3$, $12(x^4 - y^4)(x - y)^2$ and $16(x^2 - y^2)^4$.

$$8(x^3 + y^3)(x - y)^3 = 2^3(x + y)(x^2 - xy + y^2)(x - y)^3;$$

$$12(x^4 - y^4)(x - y)^2 = 2^2 \times 3(x^2 + y^2)(x + y)(x - y)^3; \text{ (Work out.)}$$

$$16(x^2 - y^2)^4 = 2^4(x + y)^4(x - y)^4;$$

$$\therefore \text{H. C. F.} = 2^2(x + y)(x - y)^3 = 4(x + y)(x - y)^3. \text{ Ans.}$$

N.B. Here notice that in finding the H. C. F. $(x - y)^3$ should be taken, and not simply $x - y$ or $(x - y)^2$, for $(x - y)^3$ is the *highest power* of $x - y$ which is *common* to the given expressions and therefore divides each of them exactly.

EXAMPLES 51.

Find the H. C. F. of

1. $a^3 - b^3$ and $a^2 - ab$; $(a - b)^2$ and $b^2 - ab$.

2. $a^3 - ab^3$ and $a^3 - b^3$; $a^3 + a^2b$ and $a^3 + b^2$.

3. $x^3 + 1$ and $(x + 1)^3$; $a^3 + x^3$ and $(a + x)^3$.

4. $x^4 - y^4$ and $x^2z^2 - y^2z^2$; $9x^3 - 4y^3$ and $6x^2 - 4xy$.

5. $ab^4 - a^2b$ and $a^2b + abx^3$; $a^3b^3 - a^2b^2c^2$ and $a^2b - ab^3$.

6. $5a^3 - 10ab$ and $3ac - 6bc$; $y - 16x^4y$ and $z^3 - 8x^3z^3$.

7. $8a^3 + x^3$ and $4a^2 - x^2$; $9x + 6y$ and $18x^2 - 8y^2$.

8. $4a^3 - b^3$ and $4a^3 + 2ab - 2b^3$; $12a^3 + ab - b^3$ and $6a^3 + 5ab + b^3$.
9. $5a^3 - 2a - 3$ and $5a^3 - 11a + 6$; $9a^3 - 4b^3$ and $9a^3 - 15ab - 14b^3$.
10. $x^4 - 2x^3$ and $x^3 - 4x + 4$; $a^2x^3 - 1$ and $a^2x^3 + 1$.
11. $x^3 + 5x^2$ and $x^3 - 25x$; $x^3 - 5x$ and $x^3 - 4x^2 - 5x$.
12. $x^3 - 2xy - 3y^2$ and $2y^3 + 3xy + x^3$; $y^3 + y - 12$ and $y^3 - 2y - 3$.
13. $x^3 + 8y^3$ and $x^3 + 4xy + 4y^3$; $3ab - 2b^3$ and $9a^3 - 12ab + 4b^3$.
14. $8(a^2x^3 - b^2y^3)$ and $12(a^2x^3 + 2abxy + b^2y^3)$; $1 - x^4$ and $1 - x^6$.
15. $9(x^3 + y^3)^3$ and $12(x^4 - y^4)$; $6(y^3 - y)$ and $7y(y^3 - 3y + 2)$.
16. $2(a^2b - ab^2)^3$ and $3(a^4b - ab^4)$; $9a^2b + 6ab^2$ and $(18a^2 + 12ab)^2$.
17. $4(a^3 + b^3)$ and $11(a^3 - a^2b - 2ab^2)$; $(a^3 + a)^3$ and $2(a^7 + a^4)$.
18. $16(a^3 - b^3)^3$ and $8(a^3 - b^3)^3$; $24(a^3 - b^3)$ and $40(a^4 + a^2b^2 + b^4)$.
19. $2x^2 - 8xy + 8y^2$ and $(x - 2y)^3$; $(xyx - yx^2)^3$ and $(x^3x - x^2x^2)^3$.
20. $x^4y^4 - y^6$ and $y^2(xy - y^2)^3$; $x^4 - 49x^2$ and $x^4 + x^3 - 42x^2$.
21. $(x^3y - 5xy)^3$, $x^6 - 8x^5 + 15x^4$; $3x^4 - 7x^3 - 20x^2$, $2x^7 - 9x^6 + 4x^5$.
22. $3x^3y^3 - 8x^2y^3 + 4xy^4$ and $2x^4y - x^2y^2 - 6x^2y^3$.
23. $mx^2 + (m+1)x + 1$ and $2\{x^3 - (n-1)x - n\}$.
24. $6a^3b - 9a^2b^2 + 3ab^3$ and $4a^4b + 2a^3b^2 - 6ab^4$.
25. $12a^3x^3 - 36a^2bx^2 + 27ab^2x^2$ and $64a^6b^2 - 216a^3b^5$.
26. $42a^4x - 7a^3x^2 - 7a^2x^3$ and $28a^2x^3 + 42a^2x^4 - 28ax^5$.
27. $48a^5x^4 - 40a^4x^5 - 48a^3x^6$ and $60a^6x^2 - 102a^5x^3 + 18a^4x^4$.
28. $(y-1)^3(y+2)$ and $(y^3-4)(y^2-1)$; y^6-1 and $y^4-y^2+y^3$.
29. $(ax+by) + (bx+ay)$ and $(ax-by) + (bx-ay)$.
30. $1+x+x^2+x^3$ and $1+2x+x^2+2x^3$; $1-x^6$ and $1+x+x^5$.
31. x^3-7x+6 and x^3-x^2+x-1 ; $1-l^3$ and $1+l^3+l^4$.
32. $36x^4-13x^2+1$ and $36x^4-60x^3+25x^2-1$; $1-x^6$ and $1-x^8$.
33. a^3-ab , a^2-b^2 and $a^3-2ab+b^3$; $1-a^2$, $1+a^3$ and $1-a^4$.
34. a^3-b^3 , a^2b-ab^2 and $a^3+ab-2b^3$; l^3+m^2 , l^4-m^4 and l^6+m^6 .
35. m^3-1 , $(m-1)^3$, $(m^3-1)(m+3)$ and m^3+m-2 .
36. a^3-4ab^2 , $a^4+4a^3b+4a^2b^2$, a^4+8ab^3 and $12b^3-3a^2b$.
37. x^3-5x+4 , x^3-6x+8 , x^3-16x and $2x^3-2x-24$.
38. $a^2b^3-a^2-b^3+1$, $(a+1)^3(b+1)^3$ and $a^2b^3+a^2+b^3+1$.
39. $(2a^4+4a^3)(a^2+2a-8)$, $(2a^2-4a)(a^2-2a^3-8a)$ and $(a^3-4a)(a^2+2a-8)$.
40. $27(l^3m^7-6lm^8+5m^9)$, $33(l^3m^3-4l^2m^4+3lm^5)$ and $36(l^3m^6-3l^2m^7+2lm^8)$.

41. $4a^2 - 2a + 1$, $8a^4 + a$ and $16a^4 + 4a^2 + 1$.
 42. $x^3 - (a-b)x - ab$, $x^2 - (a+b)x + ab$ and $x^4 - 2a^2x^2 + a^4$.
 43. $(y^3 - x^3)^2$, $(y-x)^2(y^3 + yx)$ and $(y^3 + yx - 2x^3)(y^3 - x^3)(y + x)$.
 44. $ac + bc - c^2$, $a^2 + b^2 - c^2 + 2ab$ and $a^2 - b^2 - c^2 + 2bc$.
 45. $a^4 - b^2$, $a^6 + b^2$, $a^3 + a^2b + ab + b^2$ and $4a^4c + 8a^2bc + 4b^2c$.
 46. $a^2 + b^2 + 3ab(a+b) - 1$, $a^3 + b^3 + 3ab - 1$ and $a^2 - b^2 - 1 + 2b$.

91. When two given expressions cannot be readily resolved into factors, the mode of finding their H. C. F. is similar to the one used in Arithmetic.

Rule. 1. Arrange the two given expressions according to descending or ascending powers of some common letter.

2. Divide the expression of the highest degree in that letter by the other; if the expressions are of the same degree, take for divisor that whose highest term has the smaller co-efficient.

3. Make the remainder a new divisor and continue the process until there is no remainder. The last divisor is the H. C. F.

Ex. 1. Find the H. C. F. of

$$2x^3 - 9x^2 + 12x - 4 \text{ and } 2x^4 - 13x^3 + 32x^2 - 35x + 11.$$

$$\begin{array}{r} 2x^3 - 9x^2 + 12x - 4 \quad 2x^4 - 13x^3 + 32x^2 - 35x + 11 \quad (x - 2 \\ \underline{2x^4 - 9x^3 + 12x^2 - 4x} \\ - 4x^3 + 20x^2 - 31x + 11 \\ \underline{- 4x^3 + 18x^2 - 24x + 8} \\ 2x^2 - 7x + 3 \end{array}$$

$$\begin{array}{r} 2x^3 - 7x + 3 \quad 2x^3 - 9x^2 + 12x - 4 \quad (x - 1 \\ \underline{2x^3 - 7x^2 + 3x} \\ - 2x^2 + 9x - 4 \\ \underline{- 2x^2 + 7x - 3} \end{array}$$

$$\begin{array}{r} 2x - 1 \quad 2x^2 - 7x + 3 \quad (x - 3 \\ \underline{2x^2 - x} \\ - 6x + 3 \\ \underline{- 6x + 3} \end{array}$$

\therefore H. C. F. = $2x - 1$.

The above work may be shown in short thus :

$$\begin{array}{r|l}
 x \cdot 2x^3 - 9x^2 + 12x - 4 & 2x^4 - 13x^3 + 32x^2 - 35x + 11 \quad x \\
 \hline
 2x^3 - 7x^2 + 3x & 2x^4 - 9x^3 + 12x^2 - 4x \quad . \\
 -1 \cdot \quad -2x^3 + 9x - 4 & \quad -4x^3 + 20x^2 - 31x + 11 \quad -2 \\
 \quad -2x^2 + 7x - 3 & \quad -4x^3 + 18x^2 - 24x + 8 \\
 \quad \quad \quad 2x - 1 & \quad \quad \quad 2x^2 - 7x + 3 \quad x \\
 & \quad \quad \quad 2x^2 - x \\
 & \quad \quad \quad -6x + 3 \quad -3 \\
 & \quad \quad \quad -6x + 3
 \end{array}$$

$$\therefore \text{H. C. F.} = \underline{2x-1}.$$

Special Rule : At any stage of the work the dividend or the divisor may be multiplied or divided by a quantity which has no factor in common with the other.

This rule should be added to the fore-going ones.

Ex. 2. Find the H. C F. of $2x^3 - (a+2)x^2 + a(a+1)x - a^2$ and $3x^3 + x^2(a-3) + a(a-1)x - a^2$.

Multiply the latter expression by 2, which is no factor of the first, and then begin by taking the product as the dividend.

$$\begin{array}{r}
 3x^3 + x^2(a-3) + a(a-1)x - a^2 \\
 \times 2 \\
 \hline
 6x^3 + 2x^2(a-3) + 2a(a-1)x - 2a^2 \\
 \hline
 2x^3 - (a+2)x^2 + a(a+1)x - a^2 \quad \text{Multiply by} \quad 6x^3 + 2x^2(a-3) + 2a(a-1)x - 2a^2 \quad (3) \\
 \hline
 6x^3 - 3(a+2)x^2 + 3a(a+1)x - 3a^2 \\
 \hline
 5ax^2 - a(a+5)x + a^2 \\
 = a\{5x^2 - (a+5)x + a\}.
 \end{array}$$

Since a is no factor of the divisor, reject it from the remainder ; since 5 is no factor of the latter, multiply the divisor by 5 for the next dividend.

$$\begin{array}{r}
 2x^3 - (a+2)x^2 + a(a+1)x - a^2 \\
 \times 5 \\
 \hline
 5x^3 - (a+5)x^2 + a(5a+3)x - 5a^2 \\
 \hline
 10x^3 - (5a+10)x^2 + a(5a+5)x - 5a^2 \quad (2x) \\
 \hline
 10x^3 - (2a+10)x^2 + 2ax \\
 \hline
 -3ax^2 + a(5a+3)x - 5a^2
 \end{array}$$

$$\begin{array}{r}
 \text{Multiply again by 5 ;} \\
 \hline
 -15ax^2 + a(25a+15)x - 25a^2 \quad (-3a) \\
 \hline
 -15ax^2 + a(3a+15)x - 3a^2 \\
 \hline
 22a^2x - 22a^2 = 22a^2(x-1).
 \end{array}$$

Reject $22a^2$, which has no common factor with $5x^3 - (a+5)x + a$.

$$\begin{array}{r}
 x-1 \quad 5x^3 - (a+5)x + a \quad (5x-a) \\
 \hline
 5x^2 - 5x
 \end{array}$$

$$\begin{array}{r}
 -ax + a \\
 -ax + a
 \end{array}$$

$$\therefore \text{H. C. F.} = \underline{5x-1}.$$

The above work may be put thus :

$2x^3 - (a+2)x^2 + a(a+1)x - a^3$	$3x^3 + (a-3)x^2 + a(a-1)x - a^3$	3
$10x^3 - (5a+10)x^2 + a(5a+5)x - 5a^3$	$6x^3 + (2a-6)x^2 + a(2a-2)x - 2a^3$	2
$10x^3 - (2a+10)x^2 + 2ax$	$6x^3 - (3a+6)x^2 + a(3a+3)x - 3a^3$	5
$-3ax^2 + a(5a+3)x - 5a^3$	$a[5ax^2 - a(a+5)x + a^3]$	5x
$-15ax^2 + a(25a+15)x - 25a^3$	$5x^3 - (a+5)x + a$	5x
$-15ax^2 + a(3a+15)x - 3a^3$	$5x^3 - 5x$	-a
$22a^3 - 22a^2x - 22a^3$	$-ax + a$	-a
$x - 1$	$-ax + a$	

H. C. F. = $x - 1$.

N. B. We begin by multiplying one of the given expressions by 2, in order that its first term may be an exact multiple of that of the other.

Sometimes it is more convenient to arrange the terms in ascending order of some quantity.

Ex. 3. Find the H. C. F. of $7x^4 - 10ax^3 + 3a^2x^2 - 4a^3x + 4a^4$ and $8x^4 - 13ax^3 + 5a^2x^2 - 3a^3x + 3a^4$. B. U. 1884.

$3a^2[3a^4 - 3a^3x + 5a^2x^2 - 13ax^3 + 8x^4]$	$4a^4 - 4a^3x + 3a^2x^2 - 10ax^3 + 7x^4$	3
$3ax[3a^3x - 6a^2x^2 + 3a^2x^3]$	$12a^4 - 12a^3x + 9a^2x^2 - 30ax^3 + 21x^4$	4
$8x^2[8a^2x^2 - 16ax^3 + 8x^4]$	$12a^4 - 12a^3x + 20a^2x^2 - 52ax^3 + 32x^4$	
$8a^2x^2 - 16ax^3 + 8x^4$	$-11a^2 - 11a^2x^2 + 22ax^3 - 11x^4$	
$8a^2x^2 - 16ax^3 + 8x^4$	$a^3 - 2ax + x^2$	

H. C. F. = $a^2 - 2ax + x^2$. Ans.

Ex. 4. Find the H. C. F. of

$$x^4 - x^2 + 6x - 9 \text{ and } x^4 + 2x^3 - 5x^2 - 6x + 9.$$

$-3[-9 + 6x - x^2 + x^4]$	$9 - 6x - 5x^2 + 2x^3 + x^4$	-1
$x[-9 + 3x + 3x^2]$	$9 - 6x + x^3$	-x^4
$3x - 4x^2 + x^4$	$-2x^2[-6x^2 + 2x^3 + 2x^4]$	3
$3x - x^3 - x^5$	$3 - x - x^3$	
$-x^3[-5x^2 + x^3 + x^4]$		
$-3x^3 + x^3 + x^4$		

H. C. F. = $3 - x - x^3$. Ans.

N. B. Work the above sum by arranging the expressions in descending powers of x .

Ex. 5. Find the H. C. F. of

$$4a^5b^3 - 4a^3b^4 - 24a^2b^5 \text{ and } 6a^4b^3 + 6a^2b^5 - 30a^2b^4 - 12ab^6.$$

$$4a^5b^3 - 4a^3b^4 - 24a^2b^5 = 4a^2b^3(a^3 - ab^2 - 6b^3).$$

$$6a^4b^3 + 6a^2b^5 - 30a^2b^4 - 12ab^6 = 6ab^3(a^3 + a^2b - 5ab^2 - 2b^3).$$

$$\text{H. C. F. of } 4a^2b^3 \text{ and } 6ab^3 = 2ab^3;$$

\therefore the H. C. F. sought $= 2ab^3 \times$ H. C. F. of the other factors.

$$\begin{array}{r|l} a & \begin{array}{r} a^3 - ab^2 - 6b^3 \\ a^3 - 4a^2b + 4ab^3 \\ \hline 4ab - 5ab^2 - 6b^3 \\ 4a^2b - 16ab^3 + 16b^3 \\ \hline 11b^3 \end{array} \\ 4b & \begin{array}{r} a^3 - ab^2 - 6b^3 \\ a^3 - 4a^2b + 4ab^3 \\ \hline 4a^2b - 16ab^3 + 16b^3 \\ \hline 11b^3 \end{array} \end{array} \quad \begin{array}{r|l} a^3 + a^2b - 5ab^2 - 2b^3 & 1 \\ a^3 & -ab^2 - 6b^3 \\ \hline b|a^2b - 4ab^2 + 4b^3 & a \\ a^2 - 4ab + 4b^2 & \\ \hline a^3 - 2ab & \\ -2ab + 4b^3 & -2b \\ \hline -2ab + 4b^3 & \end{array}$$

$$\therefore \text{H. C. F. sought} = 2ab^3(a - 2b) \quad \text{Ans.}$$

EXAMPLES 52.

Find the H. C. F. of

- $2x^3 - 3x + 1$ and $2x^3 - 6x + 4$; $2x^3 + 3x - 2$ and $6x^2 - 5x + 1$.
- $12x^3 - x - 6$ and $18x^2 - 9x - 14$; $24x^3 - 6x - 45$ and $28x^3 + 23x - 15$.
- $x^3 + 1$ and $x^3 - 4x^2 + 5$; $x^3 - 39x + 70$ and $x^3 - 3x - 70$.
- $x^3 - 19x^2 + 119x - 245$ and $3x^3 - 38x + 119$.
- $2x^3 + 8x^2y + 16xy^2 + 16y^3$ and $8x^2 + 4xy - 24y^2$.
- $12x^4 + 4x^3 - 3x^2 - x$ and $8x^3 - 4x^2 - 2x + 1$.
- $x^3 - 3a^2x - 2a^3$ and $x^4 - ax^3 + a^2x - 10a^4$.
- $2a^2x + 3a^2x^2 - 9ax^3$ and $6a^2x - 17a^2x^2 + 14ax^3 - 3x^4$.
- $x^4 - 4x^3 + 9x^2 - 10x$ and $x^3 + 2x^2 - 3x + 20$.
- $2x^4 - 16x^3 + 42x^2 - 36x$ and $3x^3 - 16x^2 + 21x$.
- $7x^3 - 12x^2 + 5x$ and $2x^3 + x^2 - 8x + 5$.
- $x^4 - 3x^3 - 10x^2 + 24x$ and $2ax^3 - 10ax^2 + 8ax$.
- $2x^3 - 8x^2y + 16xy^2 - 16y^3$ and $8x^2 - 4xy - 24y^2$.
- $5x^3 + 2x^2 - 15x - 6$ and $-7x^3 + 4x^2 + 21x - 12$.
- $18x^3 - 47x^2 - 5x + 25$ and $22x^3 - 57x^2 - 5x + 25$. (Better arrange in ascending powers of x).
- $15x^3 - 32x^2y + 3xy^2 + 2y^3$ and $16x^3 - 31x^2y + xy^2 - 6y^3$.
- $2x^4 + 3x^3 - 2x^2 - 4x - 5$ and $x^4 + 2x^3 - x^2 - 2x - 3$.

18. $6x^3 + 33x^2 + 36x - 27$ and $6x^4 + 31x^3 + 33x^2 - 19x - 3$.
 19. $x^4 - 4x^3 - 29x^2 - 26x + 16$ and $3x^4 - 11x^3 - 85x^2 - 79x + 48$.
 20. $2x^4 - 7x^3 - 56x^2 - 53x + 32$ and $4x^4 - 15x^3 - 116x^2 - 109x + 66$.
 21. $4x^4 - 33x^3 + 77x^2 - 42x$ and $6x^5 - 47x^4 + 96x^3 - 26x^2 - 5x$.
 22. $9x^4 + 21x^3 - 26x^2 - 132x - 84$ and $6x^4 + 22x^3 - 12x^2 - 123x - 105$.
 23. $18x^4 + 19x^3 - 133x^2 - 13x + 9$ and $24x^4 + 26x^3 - 181x^2 - 12x + 11$.
 24. $48a^4b^4 - 40a^4b^5 - 48a^3b^5$ and $60a^4b^3 - 102a^5b^4 + 18a^4b^5$.
 25. $a^5 - 11a + 10$ and $a^4 - 5a^3 + 8$; $a^5 - 6a + 5$ and $2a^3 - 3a + 1$.
 26. $x^5 - 10x^3 + 4x^2 + 12x + 8$ and $x^4 - 3x^3 - 2x^2 + 12x - 8$.
 27. $4x^5 - 15x^3 + 13x^2 - x - 1$ and $6x^5 - 7x^4 + 6x^3 - 6x^2 + 1$.
 28. $4x^7 - 24x^6 + 52x^5 - 48x^4 + 16x^3$ and $6x^7 - 30x^6 + 48x^5 - 24x^4$.
 29. $2x^4 + 9x^3y - 26x^2y^2 + 14xy^3 - 3y^4$ and $6x^3y - 17x^2y^2 + 16xy^3 - 6y^4$.
 30. $2x^5 - 3x^3 - 39x^2 - 5x - 39$ and $3x^5 + x^4 - 63x^2 - 75x - 18$.
 31. $6x^5 - 9x^4 + 19x^3 - 12x^2 + 19x - 15$ and $4x^4 - 2x^3 + 14x^2 - 5x + 25$.
 32. $6x^5 + 26x^4 + 15x^3 - 16x^2 - 10x$ and
 $30x^5 + 136x^4 + 95x^3 - 73x^2 - 63x - 5$.
 33. $x^4 + 4x^2y^2 + 16y^4$ and $20y^5 - 10xy^4 + 17x^2y^3 - 2x^3y^2 + x^4y + x^5$.
 34. $4x^6 + 6x^4y - 7x^2y^2 + 8x^2y^3 - 29xy^4 - 36y^5$ and
 $5x^6 + 7x^4y - 9x^2y^2 + 11x^2y^3 - 38xy^4 - 40y^5$.
 35. $2x^3 - x - 3$ and $x^{10} + x^0 + x^5 + 2x^7 + 2x^4 + 2x^2 + x^2 + x + 1$.
 36. $a^5b^5 - 3a^3b^3 - 8$ and $a^5b^5 - 5a^2b^3 - 12$.
 37. $ax^3 - (a^2 - b^2)x^2 - a(a^2 + b^2)x + a^4$ and $ax^3 - 2a^2x^2 + (a^3 + b^3)x - ab^3$.
 38. $m^2x^3 + m(n-1)x^2 + (m-n)x - 1$ and
 $m^2x^3 - m(n+1)x^2 + (m+n)x - 1$.
 39. $ax^3 + b(1-a)x^2 + (c-b^2)x - bc$ and $cx^3 + (a-bc)x^2 + b(1-a)x - b^2$.
 40. $x^4 - (m+2)x^3 + (m+1)^2x^2 - m(2m+1)x + m^2$ and
 $x^4 - 2(m+1)x^3 + (m+1)(3m+1)x^2 - 2m(3m+1)x + 3m^2$.
 41. $(a^2+2a)x^3 + 2(a^2+a+1)x + a^2 - 1$ and
 $(a^2+3a)x^3 + (2a^2+5a+3)x + a^2+2a+1$.
 42. $(2a^2+3a+1)x^3 + (3a^2+2a+1)x + a^2+a-2$ and
 $(3a^2+5a+2)x^3 + (5a^2+2a-1)x + 2a^2-a-1$.

92. H. C. F. of more than two polynomials.

Let A, B, C , &c. denote the polynomials. Find the H. C. F. of any two, A and B ; let it be X . Then find the H. C. F. of X and C ; let it be Y . Now find the H. C. F. of Y and D , and so on.

Ex. Find the H. C. F. of $x^3 - 2x^2 - 19x + 20$, $x^3 + 2x^2 - 23x - 60$, and $x^4 + 7x^3 - 4x^2 - 52x + 48$. B. U 1891.

First find the H. C. F. of $x^3 - 2x^2 - 19x + 20$ and

$$x^3 + 2x^2 - 23x - 60$$

$$\begin{array}{r|l} 1 & x^3 - 2x^2 - 19x + 20 \\ -1 & x^3 + 2x^2 - 23x - 60 \\ \hline & -4x^2 + 42x + 80 \\ & 4 \overline{) -4x^2 + 4x - 80} \\ & \quad \quad \quad x^3 - x - 20 \end{array} \quad \begin{array}{l} 1 \\ \\ \\ \end{array} \quad \begin{array}{l} \\ \\ \text{H. C. F.} = x^3 - x - 20 \end{array}$$

Now find the H. C. F. of $x^3 - x - 20$ and $x^4 + 7x^3 - 4x^2 - 52x + 48$.

$$\begin{array}{r|l} 1 & x^3 - x - 20 \\ -5 & x^4 + 7x^3 - 4x^2 - 52x + 48 \\ \hline & x^3 + 4x^2 - 5x - 20 \\ & 8x^3 + 16x^2 - 52x \\ & 8x^3 - 8x^2 - 160x \\ & \quad \quad \quad 24x^2 + 108x + 48 \\ & \quad \quad \quad 24x^2 - 24x - 480 \\ & \quad \quad \quad \quad \quad \quad 132x + 528 \\ & \quad \quad \quad \quad \quad \quad \quad \quad \quad x + 4 \end{array} \quad \begin{array}{l} x^3 \\ \\ 81 \\ 24 \\ \\ \end{array} \quad \begin{array}{l} \\ \\ \text{The final H. C. F.} \\ = x + 4 \text{ Ans.} \end{array}$$

EXAMPLES 53

Find the H. C. F. of

- $8x^2 + 7x - 46$, $8x^2 - 39x + 46$ and $9x^2 - 22x + 8$
- $9x^3 - x - 2$, $3x^3 - 10x^2 - 7x - 4$ and $6x^3 + x^2 - 1$.
- $3x^3 + 10x^2 + 7x - 2$, $3x^3 + 13x^2 + 17x + 6$ and $x^3 + 9x^2 - 4x - 36$.
- $4x^4 - 9x^2 + 6x - 1$, $6x^3 - 7x^2 + 1$ and $10x^3 - 17x^2 + 8x - 1$
- $2x^3 - 5x - 39$, $2x^3 - 9x^2 + 10x - 3$ and $x^4 - 21x - 18$.
- $2x^3 + 3x^2 - 3x - 2$, $5x^3 - 14x^2 + 7x + 2$ and $3x^3 + 2x^2 - 7x + 2$.
- $x^4 - 15x^3 - 10x + 24$, $x^4 - x^3 - 19x^2 - 11x + 30$,
 $x^4 + 2x^3 - 13x^2 - 14x + 24$ and $x^4 + 5x^3 - 13x^2 - 53x + 60$
- $2x^4 + 5x^3 - 25x^2 - 92x - 72$, $3x^4 + 5x^3 - 40x^2 - 100x - 48$,
 $5x^4 - 9x^3 - 56x^2 + 48x$ and $7x^4 - 4x^3 - 87x^2 - 36x$
- $6a^3 - 23a^2b + 29ab^2 - 12b^3$, $15a^3 - 25a^2b + 34ab^2 - 24b^3$
and $10a^3 - 37a^2b + 45ab^2 - 18b^3$
- $x^6 - 15x^4y + 85x^2y^2 - 225xy^3 + 274y^4 - 120y^5$,
 $x^4 - 13x^2y + 53xy^2 - 83y^3 + 42y^4$, $x^4 - 12x^2y + 49xy^2 - 7x^2y^3 + 40y^4$
and $x^4 - 15x^2y + 65xy^2 - 105y^3 + 54y^4$.

93. Theorem. Every common factor of A and B is a factor of the sum or difference of any multiples of them.

Let C be a common factor of A and B , and let $A = pC$, and $B = qC$.

To prove that C is also a factor of $mA \pm nB$.

$$\begin{array}{l} \therefore A = pC, \quad mA = mpC; \\ \therefore B = qC, \quad nB = nqC. \end{array} \quad \left. \vphantom{\begin{array}{l} A = pC \\ B = qC \end{array}} \right\}$$

$$\therefore mA \pm nB = mpC \pm nqC = (mp \pm nq)C.$$

$\therefore C$ is also a factor of $mA \pm nB$.

Cor. 1. Any common factor of A and B is also a factor of $A + B$ and $A - B$. ($m = 1, n = 1$).

Cor. 2. Any common factor of A and B is also a factor of $A - pB$ and $A + pB$. ($m = 1, n = p$).

94. Proof of the rule for finding the H. C. F. of any two compound expressions.

Let A and B be the two expressions arranged in descending or ascending powers of some common letter; also let the highest power of that letter in A be not less than the highest power of the same in B .

Let p be the quotient and C the remainder, when A is divided by B .

We shall first show that the H. C. F. of A and B is the same as that of B and C .

$$\begin{array}{r} R) A (p \\ \quad \quad \quad \phi B \\ \quad \quad \quad \hline \quad \quad \quad C \end{array}$$

From the division work, we have $A - pB = C$,..... (1)

and therefore, by transposition, $A = pB + C$... (2)

Now, every common factor of A and B is also a factor of $A - pB$ or CCor. 2, Art. 93.

Since the same is a factor of B ,.....Hyp.

therefore, every common factor of A and B is also a common factor of B and C .

Conversely, every common factor of B and C is also a common factor of A and B ; for, every common factor of B and C is also a factor of $pB + C$ or A .

It now follows that the common factors of A and B are identically the same as those of B and C .

Therefore, the highest common factor of A (dividend) and B (divisor) is the same as that of B (divisor) and C (remainder).

Now divide B by C , and let the remainder be D .

Divide C by D , and suppose there is no remainder.

Then will D be the H. C. F. of A and B .
For, by what has already been proved,

H.C.F. of A (divd.), B (divr.) = H.C.F. of B (divr.), C (rem.);

H.C.F. of B (divd.), C (divr.) = H.C.F. of C (divr.), D (rem.);

and H. C. F. of C and $D = D$ evidently, for D divides C exactly, and no quantity higher than D will divide D .

\therefore H. C. F. of A and $B = D$.

At any stage of the process, the dividend may be multiplied by a quantity which has no factor in common with the corresponding divisor.

For, take C and B . If m and C have no common factors, those of C and B are evidently the same as those of C and mB ; therefore, the highest common factor of C and mB is also the same as that of C and B , and therefore of A and B .

For a like reason, at any stage of the work a factor may be rejected out of the divisor, if it has no factor in common with the dividend.

95. Factors of H. C. F. The H. C. F. of any two expressions contains all their common factors, and none else.

According to the last article, D is the H. C. F. of A and B . Let X be any common factor of A and B .

Then also is X a factor of $A - pB$ or C . Art. 93, Cor. 2.

Since X is a factor of B ,..... Hyp.

$\therefore X$ is a common factor of B and C ;

$\therefore X$ is a factor of $B - qC$ or D ; Art. 93, Cor. 2.
i.e., any common factor of A and B is a factor of their H.C.F.

Also, D cannot contain a factor which is not in A and B , for then it will not divide them exactly.

Hence D contains all the common factors of A and B , and none else.

96. Proof of the rule for finding the H. C. F. of more than two expressions, A, B, C, D , etc.

Let H_1 be the H. C. F. of A and B . Since H_1 contains all the common factors of A and B , and none else, the common factors of A, B and C are identically the same as those of H_1 and C . Therefore the H. C. F. of A, B and C is the same as that of H_1 and C .

Let this H. C. F. be denoted by H_2 . Then H_2 contains all the common factors of H_1 and C , and therefore of A , B and C , and none else. Therefore the common factors of H_2 and D are identically the same as those of A , B , C and D . Hence the H. C. F. of A , B , C and D is the same as that of H_2 and D , and so on.

Otherwise thus : Let H_1 be the H. C. F. of A and B , and H_2 that of H_1 and C . Then will H_2 be the H. C. F. of A , B and C . If not, let G be the H. C. F. of A , B and C .

Then G is a common factor of A and B ;

$\therefore G$ is a factor of H_1Art. 95

And G „ „ „ of C ; Hyp.

$\therefore G$ is a common factor of H_1 and C .

$\therefore G$ is a factor of H_2 , the H. C. F. of H_1 and C .

Now, H_2 is the H. C. F. of H_1 and C . Hyp.

$\therefore H_2$ is a factor of H_1 .

Again, H_1 is a factor of A and of B , being their H. C. F.

$\therefore H_2$ is also a common factor of A and B ;

also, H_2 is a factor of C , being the H. C. F. of H_1 and C .

$\therefore H_2$ is a common factor of A , B and C .

Now, it has been already shown that G is a factor of H_2 .

$\therefore G$ cannot be a common factor of A , B and C , higher than H_2 , which is also a common factor.

But this is absurd, since G is by hypothesis the highest common factor of A , B and C . $\therefore G$ is not their H. C. F.

In like manner, no expression other than H_2 is the H. C. F. Hence H_2 is the H. C. F. of A , B and C .

The proof can be extended to four or more expressions.

97. Nomenclature. An objection is urged against the use in Algebra of the term, Greatest Common Measure, which ought to be restricted to Arithmetical quantities : for, the Highest Common Factor is not always the Greatest Common Measure, as can be easily shown thus :

$$x^2 - 4x + 3 = (x-1)(x-3),$$

$$\text{and } x^2 - 8x + 15 = (x-3)(x-5) ;$$

$$\therefore \text{ the H. C. F. of } x^2 - 4x + 3 \text{ and } x^2 - 8x + 15 = x - 3.$$

Now put $x = 13$; then

$$x^2 - 4x + 3 = 169 - 4 \times 13 + 3 = 120,$$

$$\text{and } x^2 - 8x + 15 = 169 - 8 \times 13 + 15 = 80.$$

Now, 40 is the Greatest Common Measure of 120 and 80, whereas 10 is the numerical value of $x - 3$, which is the algebraical Highest Common Factor. Thus the numerical values of the

EX. 3. Find the condition that x^2+ax+b and x^2+cx+d may have a common factor in x .

The common factor must be a factor

of $(x^2+ax+b)-(x^2+cx+d)$ Art. 93, Cor. I.

i.e., of $(a-c)x+b-d$, or $(a-c)\left(x+\frac{b-d}{a-c}\right)$.

$\therefore x+\frac{b-d}{a-c}$ is the factor sought.

Again, the common factor must be a factor of

$$b(x^2+cx+d)-d(x^2+ax+b)$$

i.e., of $(b-d)x^2+(bc-ad)x$, or $(b-d)x\left(x+\frac{bc-ad}{b-d}\right)$.

Since x cannot be a common factor of the given expressions, this factor must be $x+\frac{bc-ad}{b-d}$.

Now we have seen that the same is $x+\frac{b-d}{a-c}$.

\therefore comparing the two forms, we find the required condition.

$$\frac{bc-ad}{b-d} = \frac{b-d}{a-c}$$

\therefore multiplying both sides by $(b-d)(a-c)$, $(bc-ad)(a-c) = (b-d)^2$.

EXAMPLES 54.

1. Find the common factor of the first degree in x of $x^3-15x-4$ and $x^3-12x-16$.

2. Shew that the H. C. F. of $(a+b)x^3-(3a-b)x+2(a-b)$ and $(a-b)x^3-(3a+b)x+2(a+b)$ is the same as that of x^3-3x+2 and x^3+x-2 , and thence find it.

For what value of a will the following expressions have a common factor of the form $x+a$, and what is that factor?

3. $4ax^3+x-1$ and $5ax^3+8x+1$. 4. x^3-ax+2 and x^3-1 .

5. x^3+2x-a and x^3+x-2a . 6. $x^3-7x+3a$ and $x^3-12x+8a$.

7. $a(a+2)x^2+(11a-4)x+8$

and $a^2(a+2)x^3+a(11a-4)x^2+a(a+10)x+10a-4$.

8. Find the condition that x^3+px+q and x^3+qx-p may have a common factor of the form $x+a$.

9. If x^3+ax+b and x^3+bx+a have a common factor, $a+b+1=0$.

10. Shew that x^3+qx+1 and x^3+px^2+qx+1 have a common factor of the form $x+a$, when $(p-1)^2-q(p-1)+1=0$.

CHAPTER XVI.

LOWEST COMMON MULTIPLE.

98. Definition. The **Lowest or Least Common Multiple** of two or more algebraical expressions is the expression of *lowest dimensions* which is divisible by each of them without remainder.

Abbreviation. The letters L. C. M. stand for the words *lowest common multiple*.

N.B. The term *least* here is sometimes objected to. $a(x-1)^2$ and $(x-1)^2$ have $a(x-1)^2$ for the lowest common multiple.

Now let $a=4$, and $x=3$. Then $a(x-1)^2 = 4 \times 2^2 = 16$, and $(x-1)^2 = 8$, and the least common multiple of 16 and 8 = 16; but the algebraical lowest common multiple, $a(x-1)^2 = 4 \times 2^2 = 32$. Thus the algebraical lowest common multiple and the arithmetical least common multiple do not agree. This is due to the fact that what is algebraically of lowest dimensions is not necessarily least numerically. Thus $x^2 - x + 2$ is algebraically *higher* than $x + 2$, but numerically, when $x=1$, the former, being equal to 2, is *less* than the latter, which is equal to 3.

As the term *least* generally refers to numerical value, it is urged that it belongs properly to Arithmetic and not to Algebra.

If, however, we take *least* to mean *least in dimensions*, we may retain it.

99. The lowest common multiple of simple expressions or of compound expressions whose factors are easily found out, can be readily obtained by inspection.

Rule. *The L. C. M. is the product of*

- (1) *the L. C. M. of the numerical co-efficients ;*
- (2) *the highest power in which each factor appears in the given expressions.*

Ex. 1. Find the L. C. M. of $8x^2y^3$, $-24x^3yz$ and $16xyz^4$.

The L. C. M. of 8, 24 and 16 = 48.

The highest power of $x = x^3$, that of $y = y^3$, and that of $z = z^4$.

\therefore the L. C. M. sought = $48x^3y^3z^4$.

N.B. The sign of -24 is immaterial in finding the L. C. M.

Can $48x^3yz$ or $48x^3y^3z^4$ be the L. C. M.?

Ex. 2. Find the L. C. M. of $2a^3 - 7ab - 4b^3$, $6a^2 - 7ab - 5b^3$ and $a^3 - 8a^2b + 16ab^2$.

Factorising each of the given expressions, we have

$$2a^2 - 7ab + 4b^2 = (2a + b)(a - 4b).$$

$$6a^2 - 7ab - 5b^2 = (2a + b)(3a - 5b).$$

$$a^3 - 8a^2b + 16ab^2 = a(a^2 - 8ab + 16b^2) = a(a - 4b)^2.$$

The highest power of $2a + b = 2a + b$,

" " " " $a - 4b = (a - 4b)^2$,

" " " " $3a - 5b = 3a - 5b$,

" " " " $a = a$.

$$\therefore \text{L. C. M.} = a(2a + b)(a - 4b)^2(3a - 5b).$$

EXAMPLES 55.

Find the L. C. M. of

1. x^3y^4 and x^4y^3 .
2. $x^4y^5z^2$ and $x^3y^2z^3$.
3. $9a^2b^3$ and $6a^3b^4$.
4. $6a^2b^2c^2$ and $8a^3bc^4$.
5. $4a^2b^2c^2$ and $6a^2b^3d^2$.
6. $4a^2b^3$ and $10a^2b^2x^3y^3$.
7. $25a^4b^4c^4$ and $30b^3c^3x^3$.
8. $48p^3q^3r^4$ and $60p^2q^2r$.
9. $\frac{1}{2}x^3y^2z^3$ and $10y^4z^4$.
10. $\frac{1}{3}x^5y^5$ and $24y^6z^7$.
11. xy^3 , x^2yz and xyz^2 .
12. $3a^3b^2$, $5ab^3$ and $15abc$.
13. $7a^2bx^2$, $21a^2x^5$, $35a^4xy^3$ and $42abcx^2y^2$.
14. $4ay$, $3bxy$, $15b^2x^2y^3$, $a^2bx^4y^4$ and $24a^3x^5y^5$.
15. $4x^2y^3$, $13x^3y^3$, $39a^2x^2y$, $52a^2x^3y^3$ and $26ax^2y^3$.
16. $25a^2b^2c$, $30b^3c^2d$, $48c^2d^2a$, $32a^2a^2c$ and $(6abcd)^2$.
17. a^2ln , $9ab^3n^2$, $12b^2m^2$, $24a^2b^2l^2$, $32c^2a^2m^2n^2$ and $(l^2m^2n^2)^3$.
18. $\frac{1}{4}(abc)^3$, $\frac{1}{2}a^2b^3c$, $\frac{3}{8}a^2b^3c^3$, $\frac{3}{2}a^3b$ and $\frac{1}{12}a^2b^3c$.
19. $\frac{1}{2}(xyz)^4$, $\frac{1}{4}xy^2z^3$, $\frac{1}{3}x^2yz^2$, $\frac{1}{12}x^3y$ and $\frac{1}{2}x^2y^2z$.
20. $\frac{3}{2}(a^2b)^3$, $\frac{5}{8}(ab^3)^3$, $(\frac{1}{2}a^2b^3c^3)^3$, $(\frac{1}{3}a^2bc^2d)^3$ and $(abc)^6$.
21. $a^2 - b^2$ and $ab - b^2$.
22. $a^2 - b^2$ and $(a + b)^2$.
23. $a^2 - ab$ and $b^2 - ab$.
24. $ax(a - x)$ and a^2x^2 .
25. $x^7 + x^6$ and $x^{11} - x^9$.
26. $x^3 - y^3$ and $xy(x^2 + xy + y^2)$.
27. $a^2 - b^2$, $a^2 + b^3$ and $a^2 + ab + b^2$.
28. $9a^2 - 1$, $6a^2 + 2a$, $3a^4 - a^2$ and $12a^5 - 4a^4$.
29. $12ab(ab - b^2)^2$, $24ab(a^2b - ab^2)$ and $36ab(a^2 - a^3b)$.
30. $a(1 + x + x^2)$, $6a^2 - 6a^2x^2$ and $2a - 2ax^2$.
31. $7a^4x^2(a - x)^2$, $21a^2x^2(a^2 - x^2)^2$ and $12a^2x^4(a + x)^2$.
32. $4(a - b)^2(p - q)^3$, $(a + b)^2(p^2 - q^2)^4$ and $6(p + q)^4(q + r)$.

33. $12x^2y^2z^2(a^3-b^3)$, $32x^4yz^2(a^3+b^3)$ and $16x^2y^2z^4(a^3-b^3)$.
 34. $25(a^3+b^3)(a^2+b^2)$, $30ab(a^3-b^3)$ and $45b^2(a^3-b^3)$.
 35. $x^2y^3-ax^2y^2$, $2x^2(a-x)^2$, $7xy^4(a-x)^2$ and $24(x^3-ax^2)$.
 36. $2x^3-6x^2$, $3x^3-27x$, x^4-27x and x^3-3x+9 .
 37. a^3-4a+3 and a^3-5a+6 38. a^2+4 and a^3-4a+4 .
 39. a^3-ab^2 and a^2b-b^3 . 40. a^3-1 and $2a^2+a-1$.
 41. $8x^3-6x+1$ and $8x^3+2x-1$.
 42. $8x^3+7xy-46y^2$ and $8x^3-39xy+40y^2$.
 43. $25x^3+5x-6$ and $25x^3+101-8$ 44. a^4-b^4 and $a^4-2a^2b^2+b^4$.
 45. $2a(a^2x^3-b^2y^3)$, $3b(a^2x^3+b^2y^3)$, and $ab(a^2x^3+abxy+b^2y^3)$.
 46. $\sqrt{x^3-5x+6}$, $x^3-7x+12$ and x^3-5x+4 .
 47. $\sqrt{a^3-3ab-10b^2}$, $a^3+2ab-35b^2$ and $a^2-8ab+15b^2$.
 48. $\sqrt{12a^3-5a-2}$, $12a^3-2a-1$ and $9a^3-9a+2$.
 49. $\sqrt{32a^7x^4-32a^6x^6+8a^5x^8}$, $48a^7x^7-48a^6x^8$ and $12a^5x^6$.
 50. $4a^3-17a^2b+4ab^2$, $ab^3-8a^2b^2+16a^3b$ and $4a^2b-9ab^2+2b^3$.
 51. $12a^2b(a^3+b^3)$, $24ab^2(a^3-b^3)$ and $30a^2b^2(a^4+a^2b^2+b^4)$.
 52. $a^2b-b(b-c)^2$, $ac^2-a(a+b)^2$ and $(a-c)^2c-b^2c$.
 53. $9x^3-4y^3$, $4x^3-36y^3$, $3x^2-71y-01$ and $6y^2-7xy-3x^2$.
 54. $25(a^2b^7-5ab^8+6b^9)$, $45(a^2b^3-4a^3b^4+3ab^5)$, $72(a^3b^6-3a^2b^7+2ab^8)$.
 55. $12xy^2(a^3-3ax^2+2x^3)$, $60x^2y(a^2+a1-2x^3)$ and $25x^2(x^3-a^3)^2$.
 56. $(x^3-2x)(x^3+x-2)$, $(x^3-4)(x^3-1)$, $(x^3+x)(x^3+3x+2)$
 and $(x^3-3x+2)(x^3+3x+2)$.

100. L. C. M. of two expressions not readily factorised.

Let H denote the H. C. F. of two expressions A and B .

Let $A=aH$, and $B=bH$.

Since H is the H. C. F. of aH and bH , a and b have no common factor.

\therefore the L. C. M. of A and B obviously, by Rule, Art. 99,
 $=abH=abH^2 \div H=aH \times bH \div H$.

But $aH=A$, and $bH=B$, by supposition ;

\therefore the L. C. M. of A and $B=A \times B \div H$.

Hence the rule : Divide the product of the two expressions by their H. C. F. ; or, which is the same thing, divide either of

Next find the L. C. M. of this expression and $1+2a-8a^2-16a^4$.

Their H. C. F. = $1+4a+8a^2+8a^3$,

$$\text{and } (1+2a-8a^2-16a^4) \div (1+4a+8a^2+8a^3) = 1-2a.$$

$$\therefore \text{ the L. C. M. sought} = (1+6a+12a^2+8a^3-16a^4-32a^5)(1-2a) \\ = 1+4a-16a^2-32a^4+64a^5. \text{ Ans.}$$

The L. C. M. in the present case may, however, be easily found out by factorisation.

$$\begin{aligned} 1+4a+8a^2+8a^3 &= (1+8a^3) + (4a+8a^2), \text{ re-arranging terms,} \\ &= \{1+(2a)^3\} + 4a(1+2a) \\ &= (1+2a)(1-2a+4a^2) + 4a(1+2a) \\ &= (1+2a)(1-2a+4a^2+4a) \\ &= (1+2a)(1+2a+4a^2) \\ 1+4a+4a^2-16a^4 &= (1+2a)^2 - (4a^2)^2 \\ &= (1+2a+4a^2)(1+2a-4a^2) \dots\dots\dots (A). \\ 1+2a-8a^2-16a^4 &= (1+2a-8a^2)(1+2a) \\ &= (1+2a)\{1-(2a)^3\} \\ &= (1+2a)(1-2a)(1+2a+4a^2). \\ \therefore \text{ L. C. M. reqd} &= (1+2a)(1-2a)(1+2a+4a^2)(1+2a-4a^2) \\ &= (1-4a^2)(1+4a+4a^2-16a^4) \text{ See (A)} \\ &= 1+4a-16a^2-32a^4+64a^5. \end{aligned}$$

EXAMPLES 56.

Find the L. C. M. of

1. $x^3-16x+24$ and $2x^3-5x^2+4$.
2. $x^3+4x^2+10x+7$ and $x^3+5x^2+13x+14$.
3. $x^3+2x^2y-xy^2-2y^3$ and $x^3-2x^2y-xy^2+2y^3$.
4. $x^4-10x^3+35x^2-50x+24$ and $x^4-2x^3-13x^2+38x-24$.
5. $x^4+4x^3-x^2-16x-12$ and $x^4-x^3-11x^2+9x+18$.
6. $6x^4+4x^3y+12x^2y^2+11xy^3+2y^4$, $4x^4+6x^3y+8x^2y^2+14xy^3+3y^4$.
7. $2a^5+3a^4+4a^3+a^2-1$ and $3a^4+8a^3+3a^2-5$.
8. $x^3-6x^2+11x-6$, $x^3-9x^2+26x-24$ and $x^3-8x^2+19x-12$.
9. $x^3-2x^2-3x+10$, $x^3-6x^2+13x-10$ and $2x^3-7x^2+6x+5$.
10. $a^4-4a^3-a^2+16a-12$, $a^4-7a^3+11a^2+7a-12$,
 $a^4-6a^3-a^2+54a-72$ and a^3+4a^2+a-6 .

103. Theorem. Every common multiple of two or more expressions is a multiple of their L. C. M.

Since every common multiple of the given expressions is exactly divisible by each of them, any such multiple must contain, besides others, all the different factors occurring in them, each raised to the highest power it has in any of them.

Now, these latter factors, and none else, make up the L. C. M.

Therefore any common multiple contains the L. C. M., and is hence a multiple of it.

Otherwise :—

Let X be a common multiple of the given expressions, and L their L. C. M.

Then will X be a multiple of L , i.e., exactly divisible by it.

If not, let q be the quotient, and R the remainder, when X is divided by L , so that R is lower than L .
$$L \nmid X \begin{matrix} (q \\ \frac{qL}{R} \end{matrix}$$

Then $X = qL + R$, and $R = X - qL$.

Since both X and L are divisible without remainder by each of the given expressions, Hyp.

so also is $X - qL$, i.e., R . Art. 93.

\therefore the given expressions have a common multiple R , which is of lower dimensions than L .

But this is impossible, since L is the lowest common multiple. Hyp.

$\therefore X$ is a multiple of L . Hence the theorem.

CHAPTER XVII.

FRACTIONS.

104. A fraction *literally* means a broken part. The fraction $\frac{3}{5}$ of a certain quantity means that that quantity is to be divided into 5 equal parts, and that 3 of such parts are to be taken.

In this view, the fraction $\frac{a}{b}$ of x , where a and b are positive integers, means that x is to be divided into b equal parts, and a of such parts are to be taken. When x is taken as the unit, we simply speak of the fraction $\frac{a}{b}$.

But even in Arithmetic we admit fractions such as $\frac{1}{2}$, $\frac{1}{3}$, &c., which are evidently not covered by the primary sense of fraction.

Algebra goes a step further ; for in the algebraical fraction $\frac{a}{b}$, a and b are not restricted to positive quantities, but may be positive or negative, integral or fractional. Thus we may have fractions of the form $\frac{-2}{-3}$, $\frac{4}{-10}$, &c. We evidently fail to interpret $\frac{-2}{-3}$ by the primary idea of fraction, for it is altogether without sense to say that the unit is to be divided into -3 equal parts, and -2 such parts are to be taken. A modification of the definition of fraction is therefore required.

Let us return to the fraction $\frac{3}{5}$. We know that the unit is divided into 5 equal parts, and that 3 of those parts are taken. To multiply it by 5 we must take each of the 3 parts 5 times. We thus get 15 parts, every five of which makes up the unit. Hence the fifteen parts are equal to 3 units. Therefore $\frac{3}{5} \times 5 = 3$. Hence when a and b are both positive integers, $\frac{a}{b} \times b = a$.

This last result is taken as the basis of our definition of fraction in Algebra. Removing all restrictions imposed upon a and b , we assume $\frac{a}{b}$ to mean a quantity such that

$$\frac{a}{b} \times b = a ;$$

or, in other words, $\frac{a}{b} = a \div b$, by the definition of division.

Hence we start with any of the following definitions of fraction, which are equivalent to one another :

Def. I. The algebraical fraction $\frac{a}{b}$, where a and b may be positive or negative, integral or fractional, is that quantity which, when multiplied by b , gives a product equal to a .

Def. II. The algebraical fraction $\frac{a}{b}$ is the quotient obtained by dividing a by b .

$$\begin{aligned} \text{Thus } \frac{-2}{-3} &= (-2) \div (-3), \text{ by the last definition,} \\ &= 2 \div 3, \text{ by the rule of signs,} \\ &= \frac{2}{3}. \end{aligned}$$

106. We proceed to prove some important theorems, which are of constant application in the present Chapter.

Theorem I. *The value of a fraction is not altered by multiplying both its numerator and denominator by any the same quantity.*

Required to prove that $\frac{a}{b} = \frac{am}{bm}$.

By definition, $\frac{a}{b} \times b = a$.

Therefore, denoting $\frac{a}{b}$ by x , we have

$$x \times b = a;$$

multiplying both sides by m , we have

$$xb \times m = am.$$

But $xb \times m = x \times bm$; Art. 40.

$$\therefore x \times bm = am.$$

\therefore by definition of division, $x = am \div bm = \frac{am}{bm}$;

$$\text{that is, } \frac{a}{b} = \frac{am}{bm}.$$

Theorem II. *The value of a fraction is not altered by dividing both its numerator and denominator by any the same quantity.*

Required to prove that $\frac{a}{b} = \frac{a \div m}{b \div m}$.

Let $\frac{a}{b} = x$, $a \div m = y$, and $b \div m = z$.

By definition of fraction, $\frac{a}{b} \times b = a$;

$$\therefore xb = a;$$

dividing both sides by m , $xb \div m = a \div m$.

$$\therefore x \times (b \div m) = a \div m.$$

$$\therefore x \times z = y$$

\therefore by definition of division, $x = \frac{y}{z}$;

$$\text{that is, } \frac{a}{b} = \frac{a \div m}{b \div m}.$$

N. B. The above theorem easily follows from the preceding one.

Let $am = A$, and $bm = B$; then $a = A \div m$, and $b = B \div m$. Then since

by theorem I, $\frac{am}{bm} = \frac{a}{b}$, we get $\frac{A}{B} = \frac{A \div m}{B \div m}$, which is theorem II.

Theorem III. *The value of a fraction is not altered by changing the sign of both its numerator and denominator.*

Required to prove that $\frac{a}{b} = \frac{-a}{-b}$, and $\frac{-a}{+b} = \frac{+a}{-b}$.

Take the fraction $\frac{a}{b}$; multiply both the numerator and denominator by -1 , then, since by theorem I the value of the fraction is unaltered, we have

$$\frac{a}{b} = \frac{a \times (-1)}{b \times (-1)} = \frac{-a}{-b}.$$

Similarly, $\frac{-a}{b} = \frac{-a \times (-1)}{b \times (-1)} = \frac{a}{-b}.$

N. B. It follows easily from this theorem that

$$\frac{a-b+c}{d+e-f} = \frac{-a+b-c}{-d-e+f} = \frac{b-c-a}{f-d-e}.$$

Theorem IV *The sign of a fraction is changed by changing the sign of either the numerator or the denominator only.*

Required to prove that

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

By definition, $\frac{-a}{b} = (-a) \div b$
 $= -(a \div b)$, by rule of signs,
 $= -\frac{a}{b}$, by definition of fraction.

Also, $\frac{a}{-b} = a \div (-b)$
 $= -(a \div b)$, by rule of signs,
 $= -\frac{a}{b}$, by definition

N. B. This proposition should be fully grasped by the student, as it is frequently required in the addition and subtraction of a certain class of fractions. It is now easy to see that

$$\frac{a-b}{c-d} = -\frac{b-a}{c-d} = -\frac{a-b}{d-c}.$$

106. Definition The numerator and denominator of a fraction are its **terms**, and when they have no factor common to both of them, the fraction is said to be in its **lowest terms**.

Reduction. To reduce a fraction to its lowest terms, we have evidently the following rule :

Divide the numerator and denominator by every factor common to them both, or, which is the same thing, by their highest common factor.

Ex. 1. Reduce to their lowest terms $\frac{6a^4b^3c^5}{9a^3b^3c^7}$ and $\frac{-36xy^3z^3}{72x^2y^2z^5}$.

The H. C. F. of $6a^4b^3c^5$ and $9a^3b^3c^7 = 3a^3b^3c^5$.

$$\begin{aligned}\therefore \frac{6a^4b^3c^5}{9a^3b^3c^7} &= \frac{6a^4b^3c^5 \div 3a^3b^3c^5}{9a^3b^3c^7 \div 3a^3b^3c^5}, \text{ Art. 105, Th. II.} \\ &= \frac{a}{3bc^2}. \text{ Ans.}\end{aligned}$$

N. B. The more common practice is to cancel out successively the common factors of the numerator and denominator. Thus,

$$\begin{aligned}\frac{6a^4b^3c^5}{9a^3b^3c^7} &= \frac{3 \times 2 \times a^4 \times a \times b^3c^5}{3 \times 3 \times a^3 \times b^3c^7} \\ &= \frac{2a^4b^3c^5}{3b^3c^7}, \text{ cancelling 3 and } a^3, \\ &= \frac{2ab^3c^5}{3b^3 \times b^0c^7} \\ &= \frac{2ac^5}{3bc^7}, \text{ cancelling } b^3, \\ &= \frac{2ac^5}{3bc^2 \times c^5} \\ &= \frac{2a}{3bc^2}, \text{ cancelling } c^5.\end{aligned}$$

More shortly,

$$\frac{6a^4b^3c^5}{9a^3b^3c^7} = \frac{3 \times 2 \times a^{4-3}}{3 \times 3 \times b^{3-3}c^{7-5}} = \frac{2a}{3bc^2}.$$

$$\begin{aligned}\text{Also } \frac{-36xy^3z^3}{72x^2y^2z^5} &= \frac{-36y^3z^3}{36 \times 2 \times x^{2-1}y^{3-2}z^{5-3}} \\ &= \frac{-y^3}{2xz^2} \\ &= -\frac{y^3}{2xz^2}, \text{ Art. 105, Th. IV.}\end{aligned}$$

Ex. 2. Reduce to its simplest form $\frac{27a+a^3}{18a-6a^3+2a^3}$

$$\begin{aligned} 27a+a^3 &= a(27+a^2) \\ &= a(3^3+a^2) \\ &= a(3+a)(3^2-3a+a^2) \\ &= a(3+a)(9-3a+a^2) \end{aligned}$$

$$18a-6a^3+2a^3 = 2a(9-3a+a^2)$$

$$\therefore \text{ the given fraction} = \frac{a(3+a)(9-3a+a^2)}{2a(9-3a+a^2)}$$

$$= \frac{3+a}{2}. \quad \text{Ans.}$$

N. B. In the case of more than one common factor of the numerator and denominator strike out those factors until no such factor is left.

EXAMPLES 57.

Reduce to their lowest terms :

1. $\frac{axy^2}{bx^2y}$ 2. $\frac{abx^2y^3z^2}{bcx^2yz^4}$ 3. $\frac{6a^2b^3c^3r^2}{8a^2b^3c^3x^2}$ 4. $\frac{15a^2b^2}{20ax^2y}$

5. $\frac{72b^3q^2r^4}{108b^3q^2r}$ 6. $\frac{66a^3b^3cx^4}{59ab^6xy^3z^2}$ 7. $\frac{a^2-b^2}{(a+b)^2}$

8. $\frac{1+x+x^2}{1-x^2}$ 9. $\frac{a^2-25}{6a^3+30a}$ 10. $\frac{a^2b^2}{a^2b+ab^2}$

11. $\frac{bm+m^2}{mc+mb}$ 12. $\frac{4bx+4by}{4(x^2-y^2)}$ 13. $\frac{x^4z^3-y^4z^3}{x^6-x^4y^2}$

14. $\frac{a^2+a^2b^2c^2}{a^3-a^2c^2}$ 15. $\frac{14x^2-7xy}{10xz-5yz}$ 16. $\frac{9a^2-4x^2}{9a^2-6ax}$

17. $\frac{10a^2-80b^2}{(a-2b)^2}$ 18. $\frac{(x^2-y^2)^2}{(x^3-y^3)^2}$ 19. $\frac{x^2+7x+10}{x^2+5x+6}$

20. $\frac{2x^2+5x+2}{2x^2+3x-2}$ 21. $\frac{2x^2+x-3}{2x^2+11x+12}$ 22. $\frac{4+12x+9x^2}{2+13x+15x^2}$

23. $\frac{a^3+4ab-77b^2}{a^3-4ab-165b^2}$ 24. $\frac{x^2+9ax-96a^2}{x^2+23ax+132a^2}$

25. $\frac{2x^2+15x-8}{2x^2+23x-12}$ 26. $\frac{11x^2-ax-12a^2}{11x^2-23ax+12a^2}$

27. $\frac{2x^3+14x^2-3^4x}{3x^3-15x^2+18x}$ 28. $\frac{a^3+b^3-c^3+2ab}{a^2-b^2-c^2-2bc}$

$$29. \frac{a^2 - 2a^2b + ab^2}{a^3 - 3a^2b + 3ab^2 - b^3}$$

$$31. \frac{x^3 - (b+c)x + bc}{x^3 - (c+a)x + ca}$$

$$33. \frac{ac - bd - bc + ad}{ac + bd - bc - ad}$$

$$35. \frac{mnx^2 - (m^2 - n^2)x - mn}{mnx^2 + (m^2 + n^2)x + mn}$$

$$37. \frac{8a^3b - b^4}{16a^4 + 4a^2b^2 + b^4}$$

$$39. \frac{a^2x^2 + b^2y^2 - c^2z^2 + 2abx}{a^2x^2 - b^2y^2 - c^2z^2 + 2bcy}$$

$$30. \frac{a(b-c) + b(c-a)}{b(b-c) + a(c-a)}$$

$$32. \frac{x^2 + (c-a)x - ca}{x^2 - (a+b)x + ab}$$

$$34. \frac{12a^2x^3 + 33ax^2 - 9x}{12a^3x^3 + 32ax^2 - 12x}$$

$$36. \frac{a^4 + a^2b^2 + b^4}{a^4 - a^2b + ab^3 - b^4}$$

$$38. \frac{a^4 - 2a^2 + a^2 - 1}{a^4 + a^2 + 1}$$

$$40. \frac{bx^2 - (b^2 - c^2)x - bc^2}{bx^2 - (b^2 + c^2)x + bc^2}$$

107. When the factors of the numerator and denominator are hard to find by inspection, the work of reduction depends upon finding the H. C. F. by the method of Art 91.

Ex. 1. Reduce to its lowest terms $\frac{7 - 15x + x^2 + 2x^3}{2x^3 - 3x^2 + 15x - 7}$

The H. C. F. of the numr. and denr. = $2x - 1$.

$$\therefore \text{the reqd. fraction} = \frac{(7 - 15x + x^2 + 2x^3) \div (2x - 1)}{(2x^3 - 3x^2 + 15x - 7) \div (2x - 1)}$$

$$= \frac{x^2 + x - 7}{x^2 - x + 7} \quad \text{Ans}$$

N. B. The work of finding the H. C. F. may be shortened here. The H. C. F. of the numerator and denominator is a factor of their sum, Art. 93. The latter is $4 - 2x^2$ or $2x^2(2x - 1)$. Since 2 or x will not divide the numerator and denominator, $2x - 1$ is the only factor which may be common to them both. The rest of the work is best given thus:

$$\frac{7 - 15x + x^2 + 2x^3}{2x^3 - 3x^2 + 15x - 7} = \frac{x^2(2x - 1) + x(2x - 1) - 7(2x - 1)}{x^2(2x - 1) - x(2x - 1) + 7(2x - 1)} = \frac{x^2 + x - 7}{x^2 - x + 7}$$

Ex. 2. Simplify $\frac{a^3 - 7ab^2 + 6b^3}{a^3 - 4a^2b - 9ab^2 + 3b^3}$

The H. C. F. of the numerator and denominator = $a + 3b$.

$$\therefore \text{the given fraction} = \frac{(a + 3b)a^2 - 3ab(a + 3b) + 2b^2(a + 3b)}{(a + 3b)a^2 - 7ab(a + 3b) + 12b^2(a + 3b)}$$

$$= \frac{a^2 - 3ab + 2b^2}{a^2 - 7ab + 12b^2}, \text{ cancelling } a + 3b.$$

N. B. When either the numerator or the denominator can be readily factorised, as in the present case, we may proceed thus :

$$\begin{aligned} a^3 - 7ab^2 + 6b^3 &= a^3 - ab^2 - 6ab^2 + 6b^3 \\ &= a(a^2 - b^2) - 6b^2(a - b) \\ f &= (a - b)\{a(a + b) - 6b^2\} \\ &= (a - b)(a^2 + ab - 6b^2) \\ &= (a - b)(a - 2b)(a + 3b) \end{aligned}$$

Upon examination $a + 3b$ will be found to be the only factor which belongs also to the denominator. The subsequent work is as given above. If some of the other factors be found common to the two, they should be successively struck out.

EXAMPLES 58

Reduce to their lowest terms.

1. $\frac{x^3 - 21x - 36}{x^3 + x^2 - 2x - 15}$
2. $\frac{2 - 3x + 5x^2 - 2x^3}{2 - 5x + 8x^2 - 3x^3}$
3. $\frac{x^3 + 2x^2 - 4x - 8}{x^3 - 8}$
4. $\frac{x^3 - 4x + 3}{x^3 - 4x^2 + 3}$
5. $\frac{x^3 - 13x - 12}{x^3 + x^2 - 5x + 3}$
6. $\frac{2a^4 - a^2b^2 - 3b^4}{3a^3 - 2a^2b + 3ab^2 - 2b^3}$
7. $\frac{4a^3 + 3a^2b + b^3}{3a^3 + 5a^2b + ab^2 - b^3}$
8. $\frac{2a^3 - 5a^2 - 2a - 3}{5a^3 - 15a^2 + 3a - 9}$
9. $\frac{6x^3 - x^2 - 10x - 3}{12x^3 + 7x^2 - 8x - 3}$
10. $\frac{3a^4 - 8a^3 + 8a}{4a^4 - 19a^3 + 6a}$
11. $\frac{8a^4 + 10a^3 - 18a}{1 - a^4 - 14a^2 + 2a}$
12. $\frac{6a^3 - 7a^2b + b^3}{4a^3 - 9a^2b + 6ab^2 - b^3}$
13. $\frac{2a^4 - 19a^3b^2 + 3a^5}{3a^4 - 10a^3b + 9ab^3}$
14. $\frac{x^4y - 4xy^4 + 3y^5}{x^5 - 4x^4y + 27xy^4}$
15. $\frac{4l^4m^3 - 35l^3m^4 + 104l^2m^5 - 9m^6}{8l^6m^3 - 48l^4m^5 + 85l^3m^7 - 48l^2m^5}$
16. $\frac{9a^3 - a^2b - 8b^3}{3a^3 - 10ab^2 - 7a^2b + 14b^3}$
17. $\frac{a^4 - ma^3 + (n-1)a^2 + ma - n}{a^4 - na^3 + (m-1)a^2 + na - m}$
18. $\frac{a^2b - a^2b - 10ab - 8b}{a^2c + 6a^2c + 11ac + 6c}$
19. $\frac{a^3 - (m-2)ab^2 + (2m-1)ab^2 + 7b^3}{a^3 - (m+1)a^2b + (m+1)ab^2 - b^3}$
20. $\frac{p^3 + 21q - 12pq^2 - 9q^3}{p^3 + p^2q - 14pq^2 - 12q^3}$

108. Sometimes we have to express a single fraction as a group of fractions. Since a fraction represents the quotient of the numerator by the denominator, we have to take the sum of the quotients with their proper signs of the several terms of the numerator by the denominator.

Ex. 1. Express $\frac{8a^3b^3 - 20a^2b^4 + 36ab^5}{4a^4b^3}$ as a group of simple fractions in lowest terms.

$$\begin{aligned}\frac{8a^3b^3 - 20a^2b^4 + 36ab^5}{4a^4b^3} &= \frac{8a^3b^3}{4a^4b^3} - \frac{20a^2b^4}{4a^4b^3} + \frac{36ab^5}{4a^4b^3}, \text{ Art. 52,} \\ &= \frac{2b}{a} - \frac{5b^4}{a^2} + \frac{9b^2}{a^3}, \text{ reducing.}\end{aligned}$$

109. In solving a certain class of equations, we have to express a fraction in a form which is partly integral and partly fractional. The process will be best illustrated by the following examples

Ex 1. Shew that $\frac{3x-4}{x+7} = 3 - \frac{25}{x+7}$

$$\begin{aligned}\frac{3x-4}{x+7} &= \frac{3(x+7) - 21 - 4}{x+7} \\ &= \frac{3(x+7) - 25}{x+7} \\ &= \frac{3(x+7)}{x+7} - \frac{25}{x+7} \\ &= 3 - \frac{25}{x+7} \quad \text{Ans}\end{aligned}$$

N. B Sometimes actual division will have to be performed.

Ex 2 Shew that $\frac{6x^3 + 11x^2 + 7x + 6}{2x^2 + 5x + 3} = 3x - 2 + \frac{4}{x+1}$.

$$\begin{aligned}2x^2 + 5x + 3 \overline{) 6x^3 + 11x^2 + 7x + 6} &\quad (3x - 2 \\ &\underline{6x^3 + 15x^2 + 9x} \\ &\quad -4x^2 - 2x + 6 \\ &\quad \underline{-4x^2 - 10x - 6} \\ &\quad \quad 8x + 12 \\ \therefore \frac{6x^3 + 11x^2 + 7x + 6}{2x^2 + 5x + 3} &= 3x - 2 + \frac{8x + 12}{2x^2 + 5x + 3} \\ &= 3x - 2 + \frac{4(2x + 3)}{(2x + 3)(x + 1)} \\ &= 3x - 2 + \frac{4}{x + 1}. \quad \text{Ans.}\end{aligned}$$

EXAMPLES 59.

Reduce to a group of simple fractions

1. $\frac{a^2 + b^2}{ab}$

2. $\frac{bc + ca + ab}{abc}$

3. $\frac{x^2 + y^2 - 3xy}{xy^2}$

4. $\frac{a^2b + b^2a}{abcd}$

5. $\frac{a^2 + b^2 - ab}{abc}$

6. $\frac{l^4 + m^4 + n^4}{l^2m^2n^2}$

7. $\frac{x^3 - 4x^2y + xy^2 - y^3}{2xy}$

8. $\frac{4a^2b + 9ab^2 - 3b^3}{9ab^2}$

9. $\frac{4x^2y - 3x^2y^2 + (xy)^2}{12xyz}$

10. $\frac{4p^2qr - 3pq^2r + 2p^2q^2r^2 - 12}{6p^2q^2r^2}$

Shew that

11. $\frac{x}{x+3} = 1 - \frac{3}{x+3}$

12. $\frac{4x-3}{2x+1} = 2 - \frac{5}{2x+1}$

13. $\frac{ax+b}{x+c} = a + \frac{b-ac}{x+c}$

14. $\frac{a^3+b^3}{(a+b)^3} = a - 2b + \frac{3b^2}{a+b}$

15. $\frac{64x^3 - 292x^2 + 400x - 159}{4x^3 - 12x + 9} = 16x - 25 - \frac{22}{2x-3}$

110. **Reduction to lowest common denominator.** In finding the algebraical sum of a number of fractions, it is necessary, as in Arithmetic, to reduce them to a common denominator. As it is most convenient to make this common denominator the lowest possible, we have the following rule :

Take the L. C. M. of the given denominators for the common denominator; divide it by the denominator of each fraction, multiply the quotient by the corresponding numerator, and take the final product as the new numerator.

Proof. Let the given fractions, reduced to their lowest terms, be $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$. Let L denote the L. C. M. of b , d and f , and let $L/b = x$, so that $L = bx$.

By Art. 105, $\frac{a}{b} = \frac{ax}{bx} = \frac{a \times (L/b)}{L}$, and so on.

N. B. In applying the above rule, it is generally convenient to have the given fractions first reduced to their lowest terms.

Ex. 1. Reduce to the lowest common denominator

$$\frac{2}{ax}, \frac{3x}{x^2y} \text{ and } \frac{5b}{ay^2}.$$

L. C. M. of denominators = ax^2y^2 .

$$\frac{2}{ax} = \frac{2 \times (ax^2y^2 \div ax)}{ax^2y^2} = \frac{2xy^2}{ax^2y^2};$$

$$\frac{3z}{x^2y} = \frac{3z(ax^2y^2 \div x^2y)}{ax^2y^2} = \frac{3ayz}{ax^2y^2};$$

and $\frac{5b}{ay^2} = \frac{5b \times (ax^2y^2 \div ay^2)}{ax^2y^2} = \frac{5bx^2}{ax^2y^2}$

N. B. It is easily seen that the above work with the first fraction is equivalent to $\frac{2}{ax} = \frac{2 \times 1y^2}{ax \times 1y^2} = \frac{21y^2}{ax^2y^2}$, and so on. The beginner is advised to verify each case in this way.

Ex. 2. Reduce $\frac{a-b}{ab-b^2}$, $\frac{b}{a^2+ab}$ and $\frac{a^2}{a^3+b^3}$ to the lowest common denominator.

$$\frac{a-b}{ab-b^2} = \frac{a-b}{b(a-b)} = \frac{1}{b}, \text{ simplifying.}$$

Therefore the given fractions are equivalent to

$$\frac{1}{b}, \frac{b}{a(a+b)} \text{ and } \frac{a^2}{a^3+b^3}.$$

L. C. M. of denominators = $ab(a^3+b^3)$.

$$\frac{1}{b} = \frac{1 \times \{ab(a^3+b^3) \div b\}}{ab(a^3+b^3)} = \frac{a(a^3+b^3)}{ab(a^3+b^3)}.$$

$$\frac{b}{a(a+b)} = \frac{b \times \{ab(a^3+b^3) \div a(a+b)\}}{ab(a^3+b^3)} = \frac{b^2(a^3-ab+b^3)}{ab(a^3+b^3)}.$$

$$\frac{a^2}{a^3+b^3} = \frac{a^2 \times \{ab(a^3+b^3) \div (a^3+b^3)\}}{ab(a^3+b^3)} = \frac{a^3b}{ab(a^3+b^3)}.$$

EXAMPLES 60.

Express with lowest common denominator

1. $\frac{x}{6}, \frac{2a}{3}, \frac{5b}{12}.$

2. $\frac{4x}{5}, \frac{9x}{20}, \frac{z}{30}.$

3. $\frac{x^2}{yz}, \frac{y^2}{xz}, \frac{z^2}{xy}.$

4. $\frac{a}{bi}, \frac{b}{ca}, \frac{c}{ab}, \frac{a^2}{abc}.$

5. $\frac{a-x}{a+x}, \frac{a+x}{a-x}.$

6. $\frac{x-y}{4}, \frac{y-z}{10}, \frac{z-x}{20}.$

7. $\frac{1+a}{5a}, \frac{2-a}{5b}, \frac{3}{18c}.$

8. $\frac{a}{a+b}, \frac{b}{a-b}, \frac{ab}{a^2-b^2}.$

9. $\frac{2}{x-a}, \frac{3}{2a-x}, \frac{4}{x^2-4a^2}$ 10. $\frac{x}{2y}, \frac{y}{x+y}, \frac{z}{xy+y^2}$
11. $\frac{a}{a^2-ab-2b^2}, \frac{b}{b^2-4ab}$ 12. $\frac{x-y}{a^2(x+y)}, \frac{x+y}{b^2(x-y)}, \frac{x^2+y^2}{a^2b^2(x^2-y^2)}$
13. $\frac{a^2-ab}{a^2b}, \frac{b-c}{bc}, \frac{2c^2-ac}{ac^2}$ 14. $\frac{2x}{(x-a)^2}, \frac{a}{x^2-a^2}, \frac{a(x+a)}{(x-a)^3}$
15. $\frac{a^2}{(a-b)(a-c)}, \frac{b^2}{(b-c)(b-a)}, \frac{c^2}{(c-a)(c-b)}$
16. $\frac{5x^2}{x^2-5xy+5y^2}, \frac{4y^2}{x^2-xy-6y^2}, \frac{3x}{x^2+2xy}, \frac{z^2}{x^2-4y^2}$

CHAPTER XVIII.

ADDITION AND SUBTRACTION OF FRACTIONS.

111. Rule. Reduce all the fractions to the lowest common denominator; take the algebraical sum of the numerators as the new numerator, and retain the common denominator. Reduce the result to the lowest terms, when possible.

$$\text{Thus } \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$$

$$\text{and } \frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad-bc}{bd}$$

N. B. If any of the given fractions be not in its lowest terms, it is convenient to first simplify it.

Ex. 1. Find the value of $\frac{a}{5} + \frac{a}{6}$.

L. C. M. of denrs = 30.

$$\frac{a}{5} + \frac{a}{6} = \frac{6a}{30} + \frac{5a}{30} = \frac{6a+5a}{30} = \frac{11a}{30} \quad \text{Ans.}$$

Ex. 2. Simplify $\frac{x}{4a} - \frac{y}{6a}$

L. C. M. of denrs. = 12a.

$$\frac{x}{4a} - \frac{y}{6a} = \frac{3x}{12a} - \frac{2y}{12a} = \frac{3x-2y}{12a} \quad \text{Ans.}$$

Ex. 3. Find the value of $\frac{ax}{x^2y} + \frac{6ay}{3y^2z} - \frac{3a}{xz} + a$.

On reducing to lowest terms, $\frac{ax}{x^2y} = \frac{a}{xy}$, $\frac{6ay}{3y^2z} = \frac{2a}{yz}$.

\therefore the reqd. value $= \frac{a}{xy} + \frac{2a}{yz} - \frac{3a}{xz} + \frac{a}{1}$.

$$\begin{aligned} &= \frac{az}{xyz} + \frac{2ax}{xyz} - \frac{3ay}{xyz} + \frac{axy}{xyz} \\ &= \frac{az + 2ax - 3ay + ayz}{xyz} \\ &= \frac{a(2x - 3y + z + yz)}{xyz}. \end{aligned} \quad \text{Ans.}$$

N. B. The step (A) is usually omitted, and we at once write down the next step. Note also that a quantity which is integral in form may be regarded as having a denominator equal to 1, e. g., $a = \frac{a}{1}$.

Ex. 4. Simplify $\frac{2x+4}{x^2+4x-5} - \frac{x+1}{x^2+3x-10} - \frac{x-5}{x^2-3x+2}$.

Factorising the denominators, the given expression

$$\begin{aligned} &= \frac{2(x+2)}{(x-1)(x+5)} - \frac{x+1}{(x-2)(x+5)} - \frac{x-5}{(x-1)(x-2)} \\ &= \frac{2(x+2)(x-2) - (x+1)(x-1) - (x-5)(x+5)}{(x-1)(x-2)(x+5)} \\ &= \frac{2(x^2-4) - (x^2-1) - (x^2-25)}{(x-1)(x-2)(x+5)} \\ &= \frac{2x^2-8-x^2+1-x^2+25}{(x-1)(x-2)(x+5)} \\ &= \frac{18}{(x-1)(x-2)(x+5)}. \quad \text{Ans.} \end{aligned}$$

Ex. 5. Simplify $\frac{a^2}{(a-b)(x-a)} + \frac{b^2}{(b-a)(x-b)} - \frac{ab}{x^2 - (a+b)x + ab}$.

Observe that $x^2 - (a+b)x + ab = (x-a)(x-b)$, and

$$\frac{b^2}{(b-a)(x-b)} = -\frac{b^2}{(b-a)(x-b)} = -\frac{b^2}{(a-b)(x-b)}.$$

$$\begin{aligned} \text{The given expression} &= \frac{a^2}{(a-b)(1-a)} - \frac{b^2}{(a-b)(x-b)} - \frac{bx}{(x-a)(x-b)} \\ &= \frac{a^2(1-b) - b^2(x-a) - bx(a-b)}{(a-b)(1-a)(x-b)} \end{aligned}$$

$$\begin{aligned} \text{The numerator} &= a^2(1-b) - b^2x + ab^2 - abx + b^2x \\ &= a^2(1-b) - ab(a-b), \text{ re-arranging,} \\ &= a(a-b)(1-b); \end{aligned}$$

$$\begin{aligned} \therefore \text{the expression} &= \frac{a(a-b)(1-b)}{(a-b)(1-a)(x-b)} \\ &= \frac{a}{1-a} \quad \text{Ans} \end{aligned}$$

EXAMPLES 61.

Find the value of :

1. $\frac{a}{4} + \frac{a}{8}$
2. $\frac{1}{5} + \frac{7}{10}$
3. $\frac{1}{3} - \frac{x}{6}$
4. $\frac{2a}{3} + \frac{a}{6}$
5. $\frac{2a}{3} - \frac{a}{6}$
6. $\frac{3x}{5} + \frac{21}{15}$
7. $\frac{b}{24} + \frac{1}{16}$
8. $\frac{2a}{9} - \frac{5b}{12}$
9. $\frac{a}{xy} - \frac{b}{xz}$
10. $\frac{3ab}{4c} + \frac{a}{b}$
11. $\frac{9ab}{7c} - \frac{3ab}{5c}$
12. $\frac{5a}{bc} - \frac{3b}{ca}$
13. $\frac{x^2}{yz} + \frac{y^2}{zx}$
14. $\frac{a^2}{b^2} + \frac{c^2}{d^2}$
15. $\frac{ab}{cd} - \frac{bc}{ad}$
16. $\frac{3a^2}{4b^2} - \frac{6a^2}{9b^2}$
17. $\frac{7a^2b^2c^2}{14a^2b^2c^2} - \frac{5a^2b^2c^2}{15a^2b^2c^2}$
18. $\frac{21^2 \cdot 1^2 \cdot 2^2}{101^2 \cdot y^2 \cdot z^2} - \frac{y}{51}$
19. $\frac{1}{4} - \frac{r}{6} + \frac{r}{8}$
20. $\frac{r}{y} + \frac{5r}{3y} + \frac{3x}{4y}$
21. $\frac{ab}{9} - \frac{4ab}{7} + \frac{7ab}{15}$
22. $\frac{a}{bi} - \frac{b}{ca} + \frac{c}{ab}$
23. $\frac{5a^2}{b} - \frac{3a}{4} - \frac{4ab}{6c}$
24. $\frac{a}{2} - \frac{a+b}{a-b}$
25. $\frac{2a+3}{3a+4} - \frac{4x+5}{5a+6}$
26. $\frac{a+b}{a-b} - \frac{a-b}{a+b}$
27. $\frac{2a+1}{3a+2} - \frac{4a-1}{6a+2}$
28. $\frac{2}{b} - \frac{1}{a+b} + \frac{1}{a-b}$
29. $\frac{x-4}{x-3} - \frac{x-3}{x-2}$
30. $\frac{a}{2a-2b} + \frac{b}{2b-2a}$
31. $\frac{1}{x^2-6x+9} - \frac{9}{9-x^2}$
32. $\frac{1}{(x+a)^2} + \frac{2}{x^2-a^2} + \frac{1}{(a-x)^2}$
33. $\frac{a+b}{a^2+ab+b^2} - \frac{a-b}{a^2-ab+b^2}$

34. $\frac{b+y}{b-y} + \frac{4by}{b^2-y^2} + \frac{b-y}{b+y}$. 35. $\frac{5}{1-2ab} - \frac{7}{1+2a} - \frac{4-20a}{4a^2-1}$.
36. $2 - \frac{3n}{m} + \frac{9n^2-2m^2}{m^2+2mn}$. 37. $2 + \frac{3x}{2y} - \frac{6x}{3x+2y} + \frac{9x^2(3x-2y)}{2y(4y^2+9x^2)}$.
38. $1 - \frac{1-x}{1+x} - \frac{1+2x^2}{x^2-1}$. 39. $\frac{y}{2(x-y)} - \frac{y}{2(x+y)} + \frac{y^4}{x^2(y^2-x^2)}$.
40. $\frac{y}{x+y} - \frac{xy}{(x+y)^2} - \frac{xy^2}{(x+y)^3}$. 41. $\frac{x^3+x^2y}{x^3-xy^2} - \frac{xy}{x^2+xy} - \frac{xy^3}{x^2y-xy^2}$.
42. $\frac{1}{11} - \frac{12}{a+b} + \frac{10}{a+5}$. 43. $\frac{3}{(x-2)(x-3)} - \frac{6}{(x-1)(x-2)(x-3)}$.
44. $\frac{x}{(x-1)^2} - \frac{1}{(1+1)^2} - \frac{x(x^2+3)}{(1-x^2)^2}$. 45. $\frac{4}{a^2+a} + \frac{2a-5}{a^2-a+1} - \frac{2a^2-11}{a^2+1}$.
46. $\frac{2}{3a-a^2-2} - \frac{2}{2-a} + \frac{2a}{1-a}$. 47. $\frac{y-2}{y^3-2y+2} - \frac{y+1}{y^3+2y+2} + \frac{8}{y^4+4}$.
48. $\frac{x+2}{x-2} - \frac{x-2}{2(x+2)} - \frac{1}{4}$. 49. $\frac{a+x}{(a-b)(x-a)} - \frac{b+x}{(a-b)(x-b)}$.
50. $\frac{a^2-4ab-5b^2}{a(a+b)^2} - \frac{6a-30b}{a^2-4ab-5b^2}$. 51. $\frac{x^3+y^3}{xy(y^2-x^2)} - \frac{y}{x^2+xy} + \frac{x}{xy-y^2}$.
52. $\frac{p}{p-q} + \frac{q}{p+q} - \frac{2pq}{q^2-p^2}$. 53. $\frac{3}{1-2b} - \frac{17}{1-4b} + \frac{17}{1-6b}$.
54. $\frac{a}{a-x} + \frac{3a}{a+x} - \frac{2ax}{a^2-x^2}$. 55. $\frac{a+b+ab+1}{a+b-ab-1} + \frac{a+b-ab-1}{a+b+ab+1}$.
56. $\frac{1+x}{1-x} + \frac{1-x}{1+x} - \frac{1+x^2}{1-x^2} - \frac{1-x^2}{1+x^2}$.
57. $\frac{108-52x}{x(3-x)^2} - \frac{4}{3-x} - \frac{12}{x} + \left(\frac{1+x}{3-x}\right)^2$.
58. $\frac{1}{x^3-3x+2} + \frac{3}{x^2-7x+10} - \frac{4}{x^2-6x+5}$.
59. $\frac{1}{2(a-b)} - \frac{a-5b}{a^2-7ab+10b^2} + \frac{1}{2} \frac{a-6b}{a^2-9ab+18b^2}$.
60. $\frac{1}{a-b} + \frac{1}{a+b} + \frac{a-x}{a^2+(a-b)x-ab} + \frac{2a}{(b-a)(a+x)}$.
61. $\frac{x-1}{(x+2)(x+5)} - \frac{2(x+2)}{(x+5)(x-1)} - \frac{x+5}{(1-x)(x+2)}$.
62. $\frac{1}{a^2-b^2-c^2+2bc} + \frac{1}{b^2-c^2-a^2+2ca} + \frac{1}{c^2-a^2-b^2+2ab}$.

CHAPTER XIX.

MULTIPLICATION AND DIVISION OF FRACTIONS.

112. Product of Fractions. Rule: Take the product of the numerators for the new numerator, and the product of the denominators for the new denominator; simplify the result, when possible.

Proof: Let $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ be the given fractions.

Let $\frac{a}{b}$ be denoted by x , $\frac{c}{d}$ by y , and $\frac{e}{f}$ by z .

Then, by definition, $a = bx$,

$$c = dy,$$

$$\text{and } e = fz.$$

\therefore Multiplying up, $ace = bdfxyz = (bdf) \times xyz$.

$$\therefore xyz = ace \div bdf = \frac{ace}{bdf};$$

$$\text{i.e., } \frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{\text{product of numerators}}{\text{,, ,, denominators}}.$$

N.B. It now easily follows that $a \times \frac{1}{b} = \frac{a}{b}$, for $a \times \frac{1}{b} = \frac{a}{1} \times \frac{1}{b}$.

Similarly $\frac{1}{a} \times b = \frac{b}{a}$.

Ex. 1. Find the value of $4a \times \frac{3a^2bx^3}{10b^2y^3} \times \frac{25b^2yz}{30a^2xz}$

Since $4a = \frac{4a}{1}$, the given expression

$$= \frac{4a \times 3a^2bx^3 \times 25b^2yz}{1 \times 10b^2y^3 \times 30a^2xz} = \frac{a^3b^3c^2x^3yz}{a^3b^3cx^3yz} = \frac{cx}{y}. \text{ Ans.}$$

Ex. 2 Simplify $\left(1 + \frac{3x-7}{x^2-6x+9}\right)\left(1 - \frac{3x-5}{x^2-2x+1}\right)$

$$\text{The 1st factor} = \frac{x^2 - 6x + 9 + 3x - 7}{x^2 - 6x + 9} = \frac{x^2 - 3x + 2}{x^2 - 6x + 9} = \frac{(x-1)(x-2)}{(x-3)^2};$$

$$\text{,, 2nd ,,} = \frac{x^2 - 2x + 1 - 3x + 5}{x^2 - 2x + 1} = \frac{x^2 - 5x + 6}{x^2 - 2x + 1} = \frac{(x-2)(x-3)}{(x-1)^2}.$$

$$\therefore \text{the product} = \frac{(x-1)(x-2)}{(x-3)^2} \times \frac{(x-2)(x-3)}{(x-1)^2} = \frac{(x-2)^2}{(x-3)(x-1)}. \text{ Ans.}$$

EXAMPLES 82.

Find the value of

1. $\frac{ab}{cd} \times \frac{ba}{ac}$
2. $\frac{xy}{ab} \times \frac{ab^2}{xy^2}$
3. $\frac{x^3}{y^2} \times \frac{y^3}{x^2}$
4. $\frac{5x^2}{9} \times \frac{3}{10xz}$
5. $\frac{10xy}{9} \times \frac{3}{5yz}$
6. $\frac{6a^2b^3}{14ax^2} \times \frac{28cx^2}{12ab}$
7. $\frac{4ab^2}{3c^2d} \times \frac{3(a^3)}{2a^2b}$
8. $\frac{7x^2}{5ax^2} \times \frac{5x^2}{14ax}$
9. $\frac{3a^2}{4bc} \times \frac{8b^2c^2}{9a^4}$
10. $4a^2b^3 \times \frac{bc^6}{32ab^4c^3}$
11. $7xy \times \frac{x^2}{42xy^3}$
12. $\frac{a^3}{b^2} \times \frac{b^3}{c^2} \times \frac{c^2}{a^3}$
13. $\frac{x^2-y^2}{a^2-b^2} \times \frac{(a-b)^2}{(x-y)^2}$
14. $\frac{4a^2-1}{12} \times \frac{2a+1}{2a-1}$
15. $\frac{a^2-1}{a^2+1} \times \frac{(a+1)^2}{(a-1)^2}$
16. $\frac{5x-10x^2}{2x^2+x^3} \times \frac{2x+x^2}{2x-1}$
17. $\frac{2+xy}{6xy-2x^2} \times \frac{x^2y-3xy^2}{x^2y^2-4}$
18. $\frac{x^2-(a+c)x+ac}{x^2+(b+c)x+bc} \times \frac{x^2-b^2}{x^2-a^2}$
19. $\frac{2yz}{x^2} \times \frac{3xz}{4y^2} \times \frac{xy}{9z^2}$
20. $\left(\frac{a}{b} - \frac{b}{a}\right) \times \frac{ab}{a-b}$

113. Division of a fraction by another.

Rule. *Invert the divisor; then go on as in multiplication.*

Proof. $\because \frac{a}{b} = \frac{ad}{bc} \times \frac{c}{d} \therefore \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}$.

N. B. Note the results, $a \div \frac{1}{b} = ab$, and $\frac{1}{a} \div b = \frac{1}{ab}$.

Observe that $a = \frac{a}{1}$, $b = \frac{b}{1}$, &c.

Ex. 1. Simplify $\frac{15y^2}{140z} \times \frac{14z^2}{xyz} \div \frac{81z}{54xy}$.

The given expression = $\frac{15y^2}{140z} \times \frac{14z^2}{xyz} \times \frac{54xy}{81z}$
 $= \frac{15y^2 \times 14z^2 \times 54xy}{140z \times xyz \times 81z}$
 $= \frac{5 \times 3 \times 14 \times 2 \times 27xy^3z^2}{14 \times 10 \times 3 \times 27xy^3z^2}$
 $= \frac{y^2}{z} \text{ Ans.}$

Ex. 2 Simplify $\frac{x^3+27}{x^3+9x+14} \div \frac{x^2-4x-21}{x^2-49}$.

$$\begin{aligned}\text{The given expression} &= \frac{x^3+27}{x^3+9x+14} \times \frac{x^2-49}{x^2-4x-21} \\ &= \frac{x^3+3^3}{x^3+9x+14} \times \frac{x^2-7^2}{x^2-4x-21} \\ &= \frac{(x+3)(x^2-3x+9)}{(x+2)(x+7)} \times \frac{(x-7)(x+7)}{(x-7)(x+3)} \\ &= \frac{x^2-3x+9}{x+2} \quad \text{Ans.}\end{aligned}$$

EXAMPLES 63.

1. $\frac{a^3}{b^3} \div \frac{m^2}{n^2}$
2. $\frac{a^4}{b^4} \div \frac{a^2b^2}{b^3a^2}$
3. $\frac{16a^3b}{3a^2} \div \frac{8ab}{15b^2}$
4. $\frac{8x^4y^3}{9a^3b^3} \div \frac{4x^3y^2}{3a^2b^2}$
5. $\frac{px^3}{qyz} \div \frac{p^2xy^2}{q^2z}$
6. $\frac{10xyz}{3a^2b} \div \frac{5yz}{10ax}$
7. $\frac{a^2+ax}{b^2} \div \frac{a+x}{b}$
8. $\frac{a^2-x^2}{a^2+ab} \div \frac{a-x}{a+b}$
9. $\frac{1-a^2}{b+b^2} \div \frac{a^2+a}{1-b^2}$
10. $\left(\frac{1}{x}+y\right) \div \left(\frac{1}{y}+x\right)$
11. $\left(\frac{1}{x}-\frac{1}{a}\right) \div (x-a)$
12. $\left(\frac{a}{x}-\frac{y}{b}\right) \div \left(\frac{y}{a}-\frac{b}{x}\right)$
13. $\frac{a}{4b} \times \frac{2bc}{a^2} \div \frac{3ac}{bc}$
14. $\frac{3xy}{10y^2z} \times \frac{4xy^3}{9x^2y} \div \frac{5y^2}{2y^2z}$
15. $\frac{2a^2b^4}{5x^2y^2} \times \frac{8ab^3}{15xy} \div \frac{a^4b^6}{x^4y^4}$
16. $\frac{8a^2n^2}{bm} \div \frac{5a^2n}{9a^2l^3} \times \frac{l^3m^3}{4mb^2}$
17. $\frac{4a^2b^3}{c^2a^3} \div \frac{ab^3c}{ada^2} \times \frac{a^4c^4}{8a^3c^6}$
18. $\frac{19a^3b^3y^4z^6}{20c^4x^4} \div \frac{133a^2by^2z^6}{100c^6x^3} \times \frac{xy}{ac}$
19. $\frac{1^2-4x-5}{x^3+4x+3} \div \frac{x^2-7x+10}{x^2+x-6}$
20. $\left(1-\frac{1}{x-5}\right) \div \left(1-\frac{2x-15}{x^2-6x+5}\right)$

CHAPTER XX.

HARDER FRACTIONS.

114. Definition. A fraction whose numerator and denominator are whole numbers is called a **simple fraction**.

A **complex fraction** has a fraction for its numerator or denominator or both.

Thus $\frac{\frac{a}{b}}{c}$, $\frac{a}{\frac{b}{c}}$, $\frac{\frac{a}{b}}{\frac{c}{d}}$ are complex fractions.

In the first of these forms, the outside quantities, a and d , are called the **extremes**, while the two middle ones, b and c , are called the **means**.

N. B. Complex fractions are sometimes conveniently written in the forms

$$\frac{a/c}{b/d}, \frac{a/b}{c/d}, \frac{a/b}{c/d}.$$

115. Simplification of complex fractions. We shall begin with the form $\frac{a/b}{c/d}$

$$\begin{aligned} \frac{\frac{a}{b}}{\frac{c}{d}} &= \frac{a}{b} \div \frac{c}{d} \text{ by definition of fraction,} \\ &= \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \dots\dots\dots (1) \end{aligned}$$

$$\text{Similarly } \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{1} \times \frac{d}{c} = \frac{ad}{c} \dots\dots\dots (2)$$

$$\text{and } \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc} \dots\dots\dots (3)$$

The student should be able to write down at once the result of simplification of the several types of complex fractions. From (1) it is evident that this result may be expressed as *product of extremes* \div *product of means*. The same is the case in (2) and (3), if we bear in mind that in (2) $a = \frac{a}{1}$, and that in (3) $c = \frac{c}{1}$.

$$\text{Ex. 1. Simplify } \frac{\frac{a-b}{a+b} - \frac{a+b}{a-b}}{\frac{a^2-b^2}{a^2+b^2} - \frac{a^2-b^2}{a^2-b^2}}$$

$$\text{The numr.} = \frac{a-b}{a+b} - \frac{a+b}{a-b} = \frac{(a-b)^2 - (a+b)^2}{(a+b)(a-b)} = \frac{-4ab}{(a+b)(a-b)}.$$

$$\text{The denr.} = \frac{a^2 - b^2}{a^2 + b^2} - \frac{a^2 + b^2}{a^2 - b^2} = \frac{(a^2 - b^2)^2 - (a^2 + b^2)^2}{(a^2 + b^2)(a^2 - b^2)} = \frac{-4a^2b^2}{(a^2 + b^2)(a^2 - b^2)}.$$

$$\begin{aligned}\therefore \text{ the reqd. value} &= \frac{-4ab}{(a+b)(a-b)} \div \frac{-4a^2b^2}{(a^2 + b^2)(a^2 - b^2)} \\ &= \frac{-4ab}{(a+b)(a-b)} \times \frac{(a^2 + b^2)(a^2 - b^2)}{-4a^2b^2} \\ &= \frac{a^2 + b^2}{ab}. \quad \text{Ans.}\end{aligned}$$

116. Continued fractions In simplifying continued fractions, we begin with the lowest fraction, and simplify step by step.

$$\text{Ex. 1. Simplify } 1 - \frac{a}{a+2 - \frac{1}{a+1}}.$$

$$a + \frac{1}{a+2} = \frac{a^2 + 2a + 1}{a+2} = \frac{(a+1)^2}{a+2}.$$

$$\therefore \frac{a+1}{a + \frac{1}{a+2}} = \frac{a+1}{\frac{(a+1)^2}{a+2}} = \frac{(a+1)(a+2)}{(a+1)^2} = \frac{a+2}{a+1}.$$

$$\begin{aligned}\therefore \text{ the given fraction} &= 1 - \frac{a}{a+2 - \frac{1}{a+1}} \\ &= 1 - \frac{a}{\frac{(a+2)(a+1) - (a+2)}{a+1}} \\ &= 1 - \frac{a}{\frac{(a+2)a}{a+1}}\end{aligned}$$

$$= 1 - \frac{a(a+1)}{(a+2)a}$$

$$= 1 - \frac{a(a+1)}{(a+2)a}$$

$$= 1 - \frac{a+1}{a+2}$$

$$= \frac{a+2-a-1}{a+2}$$

$$= \frac{1}{a+2}. \quad \text{Ans.}$$

EXAMPLES 64.

Simplify

1. $\frac{\frac{a}{b}}{\frac{ac}{bd}}$

2. $\frac{\frac{x}{a}}{\frac{a}{x}}$

3. $\frac{\frac{a}{b^2}}{\frac{1}{b}}$

4. $\frac{\frac{b^2}{ac}}{\frac{a^2}{bc}}$

5. $\frac{\frac{a-b}{b-a}}{\frac{1}{a}}$

6. $\frac{\frac{1}{a^2} - \frac{1}{b^2}}{\frac{a}{b} - a}$

7. $\frac{\frac{a+c}{b+d}}{\frac{a}{c} + \frac{d}{b}}$

8. $\frac{\frac{x^3-y}{x^2+y^2}}{\frac{y^3-x}{y^2+x^2}}$

9. $\frac{\frac{x^2}{5} + \frac{x}{10} - \frac{3}{2}}{\frac{x^2}{3} - \frac{11}{6}x + \frac{5}{2}}$

10. $\frac{\frac{a-b}{b-a}}{\frac{a^2-b^2}{b^2+a^2}-2}$

11. $\frac{\frac{x-y}{4y} - \frac{2x}{3y} - \frac{12}{4x}}{\frac{x}{3y} - \frac{y}{4x} + \frac{12}{x}}$

12. $\frac{\frac{a}{x} - \frac{2ax}{1+x^2}}{\frac{x}{2} + \frac{1}{2x} - 1}$

13. $\frac{\frac{x-y}{1+xy}}{\frac{x^2-1}{1+x^2}}$

14. $\frac{\frac{x-7+\frac{6}{x}}{x-11+\frac{10}{x}}}{\frac{x-7+\frac{6}{x}}{x-11+\frac{10}{x}}}$

15. $\frac{\frac{x-y}{1-x}}{\frac{1}{y} - x}$

16. $\frac{\frac{x^2-a^2}{x+a^2-2}}{\frac{x^2-1}{x^2+xy+y^2}}$

17. $\frac{\frac{x-1}{x+1}}{\frac{x-1}{x+1}-2}$

18. $\frac{x+15+\frac{95}{x-5}}{x+3}$

19. $\frac{\frac{x^2-1}{y-x}}{x^2+xy+y^2}$

20. $\frac{\frac{x^4-y^4}{y^2-x^2}}{x^4+x^2y^2+y^4}$

21. $\frac{\frac{a-b}{a^2-b^2} - \frac{a+b}{a^2+b^2}}{\frac{a-b}{a^2-b^2} - \frac{a+b}{a^2+b^2}}$

22. $\frac{\frac{2x-3y}{2x-xy} - \frac{3y}{2x-xy}}{\frac{2x-xy}{2x-xy} + \frac{3y}{2x-6y}}$

23. $\frac{\frac{p-q-2q}{q} \cdot \frac{(p-q)(p+q)}{p+q}}{\frac{p-q}{q} \cdot \frac{p+q}{p}}$

24. $\frac{\frac{a}{2+\frac{a^4}{3+\frac{4}{a^4}}}}$

25. $\frac{\frac{x+4}{x+\frac{4}{\frac{1}{3}-\frac{16}{3x+4}}}}$

26. $\frac{\frac{x-1}{x+3+\frac{5}{1-x+1}}}$

27. $\frac{\frac{a+1}{a-1}}{a-1+\frac{1}{a-\frac{1}{a}}}$

28. $\frac{\frac{\frac{3x}{2}}{2x-1+\frac{1}{1-\frac{x}{x-2}}}}$

29. $2 - \frac{\frac{2}{3}}{1+\frac{1}{1+\frac{6}{x}}}$

$$30. \quad 1 - \frac{1}{3 - \frac{1}{2 - \frac{1}{1 - \frac{1}{4 - \frac{1}{x+4}}}}}$$

$$31. \quad 2 - \frac{1}{1 + \frac{2}{3 + \frac{1}{2 - \frac{1}{1 + \frac{5}{x+4}}}}}$$

$$32. \quad \frac{1-2}{2 - \frac{1}{2 - \frac{1}{2 - \frac{1}{1 - \frac{2}{x+6}}}}}$$

$$33. \quad 3 + \frac{1}{1 - \frac{1}{3 - \frac{1}{x}}}$$

EXAMPLES WORKED OUT.

Ex. 1. Simplify $\frac{1 - \frac{y+y^2}{1+y} \cdot \frac{x^3-1}{x^2}}{1 + \frac{y+y^2}{1+y} \cdot \frac{x^3-1}{x^2}} \div \frac{\left(\frac{1}{x} - \frac{1}{y}\right)^2}{\left(\frac{1}{x} + \frac{1}{y}\right)^2}$

$$\frac{1 - \frac{y+y^2}{1+y} \cdot \frac{x^3-1}{x^2}}{1 + \frac{y+y^2}{1+y} \cdot \frac{x^3-1}{x^2}} = \frac{x^2 - xy + y^2}{x^2 + xy + y^2} = \frac{x^2 - xy + y^2}{x^2 + xy + y^2}$$

$$\frac{x^3-1}{x^3-1} = \frac{x^3-1}{x^3-1} = \frac{x^3-1}{x^3-1} = \frac{x^3-1}{x^3-1}$$

$$\left(\frac{1}{x} - \frac{1}{y}\right)^2 \div \left(\frac{1}{x} + \frac{1}{y}\right)^2 = \frac{(y-x)^2}{(x+y)^2} \div \frac{(x+y)^2}{(x+y)^2} = \frac{(y-x)^2}{(x+y)^2}$$

$$\begin{aligned} \therefore \text{the reqd value} &= \frac{x^2-xy+y^2}{x^2+xy+y^2} \div \frac{x^3-1}{x^3-1} \div \frac{(y-x)^2}{(y+x)^2} \\ &= \frac{x^2-xy+y^2}{x^2+xy+y^2} \times \frac{x^3-1}{x^3-1} \times \frac{(y+x)^2}{(y-x)^2} \\ &= \frac{x^2-xy+y^2}{x^2+xy+y^2} \times \frac{(x-y)(x^2+xy+y^2)}{(x+y)(x^3-1)} \times \frac{(x+y)^2}{(y-x)^2} \\ &= \frac{x-y}{x+y} \times \frac{(x+y)^2}{(x-y)^3} \because (y-x)^2 = (x-y)^2 \\ &= \frac{x+y}{x-y} \quad \text{Ans.} \end{aligned}$$

N. B. The above work is usually cut short. If we multiply the

numerator and denominator of $\frac{1 - \frac{y}{x} + \frac{y^2}{x^2}}{1 + \frac{y}{x} + \frac{y^2}{x^2}}$ by x^2 , those of $\frac{\frac{x^3}{y^3} + 1}{\frac{x^3}{y^3} - 1}$ by y^3 ,

and those of $\left(\frac{1}{x} - \frac{1}{y}\right)^2$ by $x^2 y^2$, we get respectively $\frac{x^2 - xy + y^2}{x^2 + xy + y^2}$, $\frac{x^3 + y^3}{x^3 - y^3}$

and $\frac{(y-x)^2}{(y+x)^2}$. All this work is, however, done mentally, and we begin at once thus: the given expression $= \frac{x^2 - xy + y^2}{x^2 + xy + y^2} \div \frac{x^3 + y^3}{x^3 - y^3} \times \frac{(y-x)^2}{(y+x)^2} = \&c.$

Ex. 2. Simplify $\frac{(y-z)(y+z)^2 + (z-x)(z+x)^2 + (x-y)(x+y)^2}{(y+z)(y-z)^2 + (z+x)(z-x)^2 + (x+y)(x-y)^2}$.

B. U. 1888—89.

Let N and D denote the numerator and denominator respectively.

First simplify N .

$$\begin{aligned}(y-z)(y+z)^2 &= \{(y-z)(y+z)\}(y+z)^2 \\ &= (y^2 - z^2)(y^2 + z^2 + 2yz) \\ &= y^4 - z^4 + 2yz(y^2 - z^2);\end{aligned}$$

similarly, $(z-x)(z+x)^2 = z^4 - x^4 + 2zx(z^2 - x^2)$,

and $(x-y)(x+y)^2 = x^4 - y^4 + 2xy(x^2 - y^2)$;

by addition, $N = 2yz(y^2 - z^2) + 2zx(z^2 - x^2) + 2xy(x^2 - y^2)$. (A)

Next simplify D .

$$\begin{aligned}(y+z)(y-z)^2 &= \{(y+z)(y-z)\}(y-z)^2 \\ &= (y^2 - z^2)(y^2 + z^2 - 2yz) \\ &= y^4 - z^4 - 2yz(y^2 - z^2).\end{aligned}$$

Similarly breaking up the other terms of D , and proceeding exactly as above, we shall have

$$\begin{aligned}D &= -2yz(y^2 - z^2) - 2zx(z^2 - x^2) - 2xy(x^2 - y^2), \\ &= -N, \text{ by (A).}\end{aligned}$$

$$\therefore \frac{N}{D} = \frac{N}{-N} = -1. \text{ Ans.}$$

Otherwise thus: Putting a for $y+z$, b for $z+x$, c for $x+y$, we have

$$\begin{aligned}N &= (c-b)a^2 + (a-c)b^2 + (b-a)c^2, \\ \text{and } D &= a(c-b)^2 + b(a-c)^2 + c(b-a)^2.\end{aligned}$$

Now, $a(c-b)^2 = a\{c^2 - b^2 - 3bc(c-b)\} = a(c^2 - b^2) - 3abc(c-b)$;

similarly, $b(a-c)^2 = b(a^2 - c^2) - 3abc(b-c)$,

and $c(b-a)^2 = c(b^2 - a^2) - 3abc(b-a)$.

\therefore adding up these results, and observing that

$3abc(c-b+a-c+b-a) = 0$, we have

$$\begin{aligned} D &= a(c^2 - b^2) + b(a^2 - c^2) + c(b^2 - a^2) \\ &= -(c-b)a^2 - (a-c)b^2 - (b-a)c^2, \text{ re arranging,} \\ &= -N. \end{aligned}$$

\therefore the fraction $= -1$. Ans.

N. B. The last method will sometimes be very useful.

Ex. 3. Shew that

$$\begin{aligned} (a+b)^2 - (b+c)^2 + (c+d)^2 - (d+a)^2 &= \frac{2}{3}(a+b+c+d). \\ (a+b)^2 - (b+c)^2 + (c+d)^2 - (d+a)^2 &= \frac{2}{3}(a+b+c+d). \end{aligned}$$

We know that

$$A^2 + B^2 + C^2 + 3(A+B)(B+C)(C+A) = (A+B+C)^2. \quad (1)$$

If we put $A = a+b$, $B = -(b+c)$, and $C = c+d$, then $A+B = a-c$, $B+C = -b+d = -(b-d)$, $C+A = a+b+c+d$, $A+B+C = d+a$, and the identity (1) becomes

$$(a+b)^2 - (b+c)^2 + (c+d)^2 - 3(a-c)(b-d)(a+b+c+d) = (d+a)^2$$

\therefore by transposition, we have

$$(a+b)^2 - (b+c)^2 + (c+d)^2 - (d+a)^2 = 3(a-c)(b-d)(a+b+c+d)$$

The denr. $= \{(a+b)^2 - (b+c)^2\} + \{(c+d)^2 - (d+a)^2\}$

$$\begin{aligned} &\{(a+b) + (b+c)\} \{(a+b) - (b+c)\} \\ &+ \{(c+d) + (d+a)\} \{(c+d) - (d+a)\} \end{aligned}$$

$$= (a+2b+c)(a-c) + (c+2d+a)(c-a)$$

$$= (a-c)\{(a+2b+c) - (c+2d+a)\}$$

$$= (a-c)\{2(b-d)\}$$

$$= 2(a-c)(b-d)$$

$$\begin{aligned} \therefore \text{ the first given fraction} &= \frac{3(a-c)(b-d)(a+b+c+d)}{2(a-c)(b-d)} \\ &= \frac{3}{2}(a+b+c+d). \text{ Ans.} \end{aligned}$$

Otherwise thus :

$$\{(a+b) + (c+d)\}^2 = \{(b+c) + (d+a)\}^2;$$

expanding each side, we have

$$\begin{aligned} (a+b)^2 + (c+d)^2 + 3(a+b)(c+d)(a+b+c+d) \\ = (b+c)^2 + (d+a)^2 + 3(b+c)(d+a)(a+b+c+d). \end{aligned}$$

∴ by transposition, we get

$$(a+b)^3 - (b+c)^3 + (c+d)^3 - (d+a)^3 \\ = 3(a+b+c+d)\{(b+c)(d+a) - (a+b)(c+d)\} \quad (1)$$

Again, $\{(a+b) + (c+d)\}^3 = \{(b+c) + (d+a)\}^3$,

expanding each side, we have

$$(a+b)^3 + (c+d)^3 + 3(a+b)(c+d) - (b+c)^3 - (d+a)^3 + 3(b+c)(d+a).$$

by transposition, we have

$$(a+b)^3 - (b+c)^3 + (c+d)^3 - (d+a)^3 \\ = 2\{(b+c)(d+a) - (a+b)(c+d)\} \quad (2)$$

by (1) and (2), the given fraction

$$= \frac{3(a+b+c+d)\{(b+c)(d+a) - (a+b)(c+d)\}}{2\{(b+c)(d+a) - (a+b)(c+d)\}} \\ = \frac{3}{2}(a+b+c+d)$$

Ex. 4. Shew that $\left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right)^2 + \left(\frac{a}{b} + \frac{b}{a}\right)^2$

$$= 4 + \left(\frac{b}{c} + \frac{c}{b}\right)\left(\frac{c}{a} + \frac{a}{c}\right)\left(\frac{a}{b} + \frac{b}{a}\right)$$

$$\left(\frac{c}{a} + \frac{a}{c}\right)^2 + \left(\frac{a}{b} + \frac{b}{a}\right)^2 = \left(\frac{c^2}{a^2} + \frac{a^2}{c^2} + 2\frac{c}{a} \times \frac{a}{c}\right) + \left(\frac{a^2}{b^2} + \frac{b^2}{a^2} + 2\frac{a}{b} \times \frac{b}{a}\right) \\ = \left(\frac{c^2}{a^2} + \frac{a^2}{c^2} + 2\right) + \left(\frac{a^2}{b^2} + \frac{b^2}{a^2} + 2\right)$$

Adding $\left(\frac{b}{c} + \frac{c}{b}\right)^2$, and re-arranging terms, we have

$$\left(\frac{b}{c} + \frac{c}{b}\right)^2 + \left(\frac{c}{a} + \frac{a}{c}\right)^2 + \left(\frac{a}{b} + \frac{b}{a}\right)^2 = 4 + \left(\frac{b}{c} + \frac{c}{b}\right)^2 + \left(\frac{b^2}{a^2} + \frac{c^2}{a^2}\right) + \left(\frac{a^2}{c^2} + \frac{a^2}{b^2}\right) \\ = 4 + \left(\frac{b}{c} + \frac{c}{b}\right)^2 + \frac{b}{a^2}\left(\frac{b}{c} + \frac{c}{b}\right) + \frac{a^2}{b}\left(\frac{b}{c} + \frac{c}{b}\right) \\ = 4 + \left(\frac{b}{c} + \frac{c}{b}\right)\left(\frac{b}{c} + \frac{c}{b} + \frac{bc}{a^2} + \frac{a^2}{bc}\right) \\ = 4 + \left(\frac{b}{c} + \frac{c}{b}\right)\left\{\left(\frac{c}{b} + \frac{b}{a^2}\right) + \left(\frac{b}{c} + \frac{a^2}{bc}\right)\right\} \\ = 4 + \left(\frac{b}{c} + \frac{c}{b}\right)\left\{\frac{c}{a}\left(\frac{a}{b} + \frac{b}{a}\right) + \frac{a}{c}\left(\frac{b}{a} + \frac{a}{b}\right)\right\} \\ = 4 + \left(\frac{b}{c} + \frac{c}{b}\right)\left(\frac{c}{a} + \frac{a}{c}\right)\left(\frac{a}{b} + \frac{b}{a}\right). \quad \text{Ans.}$$

Otherwise thus :

$$\left(\frac{b}{c} + \frac{c}{b}\right)^2 + \left(\frac{c}{a} + \frac{a}{c}\right)^2 + \left(\frac{a}{b} + \frac{b}{a}\right)^2 = \left(\frac{b^2 + c^2}{bc}\right)^2 + \left(\frac{c^2 + a^2}{ca}\right)^2 + \left(\frac{a^2 + b^2}{ab}\right)^2$$

$$= \frac{a^2(b^2 + c^2)^2 + b^2(c^2 + a^2)^2 + c^2(a^2 + b^2)^2}{a^2b^2c^2}$$

The numr. = $a^2(b^4 + c^4 + 2b^2c^2) + b^2(c^4 + a^4 + 2c^2a^2) + c^2(a^4 + b^4 + 2a^2b^2)$
 $= 4a^2b^2c^2 + \{a^4(b^2 + c^2) + b^4(c^2 + a^2) + c^4(a^2 + b^2) + 2a^2b^2c^2\}$
 $= 4a^2b^2c^2 + (b^2 + c^2)(c^2 + a^2)(a^2 + b^2)$. Formula, Art 73.

$$\therefore \left(\frac{b}{c} + \frac{c}{b}\right)^2 + \left(\frac{c}{a} + \frac{a}{c}\right)^2 + \left(\frac{a}{b} + \frac{b}{a}\right)^2 = \frac{4a^2b^2c^2 + (b^2 + c^2)(c^2 + a^2)(a^2 + b^2)}{a^2b^2c^2}$$

$$= 4 + \frac{b^2 + c^2}{bc} \cdot \frac{c^2 + a^2}{ca} \cdot \frac{a^2 + b^2}{ab}$$

$$= 4 + \left(\frac{b}{c} + \frac{c}{b}\right)\left(\frac{c}{a} + \frac{a}{c}\right)\left(\frac{a}{b} + \frac{b}{a}\right).$$

Ex 5. Simplify $\frac{(2k-3l)^2 - k^2}{4k^2 - (3l+k)^2} + \frac{4k^2 - (3l-k)^2}{9(k^2 - l^2)} + \frac{9l^2 - k^2}{(2k+3l)^2 - k^2}$
 C. U. 1894.

First reduce each fraction.

$$\frac{(2k-3l)^2 - k^2}{4k^2 - (3l+k)^2} = \frac{(2k-3l+k)(2k-3l-k)}{(2k+3l+k)(2k-3l-k)}$$

$$= \frac{3l-k-l)(k-3l)}{3(k+l)(k-3l)} = \frac{k-l}{k+l}$$

$$\text{Similarly, } \frac{4k^2 - (3l-k)^2}{9(k^2 - l^2)} = \frac{(k+3l)3(k-l)}{9(k^2 - l^2)} = \frac{k+3l}{3(k+l)},$$

$$\text{and } \frac{9l^2 - k^2}{(2k+3l)^2 - k^2} = \frac{(3l-k)(3l+k)}{3(k+l)(k+3l)} = \frac{3l-k}{3(k+l)}.$$

$$\therefore \text{ the reqd. value} = \frac{k-l}{k+l} + \frac{k+3l}{3(k+l)} + \frac{3l-k}{3(k+l)}$$

$$= \frac{3(k-l) + k+3l+3l-k}{3(k+l)}$$

$$= \frac{3(k+l)}{3(k+l)} = 1. \text{ Ans.}$$

Ex. 6. Find the value of $\frac{1}{a-2x} + \frac{1}{a+2x} + \frac{2a}{a^2+4x^2} + \frac{4a^3}{a^4+16x^4}$.

We adopt here the method of successive addition.

$$\frac{1}{a-2x} + \frac{1}{a+2x} = \frac{(a+2x) + (a-2x)}{a^2 - 4x^2} = \frac{2a}{a^2 - 4x^2};$$

adding $\frac{2a}{a^3+4x^3}$ to each side, we have

$$\text{the result} = \frac{2a}{a^3-4x^3} + \frac{2a}{a^3+4x^3} = \frac{2a(a^3+4x^3) + 2a(a^3-4x^3)}{a^6-16x^6} = \frac{4a^3}{a^6-16x^6}.$$

Again, adding $\frac{4a^3}{a^6+16x^6}$ to the last result, we have

$$\begin{aligned} \text{the final result} &= \frac{4a^3}{a^6-16x^6} + \frac{4a^3}{a^6+16x^6} \\ &= \frac{4a^3(a^6+16x^6) + 4a^3(a^6-16x^6)}{a^{12}-256x^{12}} = \frac{8a^7}{a^8-256x^8}. \quad \text{Ans.} \end{aligned}$$

N. B. The present example is intended to illustrate the nice method of *judicious combination of terms*, which often makes our work far easier than it could otherwise be. The next example will illustrate the opposite method of *judicious breaking up of terms*.

$$\text{Ex. 7. Shew that } \frac{a}{a^3-1} + \frac{a^2}{a^4-1} + \frac{a^4}{a^8-1} = \frac{1}{2} \left(\frac{a+1}{a-1} - \frac{a^8+1}{a^8-1} \right).$$

$$\frac{a}{a^3-1} = \frac{1}{2} \cdot \frac{2a}{a^3-1} = \frac{1}{2} \cdot \frac{(a+1)^2 - (a^2+1)}{a^3-1} = \frac{1}{2} \left(\frac{a+1}{a-1} - \frac{a^2+1}{a^2-1} \right).$$

$$\text{Similarly } \frac{a^2}{a^4-1} = \frac{1}{2} \left(\frac{a^2+1}{a^2-1} - \frac{a^4+1}{a^4-1} \right), \text{ and } \frac{a^4}{a^8-1} = \frac{1}{2} \left(\frac{a^4+1}{a^4-1} - \frac{a^8+1}{a^8-1} \right).$$

\therefore adding up and cancelling terms, we have

$$\frac{a}{a^3-1} + \frac{a^2}{a^4-1} + \frac{a^4}{a^8-1} = \frac{1}{2} \left(\frac{a+1}{a-1} - \frac{a^8+1}{a^8-1} \right). \quad \text{Ans.}$$

Otherwise thus :

$$\frac{a}{a^3-1} = \frac{a+1-1}{a^3-1} = \frac{a+1}{a^3-1} - \frac{1}{a^3-1} = \frac{1}{a-1} - \frac{1}{a^3-1};$$

$$\text{similarly } \frac{a^2}{a^4-1} = \frac{1}{a^2-1} - \frac{1}{a^4-1}, \text{ and } \frac{a^4}{a^8-1} = \frac{1}{a^4-1} - \frac{1}{a^8-1}.$$

Adding up and cancelling terms, we have

$$\begin{aligned} \frac{a}{a^3-1} + \frac{a^2}{a^4-1} + \frac{a^4}{a^8-1} &= \frac{1}{a-1} - \frac{1}{a^3-1} \\ &= \frac{1}{2} \left(\frac{2}{a-1} - \frac{2}{a^3-1} \right) \\ &= \frac{1}{2} \left\{ \left(1 + \frac{2}{a-1} \right) - \left(1 + \frac{2}{a^3-1} \right) \right\}, \quad \because 1-1=0, \\ &= \frac{1}{2} \left(\frac{a+1}{a-1} - \frac{a^8+1}{a^8-1} \right). \end{aligned}$$

Ex. 8. Simplify $\frac{bc(x-a)}{(a-b)(a-c)} + \frac{ca(x-b)}{(b-a)(b-c)} + \frac{ab(x-c)}{(c-a)(c-b)}$. C.U. 1896.

L. C. M. of denrs. = $(b-c)(c-a)(a-b)$.

∴ the given expression

$$= \frac{-bc(x-a)(b-c) - ca(x-b)(c-a) - ab(x-c)(a-b)}{(b-c)(c-a)(a-b)}.$$

Now, $bc(x-a)(b-c) = (-bcx + abc)(b-c) = -bc(b-c)x + abc(b-c)$.

Similarly $-ca(x-b)(c-a) = -ca(c-a)x + abc(c-a)$,

and $-ab(x-c)(a-b) = -ab(a-b)x + abc(a-b)$.

∴ adding and observing that $abc(b-c+c-a+a-b)=0$,

$$\begin{aligned} & -bc(x-a)(b-c) - ca(x-b)(c-a) - ab(x-c)(a-b) \\ &= -\{bc(b-c) + ca(c-a) + ab(a-b)\}x \\ &= -(b-c)(c-a)(a-b)x. \end{aligned}$$

∴ the given expression = $\frac{(b-c)(c-a)(a-b)x}{(b-c)(c-a)(a-b)} = x$. Ans.

N. B. The above is one of a class of examples in which the simplification depends upon certain identities, which have been dealt with at large in Chapter XII, viz.,

$$(b-c) + (c-a) + (a-b) = 0;$$

$$a(b-c) + b(c-a) + c(a-b) = 0;$$

$$\begin{aligned} a^2(b-c) + b^2(c-a) + c^2(a-b) &= -a(b^2 - c^2) - b(c^2 - a^2) - c(a^2 - b^2) \\ &= bc(b-c) + ca(c-a) + ab(a-b) \\ &= -(b-c)(c-a)(a-b); \end{aligned}$$

$$a^3(b-c) + b^3(c-a) + c^3(a-b) = -(a+b+c)(b-c)(c-a)(a-b).$$

$$(b-c)^2 + (c-a)^2 + (a-b)^2 = 3(b-c)(c-a)(a-b), \text{ \&c}$$

The following useful results, which follow easily from the above identities, should be noted :

$$\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)} = 0;$$

$$\frac{a}{(a-b)(a-c)} + \frac{b}{(b-a)(b-c)} + \frac{c}{(c-a)(c-b)} = 0;$$

$$\text{and } \frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(b-c)} + \frac{c^2}{(c-a)(c-b)} = 1.$$

Ex. 9. Simplify

$$\frac{a^2}{(a-b)(a-c)(x+a)} + \frac{b^2}{(b-a)(b-c)(x+b)} + \frac{c^2}{(c-a)(c-b)(x+c)}. \text{ P. U. 1875.}$$

The given expression

$$= \frac{-a^2(b-c)(x+b)(x+c) - b^2(c-a)(x+c)(x+a) - c^2(a-b)(x+a)(x+b)}{(a-b)(b-c)(c-a)(x+a)(x+b)(x+c)} \quad (1)$$

Now simplify the numerator, we easily find that

$$-a^2(b-c)(x+b)(x+c) = -a^2(b-c)x^2 - a^2(b^2 - c^2)x - a^2bc(b-c), \quad (1)$$

$$-b^2(c-a)(x+c)(x+a) = -b^2(c-a)x^2 - b^2(c^2 - a^2)x - ab^2c(c-a), \quad (2)$$

$$-c^2(a-b)(x+a)(x+b) = -c^2(a-b)x^2 - c^2(a^2 - b^2)x - abc(c-a-b). \quad (3)$$

In adding (1), (2) and (3) column by column, observe that the 2nd column = $-\{a^2(b^2 - c^2) + b^2(c^2 - a^2) + c^2(a^2 - b^2)\}x - o$, and „ 3rd „ = $-abc\{a(b-c) + b(c-a) + c(a-b)\} = o$.

$$\therefore \text{the numr. of } (A) = -\{a^2(b-c) + b^2(c-a) + c^2(a-b)\}x \\ = (a-b)(b-c)(c-a)x^2$$

$$\therefore \text{the given expn.} = \frac{x^2}{(x+a)(x+b)(x+c)} \quad Ans$$

Ex. 10 Find the value of

$$\frac{b+c-a}{(b+c)(c-a)(a-b)} + \frac{c+a-b}{(c+a)(a-b)(b-c)} + \frac{a+b-c}{(a+b)(b-c)(c-a)} \\ = \frac{b+c-a}{(b+c)(c-a)(a-b)} = \frac{b+c}{(b+c)(c-a)(a-b)} - \frac{a}{(b+c)(c-a)(a-b)} \\ = \frac{1}{(c-a)(a-b)} - \frac{a}{(b+c)(c-a)(a-b)}$$

Similarly breaking up the other fractions, we find the whole expn

$$= \left\{ \frac{1}{(c-a)(a-b)} + \frac{1}{(a-b)(b-c)} + \frac{1}{(b-c)(c-a)} \right\} \\ - \left\{ \frac{a}{(b+c)(c-a)(a-b)} + \frac{b}{(c+a)(a-b)(b-c)} + \frac{c}{(a+b)(b-c)(c-a)} \right\}.$$

Now, it is easy to see that $\frac{1}{(c-a)(a-b)} + \&c. = 0$

$$\therefore \text{the required value} = - \left\{ \frac{a}{(b+c)(c-a)(a-b)} + \&c \right\} \\ = - \frac{a(c+a)(a+b)(b-c) + \&c}{(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)}.$$

• To simplify the numerator of the last result, observe that

$$a(c+a)(a+b)(b-c) = a^2(b-c) + a^2(b^2 - c^2) + abc(b-c),$$

$$b(a+b)(b+c)(c-a) = b^2(c-a) + b^2(c^2 - a^2) + abc(c-a),$$

$$\text{and } c(b+c)(c+a)(a-b) = c^2(a-b) + c^2(a^2 - b^2) + abc(a-b).$$

Adding up these three results column by column, and observing that the 2nd and 3rd columns are each zero, we find

$$\text{the sum} = a^3(b-c) + b^3(c-a) + c^3(a-b).$$

$$= -(a+b+c)(b-c)(c-a)(a-b). \quad [\text{Ex. 2, Page 97.}]$$

$$\therefore \text{the reqd. value} = \frac{-(a+b+c)(b-c)(c-a)(a-b)}{(b^3-c^3)(c^3-a^3)(a^3-b^3)}$$

$$= \frac{a+b+c}{(b+c)(c+a)(a+b)}. \quad \text{Ans.}$$

$$\text{Ex. 11. If } x = \frac{b^2+c^2-a^2}{2bc}, y = \frac{c^2+a^2-b^2}{2ca}, z = \frac{a^2+b^2-c^2}{2ab},$$

show that $(b+c)x + (c+a)y + (a+b)z = a+b+c$.

$$\begin{aligned} (b+c)x &= b \times \frac{b^2+c^2-a^2}{2bc} + c \times \frac{b^2+c^2-a^2}{2bc} \\ &= \frac{b^2+c^2-a^2}{2b} + \frac{b^2+c^2-a^2}{2c}, \end{aligned}$$

$$\begin{aligned} (c+a)y &= c \times \frac{c^2+a^2-b^2}{2ca} + a \times \frac{c^2+a^2-b^2}{2ca} \\ &= \frac{c^2+a^2-b^2}{2a} + \frac{c^2+a^2-b^2}{2c}, \end{aligned}$$

$$\begin{aligned} (a+b)z &= a \times \frac{a^2+b^2-c^2}{2ab} + b \times \frac{a^2+b^2-c^2}{2ab} \\ &= \frac{a^2+b^2-c^2}{2b} + \frac{a^2+b^2-c^2}{2a}. \end{aligned}$$

\therefore adding and arranging terms, $(b+c)x + (c+a)y + (a+b)z$

$$\begin{aligned} &= \left(\frac{c^2+a^2-b^2}{2a} + \frac{a^2+b^2-c^2}{2a} \right) + \left(\frac{b^2+c^2-a^2}{2b} + \frac{a^2+b^2-c^2}{2b} \right) \\ &\quad + \left(\frac{b^2+c^2-a^2}{2c} + \frac{c^2+a^2-b^2}{2c} \right) \end{aligned}$$

$$= \frac{2a^2}{2a} + \frac{2b^2}{2b} + \frac{2c^2}{2c} = a+b+c. \quad \text{Ans.}$$

EXAMPLES 65.

Simplify

$$1. \frac{\frac{1}{c^2} + \frac{1}{d^2}}{\frac{1}{c^2} - \frac{1}{d^2}} \div \frac{\frac{1}{c} + \frac{1}{d}}{\frac{1}{c} - \frac{1}{d}}.$$

$$2. \left\{ \frac{\frac{x}{y} + 2}{\frac{x}{y} + 1} + \frac{x}{y} \right\} \div \left\{ \frac{x}{y} + 2 - \frac{\frac{x}{y}}{\frac{x}{y} + 1} \right\}.$$

3. $\frac{\left(\frac{a}{b}+2\right)\left(\frac{b}{a}+1\right)}{\frac{a}{b}+\frac{b}{a}+1} + \frac{\frac{a}{b}+1}{\frac{1}{3}\left(\frac{a^3}{b^3}-1\right)} \cdot 4. \frac{\frac{ac}{c^2-d^2}+\frac{b}{c-d}-\frac{a}{c+d}}{\frac{ac}{a^2-b^2}+\frac{d}{a-b}-\frac{c}{a+b}}$
5. $\frac{\frac{x}{y+z} \div \frac{1+\frac{x+y}{x+y}}{\frac{y+z}{x+y}}}{1+\frac{y+z}{x+y} \div \frac{1+\frac{x+y}{x+y}}{\frac{y+z}{x+y}}}$ 6. $\frac{\frac{a^3+b^3}{b-a} \cdot a}{\frac{1}{b}-\frac{1}{a}} \times \frac{a^2-b^2}{a^3+b^3}$
7. $\frac{a^3-b^3}{a^3+b^3} \div \frac{(a+b)^3-ab^3}{(a-b)^3+ab^3} \div \frac{a-b}{a+b}$ 8. $\frac{l^2+m^2-n^2+2lm}{n^2-l^2-m^2+2lm} \div \frac{l+m+n}{m+n-l}$
9. $\frac{x^3-(y-z)^2}{(y+z)^3-x^3} \times \frac{y^3-(z-x)^2}{(z+x)^2-y^3} \div \frac{(x+y)^2-z^2}{z^4-(x-y)^2}$
10. $\frac{(x+2)^3+4}{(x+1)^2+1} \div \frac{(x-1)^2+1}{(x-2)^2+4} \times \frac{x^4+4}{x^4+64}$
11. $\frac{x^3+2xy+y^3}{x^3+2zx+x^2} \div \frac{x^3-y^3}{x^2-z^2} \div \frac{yz-x(y-z)+x^2}{yz+x(y-z)-x^2}$
12. $\frac{a^3+ab+b^3}{(a+b)^3} \times \frac{a^2-ab+b^2}{(a-b)^2} \times \frac{a^4-2a^2b^2+b^4}{a^4+a^2b^2+b^4}$
13. $\frac{x^3+y^3}{(x+y)^2} \div \frac{(x+y)^4}{x^3-y^3} \times \frac{x^4-y^4}{(x^2+y^2)^3} \div \frac{(x^4+y^4)^2}{x^8-y^8} \times \frac{x^{10}-y^{10}}{x^8+y^8}$
14. $\frac{\left\{ \frac{x^3+3x+2}{y^3+3y+2} \times \frac{x^3-3x+2}{y^3-3y+2} \right\}}{\left\{ \frac{x^2+x-2}{y^2+y-2} \times \frac{x^3-x-2}{y^3-y-2} \right\}}$
15. $\frac{b^3-c^3-3bc(b-c)}{a^2+b^2-2ab} \times \frac{c^3-a^3-3ca'(c-a)}{b^2+c^2-2bc} \times \frac{a^3-b^3-3ab(a-b)}{c^2+a^2-2ca}$
16. $\frac{x^3+y^3+z^3-3xyz}{(x-y)^2+(y-z)^2+(z-x)^2} \times \frac{2(x-y)}{x+y+z} \div \frac{x-2y}{x^2-3xy+2y^2}$
17. $\frac{(x+y)^3+(y+z)^3+(z+x)^3+x^3+y^3+z^3}{(x+y+z)^3} \times \frac{x+y+z}{x^2+y^2+z^2}$
18. $\frac{x^3}{y^3} \left(1 + \frac{y^3}{x^3} \right) \left(1 - \frac{y^3}{x^3} \right) \div \left\{ \left(1 + \frac{y}{x} + \frac{y^2}{x^2} \right) \left(1 - \frac{x}{y} + \frac{x^2}{y^2} \right) \right\}$
19. $\left(\frac{a}{b} - \frac{b}{a} + \frac{b}{c} - \frac{c}{b} + \frac{c}{a} - \frac{a}{c} \right) \div \{ (a-b)(b-c)(c-a) \}$
20. $\left(\frac{a^2}{b} - \frac{b^2}{a} + \frac{b^2}{c} - \frac{c^2}{b} + \frac{c^2}{a} - \frac{a^2}{c} \right) \div \{ (a+b+c)(b-c)(c-a) \}$

$$21. \left(\frac{a^3}{b} - \frac{b^3}{a} + \frac{b^3}{c} - \frac{c^3}{b} + \frac{c^3}{a} - \frac{a^3}{c} \right) \div \frac{(a+b)^2 + (b+c)^2 + (c+a)^2}{abc}.$$

$$22. \left(\frac{a}{b^2} - \frac{b}{a^2} + \frac{b}{c^2} - \frac{c}{b^2} + \frac{c}{a^2} - \frac{a}{c^2} \right) \div \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

$$23. \left(\frac{a^2}{b^2} - \frac{b^2}{a^2} + \frac{b^2}{c^2} - \frac{c^2}{b^2} + \frac{c^2}{a^2} - \frac{a^2}{c^2} \right) \div \frac{(a+b)(b+c)(c+a)}{a^2 b^2 c^2}.$$

$$24. \frac{a^2 \left(\frac{1}{b} - \frac{1}{c} \right) + b^2 \left(\frac{1}{c} - \frac{1}{a} \right) + c^2 \left(\frac{1}{a} - \frac{1}{b} \right)}{a \left(\frac{1}{b} - \frac{1}{c} \right) + b \left(\frac{1}{c} - \frac{1}{a} \right) + c \left(\frac{1}{a} - \frac{1}{b} \right)}.$$

$$25. \frac{(a-b)(b-c) + (b-c)(c-a) + (c-a)(a-b)}{a(c-a) + b(a-b) + c(b-c)}.$$

$$26. \frac{x-12 + \frac{45}{x+2}}{x+9 + \frac{12}{x+2}} \times \frac{x+11 + \frac{21}{x+1}}{x-12 + \frac{36}{x+1}} \div \frac{x+2 - \frac{18}{x-1}}{x-1 - \frac{49}{x-1}}.$$

$$27. \frac{9a^2b^2}{16(x+y)} \div \left[\frac{3a(x-y)}{7(c+d)} \div \left\{ \frac{4(c-d)}{21ab^2} \div \frac{c^2-a^2}{4(1^2-y^2)} \right\} \right].$$

$$28. \left(1 - \frac{1}{1+x} \right) \left(x + \frac{1}{2+x} \right) \times \frac{x^{\frac{1}{2}-x}}{1+\frac{1}{x}} \div \left(1 + x + \frac{1}{x} \right).$$

$$29. \frac{\frac{a+b+c}{a+b-c} + \frac{c+a-b}{b+c-a} + \frac{a+b+c}{a+b-c}}{\frac{a+b}{c+a-b} + \frac{c}{a+b+c} + \frac{b+c-a}{c+a-b}} \div \frac{a+b-c}{b+c-a}.$$

$$30. \frac{\frac{1}{x} - \frac{x+a}{x^2+a^2}}{\frac{1}{a} - \frac{a+x}{a^2+x^2}} + \frac{\frac{1}{x} - \frac{x-a}{x^2+a^2}}{\frac{1}{a} - \frac{a-x}{a^2+x^2}}.$$

$$31. \frac{1 - \frac{2}{1 + \frac{a}{b}}}{1 - \frac{3}{2 + \frac{a}{b}}} \div \left(1 + \frac{b}{a+b} \right).$$

$$32. \frac{1 + \frac{a-bx}{c+bx}}{x-2(a+c)} - \frac{1 + \frac{a-bx}{c+bx}}{x-2(a+c)} \div \frac{1 + \frac{x}{a+c-x}}{1 + \frac{x}{a+c-x}}.$$

$$33. \frac{\frac{a}{a-b} - \frac{a}{a+b}}{\frac{b}{a-b} - \frac{b}{a+b}} \div \frac{\frac{a+b}{a-b} + \frac{a-b}{a+b}}{\frac{a+b}{a-b} - \frac{a-b}{a+b}} \times \frac{a^2}{a^2+b^2}.$$

$$34. \frac{a^2b - b^4}{ab^2 + a^2b} \div \left\{ \frac{a^4 + a^2b + a^2b^2}{(a^2 - b^2)^2} \times \left(1 + \frac{b}{a} \right) \right\}.$$

$$35. \left\{ \frac{1}{q} + \frac{10}{3p-q} - \frac{b}{p-3q} \right\} \div \left\{ \frac{3(p+q)}{3p-q} - \frac{p-q}{p-q-\frac{8q^2}{p+q}} \right\}.$$

$$36. \left(\frac{mx^2 - my^2 + 2mxy}{x^2 + y^2} \right)^2 + \left(\frac{ny^2 - nx^2 + 2nxy}{x^2 + y^2} \right)^2.$$

$$37. \frac{(x+1)^3(x+2)^2 - (x-1)^3(x-2)^2}{(x+1)^5 + x^5 + (x-1)^5}.$$

$$38. \frac{(a^4 - b^4)^2 + 2a^6b^2 + 5a^4b^4 + 2a^2b^6}{(a^3 + ab + b^3)^2(a^2 - ab + b^2)^2}.$$

$$39. \frac{(a-b)^3 + (b-c)^3 + (c-a)^3}{(a-b)(b-c)(c-a)}. \quad 40. \frac{a^3(b-c) + b^3(c-a) + c^3(a-b)}{a^2(b-c) + b^2(c-a) + c^2(a-b)}.$$

$$41. \frac{(b+c)^3(b-c) + (c+a)^3(c-a) + (a+b)^3(a-b)}{(b+c)^2(b-c) + (c+a)^2(c-a) + (a+b)^2(a-b)}.$$

$$42. \frac{(b+c)(b^2-c^2) + (c+a)(c^2-a^2) + (a+b)(a^2-b^2)}{(b-c)^3 + (c-a)^3 + (a-b)^3}.$$

$$43. \frac{(b^3-c^3)^3 + (c^3-a^3)^3 + (a^3-b^3)^3}{(b-c)^3 + (c-a)^3 + (a-b)^3} - \{a^2(b+c) + b^2(c+a) + c^2(a+b)\}.$$

$$44. \frac{(b+2c-3a)^7 + (c+2a-3b)^7 + (a+2b-3c)^7}{(b+2c-3a)(c+2a-3b)(a+2b-3c)}.$$

$$45. \frac{8(a+b+c)^3 - (b+c)^3 - (c+a)^3 - (a+b)^3}{3(2a+b+c)(a+2b+c)(a+b+2c)}.$$

$$46. \frac{9y^2 - (4x-2x)^2}{(2x+3y)^2 - 10x^2} + \frac{16x^2 - (2x-3y)^2}{(3y+4x)^2 - 4x^2} + \frac{4x^2 - (3y-4x)^2}{(4x+2x)^2 - 9y^2}.$$

$$47. \frac{ab}{(c-a)(c-b)} + \frac{bc}{(a-b)(a-c)} + \frac{ca}{(b-c)(b-a)}.$$

$$48. \frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-c)(b-a)} + \frac{1}{c(c-a)(c-b)}.$$

$$49. \frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-c)(b-a)} + \frac{c^2}{(c-a)(c-b)}.$$

$$50. \frac{a+b}{(c-a)(c-b)} + \frac{b+c}{(a-c)(a-b)} + \frac{c+a}{(b-a)(b-c)}.$$

51. $\frac{1}{a^3(a-b)(a-c)} + \frac{1}{b^3(b-a)(b-c)} + \frac{1}{c^3(c-a)(c-b)}$
52. $\frac{a^2-bc}{(a-b)(c-a)} + \frac{b^2-ca}{(b-c)(a-b)} + \frac{c^2-ab}{(c-a)(b-c)}$
53. $\frac{a^2-bc}{(a+b)(a+c)} + \frac{b^2-ca}{(b+c)(b+a)} + \frac{c^2-ab}{(c+a)(c+b)}$
54. $\frac{a^2+a+1}{(c-a)(a-b)} + \frac{b^2+b+1}{(a-b)(b-c)} + \frac{c^2+c+1}{(b-c)(c-a)}$
55. $\frac{a^2-bc}{(a-b)(a-c)} + \frac{b^2+ca}{(b+c)(b-a)} + \frac{c^2+ab}{(c-a)(c+b)}$
56. $\frac{a}{(a+b)(a-c)} - \frac{b}{(a+b)(b+c)} + \frac{c}{(c-a)(b+c)}$
57. $\frac{a+b}{c(c-a)(c-b)} + \frac{b+c}{a(a-b)(a-c)} + \frac{c+a}{b(b-c)(b-a)}$
58. $\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-a)(b-c)} + \frac{c^3}{(c-a)(c-b)}$
59. $\frac{b^3+c^2-2a^2}{(a-b)(a-c)} + \frac{c^2+a^2-2b^2}{(b-c)(b-a)} + \frac{a^2+b^2-2c^2}{(c-a)(c-b)}$
60. $\frac{a^3-(b-c)^2}{(a-b)(a-c)} + \frac{b^3-(c-a)^2}{(b-a)(b-c)} + \frac{c^3-(a-b)^2}{(c-a)(c-b)}$
61. $\frac{1}{(a+b)(a-c)(x-a)} + \frac{1}{(b+c)(b+a)(x+b)} + \frac{1}{(c-a)(c+b)(x-c)}$
62. $\frac{a}{(a-b)(a-c)(x+a)} + \frac{b}{(b-a)(b-c)(x+b)} + \frac{c}{(c-a)(c-b)(x+c)}$
63. $\frac{(la+m)^2}{(a-b)(a-c)} + \frac{(lb+m)^2}{(b-a)(b-c)} + \frac{(lc+m)^2}{(c-a)(c-b)}$
64. $\frac{(a+1)^3}{(a-b)(a-c)} + \frac{(b+1)^3}{(b-a)(b-c)} + \frac{(c+1)^3}{(c-a)(c-b)}$
65. $\frac{a(b-c)^2}{(a-b)(a-c)} + \frac{b(c-a)^2}{(b-a)(b-c)} + \frac{c(a-b)^2}{(c-a)(c-b)}$
66. $\frac{(1+ab)(1+ac)}{(a-b)(a-c)} + \frac{(1+bc)(1+ac)}{(b-a)(b-c)} + \frac{(1+ca)(1+bc)}{(c-b)(c-a)}$
67. $\frac{1}{a^3(a^2-b^2)(a^2-c^2)} + \frac{1}{b^3(b^2-a^2)(b^2-c^2)} + \frac{1}{c^3(c^2-a^2)(c^2-b^2)}$

$$68. \frac{(a+m)^2}{(a-b)(a-c)(x+a)} + \frac{(b+m)^2}{(b-a)(b-c)(x+b)} + \frac{(c+m)^2}{(c-a)(c-b)(x+c)}.$$

$$69. \frac{(b+c-a)^2}{(c-a)(a-b)} + \frac{(c+a-b)^2}{(a-b)(b-c)} + \frac{(a+b-c)^2}{(b-c)(c-a)}.$$

$$70. \frac{a^2(b+c)^2}{(c-a)(a-b)} + \frac{b^2(c+a)^2}{(a-b)(b-c)} + \frac{c^2(a+b)^2}{(b-c)(c-a)}.$$

$$71. \frac{b+c-ma}{(b+c)(c-a)(a-b)} + \frac{c+a-mb}{(c+a)(a-b)(b-c)} + \frac{a+b-mc}{(a+b)(b-c)(c-a)}.$$

$$72. \frac{1}{(a-b)(a-c)(a-d)} + \frac{1}{(b-a)(b-c)(b-d)} + \frac{1}{(c-a)(c-b)(c-d)}.$$

$$73. \frac{a^2}{(a-b)(a-c)(a-d)} + \frac{b^2}{(b-c)(b-a)(b-d)} + \frac{c^2}{(c-a)(c-b)(c-d)}.$$

$$74. \frac{bcd}{(a-b)(a-c)(a-d)} + \frac{cda}{(b-c)(b-a)(b-d)} + \frac{dab}{(c-a)(c-b)(c-d)}.$$

Find the product of

$$75. \frac{x^2}{a^2} + \frac{2xy}{ab} + \frac{y^2}{b^2} \text{ by } \frac{a^2x^2}{b^2y^2} + \frac{3ax}{by} + 1.$$

$$76. \frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{a}{b} + \frac{b}{a} \text{ by } \frac{a^2}{b^2} + \frac{b^2}{a^2} - \frac{a}{b} - \frac{b}{a}.$$

Divide

$$77. 2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - \frac{5xy}{ab} - x - \frac{ay}{b} - a^2 \text{ by } \frac{2y}{b} - \frac{1}{a} + a.$$

$$78. \frac{a^4}{b^4} + \frac{b^4}{c^4} - \frac{c^4}{a^4} + \frac{2a^2}{c^2} - \frac{2c^2}{a^2} - 1 \text{ by } \frac{a^2}{b^2} + \frac{b^2}{c^2} - \frac{c^2}{a^2} - 1.$$

Find the value of

$$79. \frac{x+a}{b-x} + \frac{x-a}{b+x} - \frac{ab}{b^2-x^2}, \text{ when } x = \frac{ab}{2(a+b)}.$$

$$80. \frac{a}{x-a} + \frac{c}{x-c}, \text{ when } x = \frac{2ac}{a+c}$$

$$81. \frac{a^2(b-c)}{a-x} - \frac{b^2(a-c)}{b-x}, \text{ when } x = 1 \div \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right).$$

$$82. \frac{yz}{bc} + \frac{zx}{ca} + \frac{xy}{ab}, \text{ when } x = \frac{b+c}{b-c}, \quad y = \frac{c+a}{c-a}, \quad z = \frac{a+b}{a-b}.$$

$$83. \frac{x+y}{1-xy}, \text{ when } x = \frac{b^2+c^2-a^2}{2bc}, \text{ and } y = \frac{(a+b-c)(a-b+c)}{(a+b+c)(b+c-a)}.$$

$$84. (p+1)(q+1), \text{ when } p = \frac{a^2+b^2-c^2}{2ba}, \text{ and } q = \frac{(b+c-a)(a-b+c)}{(a+b+c)(a+b-c)}.$$

Shew that

$$85. x+y+z+xyz=0, \text{ if } ax=b-c, by=c-a, cz=a-b.$$

$$86. \left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = n(n-1), \text{ when } x^2 = \frac{n-1}{n+1}.$$

$$87. a^2+b^2+c^2=4+ab^2c, \text{ if } y^2+z^2=ayz, z^2+x^2=bzx, x^2+y^2=cxy.$$

$$88. \frac{bc}{(x-b)(x-c)} + \frac{ca}{(x-c)(x-a)} + \frac{ab}{(x-a)(x-b)} = 0, \\ \text{when } \frac{1}{x} = \frac{1}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

$$89. \frac{x+yz}{bc} = \frac{y+zx}{ca} = \frac{z+xy}{ab},$$

$$\text{if } x = \frac{b^2+c^2-a^2}{2bc}, y = \frac{c^2+a^2-b^2}{2ca}, z = \frac{a^2+b^2-c^2}{2ab}.$$

$$90. \frac{(x^2+1)(y^2+1)(z^2+1)}{(xy+1)(yz+1)(zx+1)} = \frac{(a^2+1)(b^2+1)(c^2+1)}{(ab+1)(bc+1)(ca+1)},$$

$$\text{if } x = \frac{a+1}{a-1}, y = \frac{b+1}{b-1}, z = \frac{c+1}{c-1}.$$

$$91. \left(\frac{1}{x} + \frac{1}{y}\right) \left(\frac{1}{y} + \frac{1}{z}\right) \left(\frac{1}{z} + \frac{1}{x}\right) = abc, \text{ being given that} \\ (b+c-a)x = (c+a-b)y = (a+b-c)z = 2.$$

$$92. \left(x - \frac{1}{x}\right) \left(y - \frac{1}{y}\right) = 4 \frac{xy+1}{x-y}, \text{ if } y = \frac{1+x}{1-x}.$$

$$93. a^2 - \left(\frac{c^2+a^2-b^2}{2c}\right)^2 = \frac{4}{c^2} s(s-a)(s-b)(s-c),$$

$$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{abc}{s(s-a)(s-b)(s-c)}, \text{ and}$$

$$\frac{s-a}{(s-b)(s-c)} + \frac{s-b}{(s-c)(s-a)} + \frac{s-c}{(s-a)(s-b)} = \frac{a^2+b^2+c^2-s^2}{(s-a)(s-b)(s-c)}, \\ \text{if } 2s = a+b+c.$$

Shew that identically

$$94. \left(1 + \frac{b}{c}\right) \left(1 + \frac{c}{a}\right) \left(1 + \frac{a}{b}\right) = (a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 1.$$

$$95. \frac{q-r}{a+qr} + \frac{r-p}{a+pr} + \frac{p-q}{d+pq} = a \frac{q-r}{a+qr} + \frac{r-p}{a+pr} + \frac{p-q}{d+pq}.$$

$$96. \left(\frac{1}{y-z} + \frac{1}{z-x} + \frac{1}{x-y}\right)^2 = \frac{1}{(y-z)^2} + \frac{1}{(z-x)^2} + \frac{1}{(x-y)^2},$$

$$97. \frac{(a-h)(a-l)}{(a-b)(a-c)} + \frac{(b-h)(b-l)}{(b-a)(b-c)} + \frac{(c-h)(c-l)}{(c-a)(c-b)} = 1.$$

$$98. b \cdot \frac{(a-h)(a-l)}{(a-b)(a-c)} + ca \cdot \frac{(b-h)(b-l)}{(b-a)(b-c)} + ab \cdot \frac{(c-h)(c-l)}{(c-a)(c-b)} = h/l.$$

$$99. a^2 \cdot \frac{(a+b)(a+c)}{(a-b)(a-c)} + b^2 \cdot \frac{(b+c)(b+a)}{(b-c)(b-a)} + c \cdot \frac{(c+a)(c+b)}{(c-a)(c-b)} = (a+b+c)^2$$

$$100. \frac{1}{x(x+1)} + \frac{2}{(x+1)(x+3)} + \frac{3}{(x+3)(x+6)} + \frac{4}{(x+6)(x+10)} \\ + \frac{5}{(x+10)(x+15)} = \frac{1+2+3+4+5}{x(x+15)}.$$

$$101. \frac{1}{(1+a)(1+a^2)} + \frac{a(1+a)}{(1+a^2)(1+a^4)} + \frac{a^2(1+a)(1+a^2)}{(1+a^4)(1+a^8)} \\ = \frac{1}{1-a} \left(\frac{1}{1+a} - \frac{a^7}{1+a^8} \right).$$

$$102. \left(\frac{y+z-2x}{y-z} + \frac{z+x-2y}{z-x} + \frac{x+y-2z}{x-y} \right) \\ \times \left(\frac{y-z}{y+z-2x} + \frac{z-x}{z+x-2y} + \frac{x-y}{x+y-2z} \right) = 9$$

CHAPTER XXI.

DIVISIBILITY.

117. **Definition.** The expression, $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n$, is called a rational and integral algebraic expression in x , provided n be a positive integer, and the coefficients, p_0, p_1, p_2 &c do not contain x .

118. **Remainder Theorem** If any rational and integral expression in x be divided by $x-a$, the remainder is found by putting a for x in the given expression.

Let the given expression be $px^n + qx^{n-1} + rx^{n-2} + \dots$, and let it be divided by $x-a$, giving Q as the quotient and R as the remainder, so that R does not contain x . To prove that $R = pa^n + qa^{n-1} + ra^{n-2} + \dots$.

Since dividend = divisor \times quotient + remainder,

$$px^n + qx^{n-1} + rx^{n-2} + \dots = Q(x-a) + R, \text{ identically.} \quad (A).$$

Since R does not contain x , it remains unchanged, whatever value be given to x .

Putting a for x in (A), we have

$$pa^n + qa^{n-1} + ra^{n-2} + \dots = (a-a)Q + R = R.$$

N. B. The student is advised to verify by actual division that the remainders left after dividing $px^2 + qx + r$ and $px^3 + qx^2 + rx + s$ by $x-a$ are respectively $pa^2 + qa + r$ and $pa^3 + qa^2 + ra + s$.

119. Divisibility. If a rational and integral expression in x vanishes identically when $x=a$, then will the expression be exactly divisible by $x-a$.

To prove that $px^n + qx^{n-1} + rx^{n-2} + \dots$ is exactly divisible by $x-a$, if $pa^n + qa^{n-1} + ra^{n-2} + \dots = 0$.

This theorem readily follows from the last. For the remainder on division = $pa^n + qa^{n-1} + ra^{n-2} + \dots$; hence, if the last expression be zero, the given expression will be divisible by $x-a$.

N. B. Some important inferences from the preceding theorems are illustrated by the following examples.

Ex. 1. Deduce immediately from Art. 119 that $x^3 - 3x - 2$ has $x+1$ for a factor, and that $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$ is divisible by each of $b-c$, $c-a$ and $a-b$.

Put $x = -1$ in $x^3 - 3x - 2$.

$$\text{Now, } (-1)^3 - 3(-1) - 2 = -1 + 3 - 2 = 0;$$

$\therefore x^3 - 3x - 2$ is exactly divisible by $x - (-1)$;

i. e., $x^3 - 3x - 2$ has $x+1$ as a factor. (Verify).

Put $b=c$ in $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$; then

$$\begin{aligned} \text{the resulting expression} &= a \times 0 + c(c-a)^3 + c(a-c)^3 \\ &= c(c-a)^3 + c\{-(c-a)\}^3 \\ &= c(c-a)^3 - c(c-a)^3 \\ &= 0. \end{aligned}$$

\therefore by Art. 119, $a(b-c)^3 + \&c.$ is divisible by $b-c$.

Similarly, putting $c=a$, and $a=b$ successively, the expression is proved to be divisible by each of $c-a$ and $a-b$.

Ex. 2. Find out without actual division the value of a for which $ax^3 - 2x^2 + 3x - 6$ is exactly divisible by $x - 2$?

By Art. 119, we must have $a \cdot 2^3 - 2 \cdot 2^2 + 3 \cdot 2 - 6 = 0$;

simplifying, $8a - 8 = 0$;

transposing, $8a = 8$;

$\therefore a = 1$. *Ans*

EXAMPLES 66.

Without actual division show that the following expressions are exactly divisible :

1. $x^2 - 9x + 14$ by $x - 2$.
2. $x^3 + x^2 - 2$ by $x - 1$.
3. $8x^2 + 13x - 6$ by $x + 2$.
4. $x^3 + 7x^2 - 36$ by $x + 6$.
5. $5x^3 - 8xy^2 + 3y^3$ by $x - y$.
6. $8x^3 + 10xy + 3y^3$ by $2x + y$.
7. $2x^4 + 3x^3 + 2x^2 - 3x - 4$ by $x - 1$ and $x + 1$.
8. $a^3(b - c) + b^3(c - a) + c^3(a - b)$ by each of $a - b$, $b - c$ and $c - a$.
9. $(b - c)^2 + (c - a)^2 + (a - b)^2$ by each of $a - b$, $b - c$ and $c - a$.
10. $a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) + 2abc$ by each of $a + b$, $b + c$, $c + a$.

Find without actual division the remainder, if any, on dividing the following expressions :

11. $3x^3 + 12x - 3$ by $x - 5$
12. $7x^5 + x^3 - 3x + 5$ by $x + 3$.

Find for what value of a the following are exactly divisible :

13. $ax^4 - x^3 - 2x^2 - 3x - 10$ by $x - 2$.
14. $x^6 - x^4 - ax^3 + 3x^2 + 2ax + 1$ by $x + 1$.

120. Important Propositions. The following propositions are of frequent use, and should be carefully remembered

When n is a positive integer,

- (i) $x^n - y^n$ is **always** divisible by $x - y$;
- (ii) $x^n - y^n$ is divisible by $x + y$, if n be **even**, but **not** if n be **odd**;
- (iii) $x^n + y^n$ is divisible by $x + y$, if n be **odd**, but **not** if n be **even**;
- (iv) $x^n + y^n$ is **never** divisible by $x - y$.

(1) To shew that $x^n - y^n$ is always divisible by $x - y$.

Let $x^n - y^n$ be divided by $x - y$, giving Q as the quotient and R as the remainder, so that R does not contain x .

Then, $x^n - y^n = Q(x - y) + R$, identically.

(2) To prove that $x^n - y^n$ is divisible by $x + y$, when n is even, but not when n is odd.

With the same notation as before,

$$x^n - y^n = Q(x + y) + R.$$

Since R does not contain x , it will remain unchanged whatever be the value given to x .

Put $x = -y$ in the last equation.

Thus, $(-y)^n - y^n = Q \times 0 + R = R$.

Now, when n is even, $(-y)^n - y^n = y^n - y^n = 0$, $[(\cdot y)^4 = y^4]$,

and " " " odd, $(-y)^n - y^n = -y^n - y^n = -2y^n$.

$$[(-y)^3 = -y^3].$$

Hence $R = 0$, when n is even, but not when n is odd, there being in the latter case a remainder $= -2y^n$.

$\therefore x^n - y^n$ is divisible by $x + y$, when n is even, but not when n is odd.

Otherwise thus :

Divide $x^n - y^n$ by $x + y$ to two terms.

$$\begin{array}{r} x + y \overline{) x^n - y^n} \quad (x^{n-1} - x^{n-2}y \\ \underline{x^n + x^{n-1}y} \\ -x^{n-1}y - y^n \\ \underline{-x^{n-1}y - x^{n-2}y^2} \\ x^{n-2}y^2 - y^n = y^2(x^{n-2} - y^{n-2}). \end{array}$$

$$\text{Thus } \frac{x^n - y^n}{x + y} = x^{n-1} - x^{n-2}y + y^2 \frac{x^{n-2} - y^{n-2}}{x + y}.$$

It is now evident that $x^n - y^n$ will or will not be divisible by $x + y$, according as $x^{n-2} - y^{n-2}$ is or is not divisible by $x + y$. That is, according as the theorem does or does not hold for any given index $(n-2)$ of the powers of x and y , it will or will not hold when the index is n , i.e., is increased by 2.

1st case. We know that $x^2 - y^2$ is divisible by $x + y$.

$$\therefore x^{2+2} - y^{2+2} \text{ or } x^4 - y^4 \text{ " " " " " " " "}$$

Again, $x^4 - y^4$ is divisible by $x + y$; [Proved]

$$\therefore x^{4+2} - y^{4+2}, \text{ or } x^6 - y^6 \text{ is divisible by } x + y.$$

Similarly, $x^6 - y^6$ being divisible, [Proved]

$x^8 - y^8$ is also divisible, and so on.

Hence the divisibility holds when the index is any one of the numbers, 2, 4, 6, 8, 10, 12, &c., i.e., in short, any even number.

$\therefore x^n - y^n$ is divisible by $x + y$, when n is even.

2nd case. Evidently $x-y$ is not divisible by $x+y$;

$\therefore x^{1+2}-y^{1+2}$, or x^3-y^3 " " " " " "

Again, since x^3-y^3 is not divisible by $x+y, \dots$. Proved.

$\therefore x^{3+2}-y^{3+2}$ or x^5-y^5 is not divisible by $x+y$.

Similarly, x^5-y^5 not being divisible, x^7-y^7 is also not divisible, and so on.

Thus the divisibility does not hold when the index is any one of the numbers 1, 3, 5, 7, 9, 11, &c., i. e., in short, any odd number.

Hence x^n-y^n is not divisible by $x+y$, when n is odd.

N. B. Thus x^n-y^n is divisible by $x+y$, as well as by $x-y$, when n is even, for x^n-y^n is always divisible by $x-y$. Therefore, n being even, x^n-y^n is divisible by $(x+y)(x-y)$ or x^2-y^2 .

(3) To prove that x^n+y^n is divisible by $x+y$ when n is odd, but not when n is even.

With the same notation as before,

$$x^n+y^n=(x+y)Q+R.$$

Since R does not contain x , it will remain unchanged whatever value be given to x .

Putting $x=-y$, $(-y)^n+y^n=Q \times 0+R=R$.

Now, when n is odd, $(-y)^n+y^n=-y^n+y^n=0$,

and " " " even, $(-y)^n+y^n=y^n+y^n=2y^n$.

$\therefore R=0$ when n is odd, but not so when n is even.

$\therefore x^n+y^n$ is divisible by $x+y$ when n is odd, but not when n is even.

Otherwise thus :

Dividing x^n+y^n to two terms, we have

$$\frac{x^n+y^n}{x+y} = x^{n-1}-x^{n-2}y+y^2-\frac{x^{n-2}+y^{n-2}}{x+y}.$$

The rest of the proof is exactly similar to the last proof in the proposition just preceding, and we leave it as a good exercise for the student.

(4) To prove that x^n+y^n is never divisible by $x-y$.

With the usual notation,

$$x^n+y^n=Q(x-y)+R.$$

Since R does not contain x , it remains unchanged whatever value be given to x .

Putting $x=y$, $2y^n = Q \times 0 + R = R$.

Since R does not vanish for any value of n , $x^n + y^n$ is never divisible by $x-y$.

Otherwise thus:

$$\frac{x^n + y^n}{x-y} = x^{n-1} + y \cdot \frac{x^{n-1} + y^{n-1}}{x-y}.$$

It now follows that $x^n + y^n$ will or will not be divisible, according as $x^{n-1} + y^{n-1}$ is or is not divisible. That is, as the theorem stands for any index, $(n-1)$, so does it stand for the next higher.

Now, we know that $x+y$ is not divisible by $x-y$;

$\therefore x^{1+1} + y^{1+1}$, or $x^2 + y^2$ " " " " "

Again, since $x^2 + y^2$ is not divisible by $x-y$, ... Proved.

$\therefore x^{2+1} + y^{2+1}$, or $x^3 + y^3$ is not divisible by $x-y$.

Similarly, since $x^3 + y^3$ is not divisible, $x^4 + y^4$ is also not divisible, and so on continually. That is, in short, $x^n + y^n$ is never divisible by $x-y$.

N. B. The present proposition very easily follows from Prop. (1). For, $x^n + y^n = (x^n - y^n) + 2y^n$. By Prop. (1), $x^n - y^n$ is always divisible by $x-y$. Hence in dividing $x^n + y^n$ by $x-y$, there will always be a remainder, $2y^n$.

121. Quotients. If the work of division in each case of the preceding article were continued, we would easily infer the following important results:

(1) *Always*, $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + y^{n-1})$.

(2) *n even*, $x^n - y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 + \dots - y^{n-1})$.

(3) *n odd*, $x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 + \dots + y^{n-1})$.

N. B. Observe that the number of terms in each case is n . The sign of each term in the first quotient, viz., $x^{n-1} + x^{n-2}y + \dots + y^{n-1}$, is +; but the sign is alternately + and - in the other two cases. Also note that in each quotient the index of the power of x diminishes and that of y increases continually by unity, so that the sum of the indices in each term is the same, viz., $n-1$.

The sign of the last term is often a puzzle to the beginner. The difficulty will be readily got over, if the student remembers that in the above result the product of the last terms of the factors on the right side must be equal to the last term on the left side. Thus, in the first result $y^n = (-y) \times (y^{n-1})$; i. e., last term of dividend = last term of divisor \times last term of quotient. Hence, last term of quotient = last term of dividend \div last term of divisor. Apply this rule to each case.

122. $(x^n - y^n) \div (x - y)$ as a fundamental case. From this case, we can deduce all the other cases that have already

been established. For example, we propose to deduce that $x^n - y^n$ is divisible by $x + y$ when n is any even positive integer. We are given that

$x^n - y^n$ is divisible by $x - y$, when n is any positive integer. Change y into $-a$. Then *always*

$x^n - (-a)^n$ is divisible by $x - (-a)$ or $x + a$.

When n is even, $x^n - (-a)^n = x^n - a^n$;

therefore, when n is an even positive integer,

$x^n - a^n$ is divisible by $x + a$.

As it is immaterial whether a or any other letter is used, it follows that $x^n - y^n$ is divisible by $x + y$, when n is an even positive integer.

In the same manner, deduce the third theorem of Art. 120, by taking n odd.

As to the last theorem of Art. 120, see Note, Art. 120.

Ex. 1. Find the complete quotient of $(x^9 - a^9) \div (x + a)$.

Since 9 is odd, $x^9 - a^9$ is not exactly divisible by $x + a$. By the Remainder Theorem, the remainder $= (-a)^9 - a^9 = -2a^9$. We, therefore, put $x^9 - a^9 = (x^9 + a^9) - 2a^9$, and apply case (3) of Art. 121.

$$\begin{aligned} \therefore \frac{x^9 - a^9}{x + a} &= \frac{x^9 + a^9}{x + a} - \frac{2a^9}{x + a} \\ &= x^8 - ax^7 + a^2x^6 - \dots + a^8 - \frac{2a^9}{x + a}. \text{ Ans.} \end{aligned}$$

Ex. 2. Assuming that $x^n - y^n$ is divisible by $x - y$ when n is any whole number, show that $(ab)^n - (bc)^n + (cd)^n - (da)^n$ is always divisible by $ab - bc + cd - da$. M. U. 1873.

$$\begin{aligned} (ab)^n - (bc)^n + (cd)^n - (da)^n &= a^n b^n - b^n c^n + c^n d^n - a^n a^n, [(ab)^2 = a^2 b^2] \\ &= b^n (a^n - c^n) + d^n (c^n - a^n) \\ &= b^n (a^n - c^n) - d^n (a^n - c^n) \\ &= (a^n - c^n)(b^n - d^n). \end{aligned}$$

Similarly $ab - bc + cd - da = (a - c)(b - d)$

$$\therefore \frac{(ab)^n - (bc)^n + (cd)^n - (da)^n}{ab - bc + cd - da} = \frac{a^n - c^n}{a - c} \times \frac{b^n - d^n}{b - d}.$$

Now, given that $x^n - y^n$ is divisible by $x - y$.

$\therefore a^n - c^n$ is divisible by $a - c$, and $b^n - d^n$ by $b - d$.

$\therefore (ab)^n - (bc)^n + (cd)^n - (da)^n$ is divisible by $ab - bc + cd - da$, the quotient being equal to the product of the quotients $(a^n - c^n) \div (a - c)$ and $(b^n - d^n) \div (b - d)$.

Ex. 3. Shew that the last digit in $3^{2n+1} + 2^{2n+1}$ is 5, if n be any whole number. M. U. 1868.

$$3^{2n+1} = 3 \times 3 \times 3 \times \dots \text{to } 2n+1 \text{ factors.}$$

Since none of the factors is divisible by 2,

$\therefore 3^{2n+1}$ is an odd number.

$$2^{2n+1} = 2 \times 2 \times 2 \times \dots = \text{an even number.}$$

$$\therefore 3^{2n+1} + 2^{2n+1} = \text{an odd number} + \text{an even number}$$

$$= \text{an odd number. } [9(\text{odd}) + 4(\text{even}) = 13(\text{odd})]$$

Again, since $2n+1$ is odd, $3^{2n+1} + 2^{2n+1}$ is divisible by 3+2 or 5.

Art. 120, Prop. (3).

$\therefore 3^{2n+1} + 2^{2n+1}$ is an odd number divisible by 5.

Now, $\therefore 3^{2n+1} + 2^{2n+1}$ is odd, the last digit must be one of the numbers 1, 3, 5, 7, 9. But since it is divisible by 5, the last digit must be none of the numbers, 1, 3, 7, 9, but 5 only

Ex. 4. n being any whole number, shew that

$$5^n = 4(5^{n-1} + 5^{n-2} + \dots + 1) + 1.$$

$$5^n = (5^n - 1) + 1$$

$$= (5^n - 1^n) + 1$$

$$= (5-1)(5^{n-1} + 5^{n-2} + \dots + 1) + 1, \text{ Art. 120, case (1),}$$

$$= 4(5^{n-1} + 5^{n-2} + \dots + 1) + 1. \quad \text{Ans.}$$

Ex. 5. Find the product $(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)$.

Let $P = (a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)$.

Multiplying each side by $a-b$ we have

$$(a-b)P = (a-b)(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)$$

$$= (a^2-b^2)(a^2+b^2)(a^4+b^4)(a^8+b^8)$$

$$= a^{10} - b^{10} \text{ by successive multiplication.}$$

$$\therefore P = (a^{10} - b^{10}) \div (a-b)$$

$$= a^{15} + a^{14}b + a^{13}b^2 + \dots + a^2b^{13} + ab^{14} + b^{15}. \quad \text{Ans.}$$

Ex. 6. Shew that $x^n - nx + n - 1$ is exactly divisible by $(x-1)^2$.

$$x^n - nx + n - 1 = (x^n - 1) - n(x-1), \text{ re-arranging,}$$

$$= (x-1)(x^{n-1} + x^{n-2} + \dots + 1) - n(x-1).$$

Hence $x^n - nx + n - 1$ is divisible by $x-1$, the quotient being

$$= (x^{n-1} + x^{n-2} + x^{n-3} + \dots + x + 1) - n.$$

Now break up n into $1+1+1+\dots$, *i. e.*, n ones. Thus the quotient $= (x^{n-1} + x^{n-2} + \dots + x + 1) - (1+1+\dots+1+1)$

$$= (x^{n-1} - 1) + (x^{n-2} - 1) + \dots + (x - 1), \text{ re-arranging.}$$

Since $x^{n-1} - 1$, $x^{n-2} - 1$, &c., are all divisible by $x - 1$,
 \therefore the quotient of $x^n - nx + n - 1$ by $x - 1$ is itself divisible by $x - 1$.
 $\therefore x^n - nx + n - 1$ is divisible by $(x - 1)^2$.

• EXAMPLES 67. •

1. Assuming that $x^n - y^n$ is divisible by $x - y$ when n is any positive integer, prove that $x^n + y^n$ is divisible by $x + y$ when n is any odd positive integer

Write down without division the quotients in the following cases :

- | | |
|--|---------------------------------------|
| 2. $(x^8 - y^8) \div (x - y)$. | 3. $(1 - y^5) \div (1 - y)$. |
| 4. $(1 - y^6) \div (1 - y)$. | 5. $(1 - y^{10}) \div (1 - y^2)$. |
| 6. $(a^8 - b^4) \div (a^2 - b)$. | 7. $(x^5 + y^5) \div (x + y)$. |
| 8. $(x^7 + y^7) \div (x + y)$. | 9. $(x^{10} - y^{10}) \div (x + y)$. |
| 10. $(x^9 + y^9) \div (x^3 + y)$. | 11. $(1 - 32x^5) \div (1 - 2x)$. |
| 12. $(1 - 64x^6) \div (2x + 1)$. | 13. $(32x + 243x^6) \div (3x + 2)$. |
| 14. $(1 - x - x^n + x^{n+1}) \div (1 - 2x + x^2)$. | |
| 15. $(a^3c^3 + b^3d^3 + b^3c^3 + a^3d^3) \div (ac + bd + bc + ad)$. | |

16. Shew that $x^7 - 7x + 6$ is divisible by $(x - 1)^2$, and find the quotient.

17. When will $mx^3 + nx^2 + rx + s$ be divisible by $x + a$?

18. Find, without division, the remainder of

- (1) $7x^3 - 3x^2 + 4x - 1$ divided by $x - 2$.
- (2) $3x^4 - 4x^2 + x - 2$ divided by $x + 1$
- (3) $x^3 + (a + b)x^2 + (a + b)^2x + (a + b)^3$ divided by $x - a + b$.

19. Shew that $(1 - x)^6 + 32x^5(1 - x)$ is divisible by $1 - x^2$.

20. Shew that $1 + x^8 - x^5 - x^3$ is divisible by $1 - x^2$, and find the quotient

21. Shew that $(2x + y)^n - (x + 2y)^n - x^n + y^n$ is divisible by $x^2 - y^2$, n being any whole number.

22. Shew that $(ab + x^2)^{2n+1} + \{x(a + b)\}^{2n+1}$ is divisible by each of $x + a$ and $x + b$, n being any positive integer.

23. Shew that $(1 - x)^2$ is a factor of $1 - (n + 1)x^n + nx^{n-1}$, n being any positive integer.

24. Shew that $6^{2n} + 7^n + 6$ is divisible by 7, n being any positive integer.

25. Shew that $(1 + x - 2x^2)^3 + (2 - 3x + x^2)^3$ is exactly divisible by $3 - 2x - x^2$.

26. Shew that $(1+x^2)^n - (1+y^2)^n$ is divisible by $x-y$, n being any whole number.

27. Shew that $2^5 + 3^5$ is divisible by 5, and $2^6 + 3^6$ by 13, without a remainder.

28. Shew that the last digit of $2^7 + 3^7$ must be 5.

29. Shew that the last digit of $9^{2n+1} + 1$ is 0, n being any whole number.

30. Shew that the value of $4^7 + 4 \cdot 3^6$ ends in two ciphers, and that of $(7^{2n+1} + 1)(6^{n+1} + 100n - 6^n - 5)$ ends in three ciphers.

31. Write down the product $(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})$

CHAPTER XXII.

THEORY OF INDICES.

123. We have hitherto supposed an index or exponent to be a *positive integer*. In the present chapter we will explain the meaning of negative and fractional indices. For this purpose it is necessary to begin with positive integral indices.

124. Fundamental Laws. The three following laws are regarded as fundamental in the case of indices that are positive whole numbers.

$$a^m \times a^n = a^{m+n} \dots \dots \dots \text{I.}$$

$$a^m \div a^n = a^{m-n}, m > n \dots \dots \dots \text{II.}$$

$$(a^m)^n = a^{mn} \dots \dots \dots \text{III.}$$

125. If m and n be any positive integers, to prove that

$$a^m \times a^n = a^{m+n}.$$

Since $a^m = a \times a \times a \dots$ to m factors,

and $a^n = a \times a \times a \dots$ to n factors,

$$\begin{aligned} \therefore a^m \times a^n &= a \times a \times a \dots \text{to } m \text{ factors} \times a \times a \times a \dots \text{to } n \text{ factors} \\ &= a \times a \times a \dots \text{to } m+n \text{ factors} \\ &= a^{m+n}. \end{aligned}$$

Cor. If p is also a positive integer, then

$$a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{m+n+p}, \text{ and so on.}$$

Hence $a^m \times a^n \times a^p \times \dots = a^{m+n+p+\dots}$

This result is called the **Index Law**.

126. If m and n be positive integers, and $m > n$, to prove that

$$\begin{aligned} a^m \div a^n &= a^{m-n}. \\ a^m \div a^n &= \frac{a^m}{a^n} = \frac{\overbrace{a \times a \times a \times \dots \text{to } m \text{ factors}}}{\underbrace{a \times a \times a \times \dots \text{to } n \text{ factors}}} \\ &= \frac{a \times a \times a \dots \text{to } n \text{ factors} \times \overbrace{a \times a \times a \times \dots \text{to } m-n \text{ factors}}}{\cancel{a \times a \times a \times \dots \text{to } n \text{ factors}}} \\ &= a \times a \times \dots \text{to } m-n \text{ factors} \\ &= a^{m-n}, \text{ by definition.} \end{aligned}$$

127. If m and n be any positive integers, to prove that

$$(a^m)^n = a^{mn}.$$

By definition, $(a^m)^n = \overbrace{a^m \times a^m \times a^m \dots \text{to } n \text{ factors}}$
 $= \overbrace{(a \times a \times \dots \text{to } m \text{ factors})}$
 $\times \overbrace{(a \times a \times \dots \text{to } m \text{ factors}) \times \dots \text{to } n \text{ groups}}$
 $= \overbrace{a \times a \times \dots \text{to } mn \text{ factors}}$
 $= a^{mn}.$

Cor. $(a^m)^n = (a^n)^m = a^{mn}$; e.g., $(a^3)^4 = (a^4)^3 = a^{4 \times 3} = a^{12}.$

128. **First Law as basis.** The second and third index laws just proved are deducible from the first. For, we have by the First Law.

$$a^m \times a^n = a^{m+n}, m, \text{ and } n \text{ being any whole numbers.}$$

$$\therefore a^{m+n} \div a^n = a^m \dots \dots \dots \text{Def. of Division.}$$

Let $m+n=M$, so that $m=M-n$.

\therefore from the above result, we have

$$a^M \div a^n = a^{M-n}, \text{ which is the Second Law,}$$

M and n being positive integers, and $M (=m+n) > n$.

Again, by the corollary to the First Law,

$$a^m \times a^n \times a^p \times \dots = a^{m+n+p+\dots}$$

Let $m=n=p=\dots$, and let the number of these quantities be x .

$$\begin{aligned} \therefore a^m \times a^m \times a^m \times \dots \text{to } x \text{ factors} \\ = a^{m+m+m+\dots \text{to } x \text{ terms}}; \end{aligned}$$

that is, $(a^m)^x = a^{mx}$, which is the Third Law.

N. B. The reason why $a^m \times a^n \times a^p \times \dots = a^{m+n+p+\dots}$ is referred to as the **Index Law**, as stated in the Cor., Art. 125, is now evident. It will be seen from the next article that it is also taken as the basis of indices other than whole numbers.

129. The base Law for Negative and Fractional Indices.

It is easy to see that the ordinary definition of an index does not apply to negative and fractional indices. For, according to it, $a^8 = a \times a \times a \times \dots$ to 8 factors, but evidently it is absurd to say that $a^{\frac{5}{4}} = a \times a \times \dots$ to $\frac{5}{4}$ factors, or $a^{-8} = a \times a \times \dots$ to -3 factors.

To find out a meaning for *negative and fractional indices*, we *assume* them to conform to the First Law, namely, $a^m \times a^n = a^{m+n}$, i.e., we assume the First Law to be *universally* true. We shall then be able to show, as in the case of positive integral indices, that the two other Index Laws also hold good in the case of negative and fractional indices.

130. The meaning of a^0 . We assume that for *all values* of m and n ,

$$a^m \times a^n = a^{m+n};$$

putting $m=0$,

$$a^0 \times a^n = a^{0+n} = a^n;$$

dividing both sides by a^n ,

$$a^0 = 1.$$

Thus *any finite quantity raised to the zero power is equal to 1.*

131. The meaning of positive fractional indices.

To find a meaning for $a^{\frac{p}{q}}$, where p and q are whole numbers. As stated in Art. 129, we assume the law $a^m \times a^n = a^{m+n}$.

$$\text{Thus } a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \dots \times a^{\frac{p}{q}} = a^{\frac{p}{q} + \frac{p}{q} + \dots + \frac{p}{q}} = a^{\frac{p}{q} \times q} = a^p,$$

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \dots \times a^{\frac{p}{q}} = a^{\frac{p}{q} + \frac{p}{q} + \frac{p}{q} + \dots + \frac{p}{q}} = a^{\frac{p}{q} \times q} = a^p,$$

and similarly $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \dots$ to four factors $= a^{\frac{4p}{q}}$,

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \dots \text{, five } = a^{\frac{5p}{q}}, \text{ and so on.}$$

$$\text{Thus } a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \dots \text{ } q \text{ times } = a^{\frac{qp}{q}} = a^p;$$

$$\text{i.e., } \left(a^{\frac{p}{q}}\right)^q = a^p;$$

$$\therefore a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Hence $a^{\frac{p}{q}}$ is the q^{th} root of a^p .

Thus $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{2}{5}} = \sqrt[5]{a^2}$, $a^{\frac{7}{4}} = \sqrt[4]{a^7}$, &c. Hence in a positive fractional index, the numerator denotes a power, and the denominator a root.

132. The meaning of negative indices. To find a meaning for a^{-m} .

For all values of m and n , we assume

$$a^m \times a^n = a^{m+n}.$$

Putting $n = -m$, $a^m \times a^{-m} = a^{m-m} = a^0 = 1$.

$$\therefore a^{-m} = 1 \div a^m = \frac{1}{a^m}.$$

That is, a^{-m} is the reciprocal of a^m .

N.B. Thus, $a^{-n} = \frac{1}{a^n}$, $a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{a^2}}$, &c.

133. Inversion of terms. We propose to show that

$$\frac{a^n}{b^n} = \frac{b^{-n}}{a^{-n}}.$$

From the last article, $\frac{1}{a^n} = a^{-n}$;

$$\therefore 1 \div \frac{1}{a^n} = 1 \div a^{-n};$$

$$\text{i.e., } a^n = \frac{1}{a^{-n}};$$

also from the last article, $\frac{1}{b^n} = b^{-n}$;

multiplying up the last two results,

$$\frac{a^n}{b^n} = \frac{b^{-n}}{a^{-n}}.$$

Hence the following **Rule**: Any quantity may be changed from the numerator of a fraction to its denominator, or vice versa, provided the sign of the index of that quantity be changed.

$$\text{Thus } \frac{x^2}{y^3} = \frac{y^{-3}}{x^{-2}} = x^2 y^{-3}; \quad a^6 = \frac{1}{a^{-6}}; \quad x^{\frac{2}{3}} = \frac{1}{x^{-\frac{2}{3}}}.$$

134. To prove that $a^m \div a^n = a^{m-n}$ for all values of m and n .

136. Proposition. $(a^m b^n)^p = a^{mp} b^{np}$ always.

Let m and n have any value.

Case I. Let p be a positive integer.

$$\begin{aligned}(a^m b^n)^p &= (a^m b^n) \times (a^m b^n) \times \dots \text{to } p \text{ groups} \\ &= (a^m \times a^m \times \dots \text{to } p \text{ factors}) (b^n \times b^n \times \dots \text{to } p \text{ factors}) \\ &= a^{mp} b^{np} \dots \text{3rd Law.}\end{aligned}$$

Case II. Let p be a positive fraction.

Let $p = \frac{x}{y}$, where x and y are positive integers,

so that $x = py$.

$$\begin{aligned}(a^m b^n)^p &= (a^m b^n)^{\frac{x}{y}} = \sqrt[y]{(a^m b^n)^x} \\ &= \sqrt[y]{a^{mx} b^{nx}} \dots \text{Case I.} \\ &= \sqrt[y]{a^{mpy} b^{npy}}, \quad \because x = py, \\ &= \sqrt[y]{(a^{mp} b^{np})^y} \dots \text{Case I.} \\ &= (a^{mp} b^{np})^{\frac{y}{y}} \\ &= a^{mp} b^{np}.\end{aligned}$$

Case III. Let p be any negative quantity.

Let $p = -x$, where x is a positive integer or fraction.

$$\begin{aligned}(a^m b^n)^p &= (a^m b^n)^{-x} = \frac{1}{(a^m b^n)^x} \\ &= \frac{1}{a^{mx} b^{nx}} \dots \text{Cases I, II.} \\ &= a^{-mx} b^{-nx} \\ &= a^{mp} b^{np}, \text{ replacing } -x \text{ by } p.\end{aligned}$$

Hence $(a^m b^n)^p = a^{mp} b^{np}$, where m , n and p may be any quantities, positive or negative, integral or fractional.

N. B. Thus $(x^{-\frac{1}{3}} y^{\frac{2}{3}})^{-15} = x^{(-\frac{1}{3}) \times (-15)} y^{\frac{2}{3} \times (-15)} = x^5 y^{-10}.$

The student should try to fully comprehend the above proposition, as it is frequently needed.

137. Transformation of indices into Positive Integral Forms. We propose here a transformation which will be very useful in dealing with expressions involving negative and fractional indices. Suppose we want to transform simultaneously the expressions

$$a^{-1} b^{\frac{2}{3}} - 5a^{-\frac{2}{3}} b^{\frac{1}{2}} + 2a^{-\frac{1}{2}} b^{\frac{1}{4}} \text{ and } a^{-\frac{3}{4}} b + 3a^{-\frac{1}{6}} b^{\frac{5}{6}}.$$

into forms in which the indices will be whole numbers. We propose the following process, which will generally answer best :

First find the G. C. M. of all the indices of the powers of each symbol, and next assume a new symbol for that power of each given symbol whose index is the corresponding G. C. M. found.

In the expressions proposed above, the indices of the powers of a are -1 , $-\frac{2}{5}$, $-\frac{1}{2}$, $-\frac{3}{2}$, $-\frac{5}{10}$, and their G. C. M. = $-\frac{1}{10}$; therefore assume $a^{-\frac{1}{10}} = x$.

$$\therefore a^{-1} = x^{10}, a^{-\frac{2}{5}} = (a^{-1})^{\frac{2}{5}} = x^{10 \times \frac{2}{5}} = x^4, \&c.$$

The indices of the powers of b are $\frac{3}{8}$, $\frac{1}{2}$, $\frac{1}{4}$, 1 , $\frac{5}{8}$, and their G. C. M. = $\frac{1}{8}$; assume $b^{\frac{1}{8}} = y$.

$$\text{Thus } b = y^8, b^{\frac{3}{8}} = y^{8 \times \frac{3}{8}} = y^3, \&c.$$

Thus the proposed expressions are transformed into

$$x^{10}y^8 - 5x^4y^6 + 2x^5y^5 \text{ and } x^{15}y^{12} + 3x^3y^{10}.$$

Ex. 1. Shew that $(x^{m^{n-1}})^m = x^{m^n}$.

$$\begin{aligned} (x^{m^{n-1}})^m &= (x^p)^m, \text{ where } p = m^{n-1}, \\ &= x^{mp} \dots \dots \dots \text{3rd Law.} \end{aligned}$$

$$\begin{aligned} \text{Now, } mp &= m \times m^{n-1} = m^{1+n-1} \dots \dots \text{1st Law,} \\ &= m^n. \end{aligned}$$

$$\therefore (x^{m^{n-1}})^m = x^{m^n}. \quad \text{Ans}$$

N. B. The student should here be careful to distinguish between m^{n-1} and $(m)^{n-1}$. The last = $m^{(n-1)} = m^{n-1}$.

Ex. 2. Simplify $[x^{\frac{1}{6}}\{x^{-\frac{1}{4}}y^{-\frac{1}{6}}(x^4y^4)^{\frac{1}{6}}\}^{-\frac{1}{2}}]^{\frac{1}{2}}$.

$$\text{Since } (x^4y^4)^{\frac{1}{6}} = x^{\frac{4}{6}}y^{\frac{4}{6}} = x^{\frac{2}{3}}y^{\frac{2}{3}},$$

$$\text{the given expression} = [x^{\frac{1}{6}}\{x^{-\frac{1}{4}}y^{-\frac{1}{6}}x^{\frac{2}{3}}y^{\frac{2}{3}}\}^{-\frac{1}{2}}]^{\frac{1}{2}}$$

$$= [x^{\frac{1}{6}}\{x^{-\frac{3}{4}}y^{-\frac{2}{3}}\}^{-\frac{1}{2}}]^{\frac{1}{2}}$$

$$= [x^{\frac{1}{6}}\{x^{\frac{3}{4}}y^{\frac{2}{3}}\}^{-\frac{1}{2}}]^{\frac{1}{2}}$$

$$= x^{\frac{1}{6}}\{x^{\frac{3}{4}}y^{\frac{2}{3}}\}^{-\frac{1}{4}}$$

$$= x(x^{\frac{3}{4}}y^{\frac{2}{3}})^{-\frac{1}{4}}$$

$$= x \cdot x^{-\frac{3}{4}}y^{-\frac{2}{12}}$$

$$= x^{1-\frac{3}{4}}y^{-\frac{1}{6}}$$

$$= x^{-\frac{1}{4}}y^{-\frac{1}{6}}. \quad \text{Ans.}$$

Ex. 3. If $m = a^x$, $n = a^y$, and $a^z = (m^y n^x)^z$, shew that $xyz = 1$.

$$\begin{aligned} a^z &= (m^y n^x)^z = \{(a^x)^y (a^y)^x\}^z; \because m = a^x, n = a^y, \\ &= \{a^{xy} a^{yx}\}^z = (a^{2xy})^z \\ &= a^{2xyz} \end{aligned}$$

\therefore equating the indices, we have

$$z = 2xyz, \text{ i.e., } xyz = 1. \text{ Ans.}$$

Ex. 4. Multiply $a^{\frac{1}{2}} + 3a^{\frac{1}{2}}x^{\frac{1}{2}} + 4x^{\frac{3}{2}}$ by $a - 3a^{\frac{1}{2}}x^{\frac{1}{2}} + 4x^{\frac{1}{2}}$.

$$\begin{array}{r} a^{\frac{1}{2}} + 3a^{\frac{1}{2}}x^{\frac{1}{2}} + 4x^{\frac{3}{2}} \\ a - 3a^{\frac{1}{2}}x^{\frac{1}{2}} + 4x^{\frac{1}{2}} \\ \hline a^{\frac{3}{2}}x^{\frac{1}{2}} + 3a^{\frac{3}{2}}x^{\frac{3}{2}} + 4ax^{\frac{3}{2}} \\ - 3a^{\frac{3}{2}}x^{\frac{1}{2}} - 9ax^{\frac{3}{2}} - 12a^{\frac{1}{2}}x^{\frac{1}{2}} \\ \hline 4ax^{\frac{3}{2}} + 12a^{\frac{1}{2}}x^{\frac{1}{2}} + 16x^{\frac{5}{2}} \\ \hline a^{\frac{3}{2}}x^{\frac{1}{2}} - ax^{\frac{3}{2}} + 16x^{\frac{5}{2}} = \text{complete product.} \end{array} \quad \begin{array}{l} = \text{product by } a; \\ = \text{ " " } -3a^{\frac{1}{2}}x^{\frac{1}{2}}; \\ = \text{ " " } 4x^{\frac{1}{2}}. \end{array} \quad \text{Ans.}$$

Otherwise thus: Let $a^{\frac{1}{2}} = c$, and $x^{\frac{1}{2}} = y$; Art. 137.

then $ax^{\frac{1}{2}} + 3a^{\frac{1}{2}}x^{\frac{3}{2}} + 4x^{\frac{3}{2}} = c^2y + 3cy^3 + 4y^5 = y(c^2 + 3cy + 4y^4)$,

and $a - 3a^{\frac{1}{2}}x^{\frac{1}{2}} + 4x^{\frac{1}{2}} = c^2 - 3cy + 4y^2$.

$$\begin{aligned} \therefore \text{ the reqd. product} &= y(c^2 + 3cy + 4y^4)(c^2 - 3cy + 4y^2) \\ &= y\{(c^2 + 4y^2)^2 - (3cy)^2\} \\ &= y(c^4 - c^2y^2 + 16y^4), \text{ simplifying,} \\ &= c^4y - c^2y^3 + 16y^5 \\ &= a^2x^{\frac{1}{2}} - ax^{\frac{3}{2}} + 16x^{\frac{5}{2}}, \because c = a^{\frac{1}{2}}, y = x^{\frac{1}{2}}. \end{aligned}$$

N. B. The alternative method given here is often very convenient.

Ex. 5. Divide $x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{1}{3}} + y$ by $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{1}{3}}$. C. U. 1860.

$$\begin{array}{r} x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{1}{3}} \overline{) x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{1}{3}} + y} \\ x^{\frac{4}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + x^{\frac{2}{3}}y^{\frac{1}{3}} \\ \hline xy^{\frac{1}{3}} + y \end{array} \quad \left(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{1}{3}} \right)$$

$$\begin{array}{r} xy^{\frac{1}{3}} + y \\ xy^{\frac{1}{3}} - x^{\frac{2}{3}}y^{\frac{1}{3}} + x^{\frac{1}{3}}y^{\frac{2}{3}} \\ \hline x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} + y \\ x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} + y \\ \hline 0 \end{array}$$

\therefore reqd. quotient $= x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{1}{3}}$. Ans.

Now, factorise each of $x^{-6} + a^6$ and $x^{-6} - a^6$.

$$\begin{aligned}x^{-6} + a^6 &= (x^{-2})^3 + (a^2)^3 \\&= (x^{-2} + a^2)\{(x^{-2})^2 - a^2x^{-2} + (a^2)^2\} \\&= (x^{-2} + a^2)(x^{-4} - a^2x^{-2} + a^4).\end{aligned}$$

Again, $x^{-6} - a^6 = (x^{-2})^3 - (a^2)^3$

$$\begin{aligned}&= (x^{-2} - a^2)(x^{-4} + a^2x^{-2} + a^4) \\&= (x^{-1} - a)(x^{-2} + ax^{-1} + a^2)(x^{-1} + a)(x^{-2} - ax^{-1} + a^2).\end{aligned}$$

\therefore using these results in (A), we have

$$\begin{aligned}x^{-12} - a^{12} &= (x^{-2} + a^2)(x^{-4} - a^2x^{-2} + a^4)(x^{-1} - a) \\&\quad \times (x^{-2} + ax^{-1} + a^2)(x^{-1} + a)(x^{-2} - ax^{-1} + a^2) \\&= (x^{-1} - a)(x^{-1} + a)(x^{-2} + a^2)(x^{-2} - ax^{-1} + a^2) \\&\quad \times (x^{-2} + ax^{-1} + a^2)(x^{-4} - a^2x^{-2} + a^4). \quad \text{Ans.}\end{aligned}$$

Ex. 8. What must be the form of m in order that $a^m - x^m$ may have both $a^n + x^n$ and $a^n - x^n$ for divisors, n being any positive integer? Shew that $2^{4n} - 1$ is divisible by 15. M. U. 1875.

Since $a^m - x^m$ is divisible by each of $a^n + x^n$ and $a^n - x^n$,

$$\therefore a^m - x^m \text{ „ „ „ } (a^n + x^n)(a^n - x^n) \text{ or } a^{2n} - x^{2n}.$$

Let $a^{2n} = c$, $x^{2n} = d$, then $a = c^{\frac{1}{2n}}$, $x = d^{\frac{1}{2n}}$,

and $a^m - x^m = c^{\frac{m}{2n}} - d^{\frac{m}{2n}}$.

\therefore by the last result, we have

$$c^{\frac{m}{2n}} - d^{\frac{m}{2n}} \text{ exactly divisible by } c - d;$$

$\therefore \frac{m}{2n}$ must be a positive whole number, Art. 120,

\neq suppose.

$\therefore m = 2np$, where p is any positive whole number

\equiv any even positive multiple of n . Ans.

$$2^{4n} - 1 = (2^4)^n - 1 = (16)^n - (1)^n, \quad \therefore 1 \times 1 \times \dots = 1.$$

Now, n being a positive integer,

$(16)^n - 1^n$ is divisible by $16 - 1$ or 15. Art. 120.

$\therefore 2^{4n} - 1$ is divisible by 15. Ans.

EXAMPLES 68.

Express in terms showing the meanings of

1. $a^{\frac{1}{2}}$.
2. $a^{\frac{1}{3}}$.
3. $x^{\frac{2}{3}}$.
4. $y^{\frac{4}{5}}$.
5. $a^{\frac{2}{3}}b^{\frac{3}{4}}$.
6. $a^{-\frac{1}{2}}$.
7. $x^{-\frac{2}{3}}$.
8. y^{-4} .
9. $x^{-\frac{3}{4}}$.
10. $(x^2y^3)^{-\frac{1}{2}}$.

Find the value of

11. $27^{\frac{1}{3}}$. 12. $16^{-\frac{3}{4}}$. 13. $2^{-3} 36^{\frac{1}{2}}$. 14. $625^{\frac{2}{5}} \times 5^{-2}$.
 15. $2^{-1} \times 16^{-\frac{3}{2}}$. 16. $8^{\frac{1}{3}} 81^{-\frac{1}{4}}$. 17. $(\frac{4}{9})^{\frac{3}{2}}$. 18. $(\frac{4}{9})^{-\frac{3}{2}}$.
 19. $(\frac{8}{125})^{-\frac{1}{5}}$. 20. $(\frac{1}{2})^{-5} 4^{-2}$. 21. $\frac{1}{6^{-3}}$. 22. $\frac{2^{-3}}{4^{-2}} \times 2^{\frac{1}{2}}$.
 23. $(\frac{1}{6})^{-\frac{1}{2}} \times 64^{-\frac{1}{3}} \times 3^{\frac{1}{2}}$. 24. $(\frac{1}{6})^{-3} \times 4^{\frac{3}{2}} \times 10^{-3}$. 25. $1^{\frac{1}{2}} \times x^2 \times x^{\frac{3}{2}}$.
 26. $x^{-\frac{1}{2}} \times x \times x^{-\frac{1}{4}} \times x^{\frac{9}{4}}$. 27. $a^{\frac{2}{3}} \times (a^3)^3 \times (a^{-1})^4$. 28. $p^{\frac{1}{2}} \times (p^{-2})^3 \times (p^3)^2$.
 29. $\sqrt[4]{a^{12}} \sqrt[3]{a^6} \sqrt[2]{a^4}$. 30. $\sqrt[3]{x^6} \cdot \sqrt{x^3} \cdot \sqrt[4]{x^8}$.
 31. $a^{\frac{1}{2}} b^{-\frac{1}{6}} \times a^{-\frac{1}{3}} b^{\frac{1}{2}}$. 32. $(2ab^2)^{-\frac{3}{2}} (2a^2b^3)^{\frac{5}{6}}$.
 33. $(xyz)^{-\frac{5}{4}} (8x^3yz^2)^{\frac{1}{4}}$. 34. $a^{\frac{1}{6}} b^{\frac{1}{3}} \div a^{\frac{2}{3}} b^{\frac{1}{4}}$.
 35. $a^2b^{-\frac{2}{3}}c \div a^{-1}b^{\frac{1}{3}}c^{\frac{1}{2}}$. 36. $(x^3yz^2)^{-1} \div (xyz^3)^{-2}$.
 37. $\sqrt[3]{a^3b} (xy^{-1} \div \sqrt{a^3b} x^{-1})$. 38. $\{a^{-\frac{1}{2}} (a^{\frac{1}{2}} b^{-\frac{1}{2}} \div ab^{-\frac{2}{3}})^{-1}\}^6$.
 39. $\frac{\sqrt[5]{32x^6y^{10}z^6}}{a^{16}y^{20}}$. 40. $\frac{\sqrt[6]{729l^6m^{18}n^{10}}}{64l^{17}m^7n^4}$.
 41. $\frac{a^{a+b} a^{a-b} a^{c-2a}}{a^{c-a}}$. 42. $\frac{b^{3a+2a} b^{3a-6a}}{b^{6a-6a}}$.
 43. $\sqrt[6]{\left\{\left(\frac{a^2-b^2}{a^4-b^4}\right)^3\right\}^{-9}} - b^2$. 44. $6xyz \left(\sqrt{4x^3} - \sqrt{\frac{y^3}{27x^3z^3}} \right)$.
 45. $\frac{[(a^3b^{\frac{1}{2}})^6 \times (a^{-2}b^{-1})^3]^{-6}}{a^{-\frac{3}{2}}(b^2)^3}$. 46. $\left(\frac{n}{p^{n-m}}\right)^{n^2-m^2} + \left(\frac{p^n}{p^m}\right)^m$.
 47. $\left(\frac{x^{\frac{2}{3}}z^{-\frac{1}{3}}}{y^{\frac{1}{2}}}\right)^2 \times \left(\frac{y^6z^2}{x^3}\right)^{\frac{1}{4}}$. 48. $\left[\frac{a^{\frac{5}{2}}b^{\frac{4}{3}}}{c^{-\frac{1}{4}}} \times \frac{c^4}{a^{-3}b^{-\frac{5}{2}}} \div \frac{b^{-2}c^{\frac{1}{2}}}{a^{-\frac{1}{2}}}\right]^{\frac{1}{6}}$.
 49. $\left\{\frac{x^{\frac{a}{b}}x^{\frac{m}{b}}y^{\frac{b}{b}}}{(\sqrt{x})^{\frac{b}{b}}(\sqrt{y})^{\frac{a}{b}}}\right\}^{mn} \div \left(\frac{x^a}{x^b}\right)^m$. 50. $\left(\frac{ax}{z}\right)^{1-m} \cdot \left(\frac{by}{x^2}\right)^{m-n} \cdot \left(\frac{cz}{y}\right)^{n-1}$.
 51. $\frac{\left(a+\frac{1}{b}\right)^m \cdot \left(a-\frac{1}{b}\right)^n}{\left(b+\frac{1}{a}\right)^m \left(b-\frac{1}{a}\right)^n}$. 52. $\frac{\left(\frac{1}{x}+\frac{1}{y}\right)^{-m} \cdot \left(\frac{1}{x^2}-\frac{1}{y^2}\right)^{-n}}{\left(\frac{x}{y}+1\right)^{-m} \left(\frac{x-y}{y}\right)^{-n}}$.

Multiply

53. $a^{\frac{1}{m}} - a^{-\frac{1}{m}}$ by $a^{\frac{1}{m}} + a^{-\frac{1}{m}}$ and $a^{\frac{2}{m}} + 1 + a^{-\frac{2}{m}}$.

54. $a^4 - a^2 - a^{\frac{1}{2}} - a$ by $a^2 - a^{\frac{1}{2}} + 1$ and $a^2 - a + a^{-\frac{1}{2}} - 1$.
 55. $2x^{-\frac{1}{2}}y^{\frac{1}{2}} + 3x^{\frac{1}{2}}y^{-\frac{1}{2}}$ by $2x^{\frac{1}{2}}y^{\frac{1}{2}} - 3x^{-\frac{1}{2}}y^{-\frac{1}{2}}$ and $2x^{\frac{1}{2}}y^{-\frac{1}{2}} - 3x^{-\frac{1}{2}}y^{\frac{1}{2}}$.
 56. $\sqrt[4]{x^5y^3} + 3x^{-\frac{1}{2}}y^{-\frac{1}{10}} - 2\sqrt{x}\sqrt[4]{y}$ by $2x^{\frac{1}{2}}y^{\frac{1}{2}} - \frac{3}{\sqrt[4]{x^{10}y} - \frac{x^{-1}}{\sqrt[5]{y^2}}}$.
 57. $a^x + 3a^{x-1}b^y - 6a^{\frac{x}{2}-2}b^{2y}$ by $a^yb^y - 7a^{y-1}b^{2y}$.

Divide

58. $a - 5a^{\frac{1}{2}} - 20 + 24a^{-\frac{3}{2}}$ by $a^{\frac{3}{2}} - 3a^{\frac{1}{2}} - 2 + 4a^{-\frac{1}{2}}$.
 59. $2\sqrt[3]{\left(\frac{a}{b^3}\right)^{\frac{5}{3}}} - \frac{\sqrt[3]{b^8}}{a}$ by $\sqrt{\frac{2a}{b^3}} - \frac{\sqrt[3]{b}}{\sqrt[3]{a}}$.
 60. $a^{2m} - 9b^{2n} + 12b^{n,m} - 4c^{12}$ by $a^m + 3b^n - 2c^6$.
 61. $x^{2n} - y^{2n}$ by $x^{2n-1} + y^{2n-1}$, and $x^{2n} - y^{2n}$ by $x^{2n-1} - y^{2n-1}$.
 62. $x^6 + x^{-6} + 6(x^4 + x^{-4}) + 15(x^2 + x^{-2}) + 20$ by $x^3 + x^{-3} + 3(x + x^{-1})$.
 63. $2\sqrt[6]{a^5b} \sqrt[4]{b^3} - 3ab \sqrt[4]{b^{-1}} - 2ac \sqrt[6]{ab^5} + 3b \sqrt[6]{a^2b^{-1}}$
 by $2\sqrt[6]{ab} - 3\sqrt[3]{a^2b}$.

Resolve into factors

64. $x^{\frac{3}{2}} + y^{\frac{3}{2}}$ 65. $a^{\frac{4}{3}} - b^{\frac{8}{3}}$ 66. $a^{-4} - 4b^{-6}$
 67. $a^{-9} - 1$ 68. $a^{\frac{2}{3}} + 5a^{\frac{1}{3}} + 4$ 69. $x + 2 + x^{-1}$.
 70. Divide $e^{\frac{5x}{2}} - e^{\frac{x}{2}}x + e^x x^{\frac{3}{2}} - 2e^{\frac{x}{2}}x^2 + 1^{\frac{5}{2}}$ by $e^{\frac{x}{2}} - e^{\frac{1}{2}}x^2 + e^{\frac{x}{2}}x - x^{\frac{3}{2}}$ by means of factors.
 71. Divide $a^{xy} - a^yb^{(x-1)y} - b^x a^{(x-1)y} + b^{xy}$ by $a^y - b^x$, and multiply the quotient by $a^{x+y} + b^{x+y}$.

Find the H. C. F. of

72. $e^{2x}y^3 + e^{2x} - y^3 - 1$ and $e^{2x}y^3 + 2e^xy^2 - 2e^x + y^2 - 2$.
 73. $\frac{1}{8}a^{\frac{1}{2}} - \frac{3}{10}a^{\frac{1}{10}} + \frac{1}{2}a^{\frac{3}{2}} - \frac{1}{3}a^{\frac{7}{2}} - \frac{2}{5}a^{\frac{9}{2}} + \frac{1}{4}a^{\frac{1}{2}}$ and $a^{\frac{1}{2}} - a^{\frac{1}{10}} + \frac{1}{2}a^{\frac{3}{2}}$.

Find the L. C. M. of

74. $(x^{\frac{1}{2}} - y^{\frac{1}{2}})^{\frac{1}{2}}$, $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^{\frac{1}{2}}$, $(x^{\frac{1}{2}} - y^{\frac{1}{2}})^{\frac{3}{2}}$, $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^{\frac{3}{2}}$, $x^{\frac{3}{2}} - y^{\frac{3}{2}}$ and $x^{\frac{3}{2}} + y^{\frac{3}{2}}$.

Simplify

75. $\frac{a^{-1}(ab^{-1} - 1)^2}{b^{-2}(1 + a^{-1}b)} \times \frac{b^2(a^{-2} + b^{-2})}{a(ab^{-1} - a^{-1}b)} \div \frac{1 - a^{-1}b}{ab^{-1} + 1}$.
 76. $\frac{x^{\frac{3}{2}} - a^{\frac{3}{2}}}{x - a} \left(\frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}} + a^{\frac{1}{2}}} + a^{\frac{1}{2}} \right) + \frac{x^{\frac{3}{2}} - a^{\frac{3}{2}}}{x + a} \left(\frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}} - a^{\frac{1}{2}}} - a^{\frac{1}{2}} \right)$.

$$77. \frac{x^{2n}}{x^n-1} - \frac{x^{2n}}{x^n+1} - \frac{1}{x^n-1} + \frac{1}{x^n+1}, \text{ when } x = \sqrt{\frac{a-b}{a+b}}$$

$$78. \frac{(ab)^{n+y}\{1+a^{-2}b^{-y}-a^{-y}b^{-2}-(ab)^{2-y}\div(ab)^{2x}\}}{(a^{-1}b)^2(ab)^y+b^{x+y}\{(ab)^y-(ab)^{-2}\}-b^{2y}}.$$

$$79. \text{ Shew that } \frac{x^{2^n}-y^{2^n}}{x-y} = (x+y)(x^2+y^2)(x^4+y^4)\dots(x^{2^{n-1}}+y^{2^{n-1}}).$$

80. Write down without actual division the following quotients :

$$\left(\frac{x^8-y^{12}}{256-645}\right) \div \left(\frac{1}{2}x^2+\frac{1}{2}y^2\right), (x^3-y^3) \div (x^{-\frac{1}{2}}-y^{-\frac{1}{2}})$$

$$\text{and } (x^3-y^3) \div (x^{-\frac{1}{2}}-y^{-\frac{1}{2}})$$

81. What must be the form of m in order that 2^m-1 may have both 9 and 7 for divisors ?

82. If $x^{2n+1}+y^{2n+1}$ be exactly divisible by x^3+y^3 , show that $n-1$ is a multiple of 3.

83. Show that $(x-1)^{2n+1}+2^{2n}(1-x)$ is divisible by $(x^2-1)(x-3)$, n being any positive integer.

84. Find the product $(1+x+x^2+\dots+x^{2n})(1-x+x^2-\dots+x^{2n})$

Prove that, n being any positive whole number,

85. The last digit of $(2 \times 4^n)^2+1$ is 5.

86. $2^{3n-1}+2^{3n-2}+2^{3n-3}+\dots+1$ is divisible by 7.

$$87. 1 - \frac{1}{7^{2n}} = 8 \left(\frac{1}{7} - \frac{1}{7^2} + \dots - \frac{1}{7^{2n}} \right).$$

CHAPTER XXIII.

ELEMENTARY SURDS.

138. **Definitions.** When any root of a quantity cannot be exactly extracted, that root is called a **surd**, or an **irrational quantity**. Thus $\sqrt{2}$, $\sqrt[3]{10}$, $\sqrt[5]{x^4}$, $\sqrt[3]{a^3+b^3}$ are surds. Hence these are only cases of fractional indices. The first two are arithmetical, and the last two are algebraic surds.

The **order** of a surd is estimated by the number denoting the root. Thus $\sqrt[4]{a}$ and $\sqrt[n]{x}$ are surds of the fourth and n th orders respectively.

The surds of the second order, *e.g.*, $\sqrt{5}$, \sqrt{x} , $\sqrt{a+b}$, &c. are often called **quadratic surds**, and are more frequently met

with than those of other orders. Surds of the third order are called **cubic surds**.

N. B. Although $\sqrt[4]{5}$, &c., cannot be exactly found, we can make a close approximation to it by continuing the usual arithmetical work to a sufficient number of decimals.

139. Apparent forms Any quantity may be expressed in a surd form, and conversely, quantities in apparent surd forms may be expressed in their true rational forms. Thus

$$2 = \sqrt[4]{2 \times 2 \times 2 \times 2} = \sqrt[4]{16}; \quad 6 = \sqrt[3]{6 \times 6 \times 6} = \sqrt[3]{216};$$

$$a = \sqrt{a^2}; \quad x + y = \sqrt[3]{(x+y)^3}, \text{ \&c.}$$

$$\sqrt{9} = \sqrt{3 \times 3} = 3; \quad \sqrt[5]{3125} = \sqrt[5]{5 \times 5 \times 5 \times 5 \times 5} = 5;$$

$$\sqrt[3]{a^3} = a; \quad \sqrt[5]{b^{10}} = b^{\frac{10}{5}} = b^2, \text{ \&c.}$$

Ex. 1. Evaluate $\sqrt[4]{81}$, and express 4 as a cubic surd.

$$\sqrt[4]{81} = \sqrt[4]{9^2} = 9^{\frac{2}{4}} = 9^{\frac{1}{2}} = \sqrt{9} = \sqrt{3^2} = 3. \quad \text{Ans.}$$

$$4 = \sqrt[3]{4 \times 4 \times 4} = \sqrt[3]{64}. \quad \text{Ans.}$$

Ex. 2. Express $\sqrt[3]{250}$ as the product of a rational quantity and a surd.

$$\sqrt[3]{250} = \sqrt[3]{5 \times 5 \times 5 \times 2} = 5 \sqrt[3]{2}. \quad \text{Ans.}$$

Ex. 3. Express $\sqrt[4]{7}$ and $\sqrt[6]{11}$ as surds of the same order.

The L. C. M. of the root figures (4, 6) = 12.

$$\text{Now, } \sqrt[4]{7} = 7^{\frac{1}{4}} = 7^{\frac{3}{12}} = \sqrt[12]{7^3} = \sqrt[12]{343};$$

$$\sqrt[6]{11} = 11^{\frac{1}{6}} = 11^{\frac{2}{12}} = \sqrt[12]{11^2} = \sqrt[12]{121}.$$

Hence the reqd surds are $\sqrt[12]{343}$ and $\sqrt[12]{121}$. Ans.

N. B. It is advisable to first reduce a surd to one of a lower order, when possible. Thus $\sqrt[6]{81} = \sqrt[4]{9}$.

Ex. 4. Which is the greater, $2\sqrt{10}$ or $3\sqrt[3]{9}$?

$$2\sqrt{10} = \sqrt{4 \times 10} = \sqrt{40}; \text{ and } 3\sqrt[3]{9} = \sqrt[3]{27 \times 9} = \sqrt[3]{243}.$$

Now reduce the surds to the same order.

$$\sqrt{40} = \sqrt[6]{40^3} = \sqrt[6]{64000}, \text{ and } \sqrt[3]{243} = \sqrt[6]{243^2} = \sqrt[6]{59049}.$$

Since 64000 is greater than 59049, $2\sqrt{10}$ is greater than $3\sqrt[3]{9}$. Ans.

Ex. 5. Find the product $\sqrt[3]{5} \times \sqrt{2}$ as one entire surd.
First reduce the factors to the same order.

$$\sqrt[3]{5} = \sqrt[6]{5^2} = \sqrt[6]{25}, \text{ and } \sqrt{2} = \sqrt[6]{2^3} = \sqrt[6]{8}.$$

$$\therefore \text{ reqd. product} = \sqrt[6]{5 \times 8}$$

$$= 25^{\frac{1}{6}} \times 8^{\frac{1}{6}}$$

$$= (25 \times 8)^{\frac{1}{6}} \dots \dots \dots \text{Art. 136.}$$

$$= (200)^{\frac{1}{6}} = \sqrt[6]{200}. \quad \text{Ans}$$

Ex. 6. Simplify $2\sqrt{3} - \sqrt{48} - \sqrt[3]{16} + 3\sqrt[3]{4}$

$$\text{Required value} = 2\sqrt{3} - \sqrt{16 \times 3} - \sqrt[3]{8 \times 2} + 3\sqrt[3]{2^2}$$

$$= 2\sqrt{3} - 4\sqrt{3} - 2\sqrt[3]{2} + 3\sqrt[3]{2}$$

$$= (3-2)\sqrt[3]{2} + (2-4)\sqrt{3}$$

$$= \sqrt[3]{2} - 2\sqrt{3} \quad \text{Ans.}$$

Ex. 7. Find the value of $(\sqrt{7} + 5\sqrt{3})(2\sqrt{7} - 4\sqrt{3})$,
and $(3\sqrt{x} + 4\sqrt{y})(4\sqrt{x} - 3\sqrt{y})$

Proceed as in ordinary algebraical multiplication.

$$(\sqrt{7} + 5\sqrt{3})(2\sqrt{7} - 4\sqrt{3}) = (\sqrt{7} + 5\sqrt{3})2\sqrt{7} - (\sqrt{7} + 5\sqrt{3})4\sqrt{3}$$

$$= 2 \times 7 + 10\sqrt{21} - 4\sqrt{21} - 20 \times 3$$

$$= (10-4)\sqrt{21} + 14 - 60, \text{ re-arranging,}$$

$$= 6\sqrt{21} - 46. \quad \text{Ans.}$$

$$(3\sqrt{x} + 4\sqrt{y})(4\sqrt{x} - 3\sqrt{y}) = (3\sqrt{x} + 4\sqrt{y})4\sqrt{x} - (3\sqrt{x} + 4\sqrt{y})3\sqrt{y}$$

$$= 12x + 16\sqrt{xy} - 9\sqrt{xy} - 12y$$

$$= 12(x-y) + 7\sqrt{xy}. \quad \text{Ans.}$$

Ex. 8. Divide $2\sqrt{3}$ by $\sqrt[3]{12}$.

$$\frac{2\sqrt{3}}{\sqrt[3]{12}} = \frac{\sqrt{4 \times 3}}{\sqrt[3]{12}} = \frac{\sqrt[6]{(4 \times 3)^3}}{\sqrt[6]{12^2}} = \sqrt[6]{\frac{12^3}{12^2}} = \sqrt[6]{12} = \sqrt[6]{12}. \quad \text{Ans.}$$

EXAMPLES 69.

Express as surds of the 2nd order

1. 3

2. 7.

3. x^2y .

4. $2a^3(b+c)$.

Express as surds of the 6th order

5. 2.

6. $2ab$.

7. \sqrt{a} .

8. $\sqrt[3]{xy}$.

Reduce to entire surds

- 9 $2\sqrt{3}$. 10. $2^3\sqrt{4}$. 11. $a^2\sqrt{b}$. 12. $3xy^6\sqrt{z}$.
 13. $\frac{3}{4}\sqrt{5}$. 14. $4^2\sqrt{12}$. 15. $\frac{a^3}{b}\sqrt{ab^2}$. 16. $\left(\frac{a}{b}\right)^2\sqrt{\frac{b}{a}}$.
 17. $2a\sqrt{\frac{b}{2a}}$. 18. $3r^2\sqrt{\frac{y}{r}}$. 19. $(ab)^n\sqrt{\frac{a}{b}}$. 20. $\left(\frac{x}{y}\right)^m\sqrt{\frac{y}{x}}$.

Express as the product of a surd and a rational quantity

- 21 $\sqrt{72}$. 22. $\sqrt[3]{72}$. 23. $\sqrt{288}$. 24. $\sqrt[3]{500}$.
 25. $\sqrt[5]{96}$. 26. $\sqrt[4]{320}$. 27. $\sqrt{a^2b}$. 28. $\sqrt[3]{a^5b^8c^6d^7}$.
 29. $\sqrt{(a+b)^2c}$. 30. $\sqrt[3]{(r+y)^4z}$. 31. $\sqrt[3]{8x^4(y-z)^6}$.
 32. $\sqrt{(x+3)(x^2-1)(1^2+r-2)(x^2+5x+6)}$.
 33. Express with rational denominators $\sqrt{\frac{1}{3}}$, $\sqrt[3]{\frac{2}{3}}$, $\frac{5}{\sqrt{2}}$.

Reduce to surds of the same order

34. $\sqrt{3}, \sqrt[3]{4}$. 35. $\sqrt[3]{4}, \sqrt[4]{5}$. 36. $\sqrt[4]{ab}, \sqrt[6]{a^2b^4}$.

Which is the greater?

37. $\sqrt{5}$ or $\sqrt[3]{11}$. 38. $\sqrt{12}$ or $\sqrt[4]{25}$. 39. $\sqrt[4]{7}$ or $\sqrt[3]{216}$.

Reduce to the simplest form

40. $\sqrt{2} + \sqrt{8} - \sqrt{27} - \sqrt{12}$. 41. $\sqrt{48} + \sqrt{12} - \sqrt{75} - \sqrt{3}$.
 42. $\sqrt{27} - \sqrt{108} + 9\sqrt{1}$. 43. $\sqrt{64} - \sqrt{24} - \sqrt{6} + \sqrt{54}$.
 44. $\sqrt{88} - \sqrt{8} - \sqrt{18} - \sqrt{50}$. 45. $2\sqrt{72} + 4\sqrt{64} - 6\sqrt{64}$.
 46. $12\sqrt{\frac{1}{3}} + 6\sqrt[3]{\frac{1}{5}} - 4\sqrt[4]{9}$. 47. $2\sqrt[6]{125} + 11\sqrt{5} - 10\sqrt[3]{1}$.
 48. $\sqrt[3]{32} - 3\sqrt{\frac{1}{2}} + \sqrt{18} + \sqrt{192} - 4\sqrt[4]{\frac{1}{2}} + 7\sqrt[4]{9}$.

Find the products

49. $\sqrt{3} \times \sqrt{6}$; $\sqrt{5} \times \sqrt{10}$. 50. $2\sqrt{3} \times \sqrt{2}$; $2\sqrt{3} \times \sqrt{27}$.
 51. $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$. 52. $(2\sqrt{3} - \sqrt{11})(2\sqrt{3} + \sqrt{11})$.
 53. $(\sqrt{x} + \sqrt{x^2 - a^2})(\sqrt{x} - \sqrt{x^2 - a^2})$. 54. $(\sqrt{a+b} + \sqrt{a})(\sqrt{a+b} - \sqrt{a})$.
 55. $(2\sqrt{3} + \sqrt{10})(2\sqrt{10} - \sqrt{3})$. 56. $(2\sqrt{3} + \sqrt{5})(2\sqrt{5} + \sqrt{3})$.

Find the quotients

57. $4\sqrt{3} \div 2\sqrt{3}$. 58. $2\sqrt{5} \div \sqrt{10}$. 59. $\sqrt{12} \div \sqrt{8}$. 60. $3\sqrt[3]{2} \div \sqrt[3]{24}$

Find the value of

61. $\frac{x^3 - y^3}{x + y} \times \frac{x^3 + y^3}{x - y}$, when $x = 2 + \sqrt{2}$, and $y = 2 - \sqrt{2}$.
 62. $(1 + \sqrt{2} + \sqrt{6})(2 - \sqrt{2} + \sqrt{3}) + (\sqrt{2} + \sqrt{3})(2\sqrt{3} - \sqrt{2})$.

$$63. \frac{\sqrt{a+b} + \sqrt{a-b}}{\sqrt{a+b} - \sqrt{a-b}} - \frac{\sqrt{a+b} - \sqrt{a-b}}{\sqrt{a+b} + \sqrt{a-b}}$$

$$64. \{(\sqrt{3} + \sqrt{2})^2 - 5\} \left(\frac{3\sqrt{2} + 2\sqrt{3}}{3\sqrt{2} - 2\sqrt{3}} - \frac{3\sqrt{2} - 2\sqrt{3}}{3\sqrt{2} + 2\sqrt{3}} \right)$$

Show that

$$65. x \left\{ 1 + \frac{x}{\sqrt{1-x^2}} \right\} + \sqrt{1-x^2} \left\{ 1 + \frac{\sqrt{1-x^2}}{x} \right\} = \frac{1}{x} + \frac{1}{\sqrt{1-x^2}}$$

140. Rationalization. To convert a surd expression into a rational one by means of a multiplying factor is the most frequent and most important work with surds. We will confine ourselves to the simplest cases of quadratic surds.

Since $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = (\sqrt{x})^2 - (\sqrt{y})^2 = x - y$, it follows that $\sqrt{x} + \sqrt{y}$ is a rationalizing factor of $\sqrt{x} - \sqrt{y}$, and conversely.

Ex. 1. Express $\frac{\sqrt{x+a} - \sqrt{x-a}}{\sqrt{x+a} + \sqrt{x-a}}$ with a rational denominator.

A rationalizing factor of $\sqrt{x+a} + \sqrt{x-a}$ is $\sqrt{x+a} - \sqrt{x-a}$.

Multiply the numerator and denominator of the given fraction by $\sqrt{x+a} - \sqrt{x-a}$.

$$\begin{aligned} \text{Thus } \frac{\sqrt{x+a} - \sqrt{x-a}}{\sqrt{x+a} + \sqrt{x-a}} &= \frac{\sqrt{x+a} - \sqrt{x-a}}{\sqrt{x+a} + \sqrt{x-a}} \times \frac{\sqrt{x+a} - \sqrt{x-a}}{\sqrt{x+a} - \sqrt{x-a}} \quad \text{Art. 105} \\ &= \frac{(\sqrt{x+a} - \sqrt{x-a})^2}{(\sqrt{x+a})^2 - (\sqrt{x-a})^2} \\ &= \frac{(\sqrt{x+a})^2 + (\sqrt{x-a})^2 - 2\sqrt{x+a} \times \sqrt{x-a}}{(x+a) - (x-a)} \\ &= \frac{(x+a) + (x-a) - 2\sqrt{x^2 - a^2}}{x+a-x+a} \\ &= \frac{2x - 2\sqrt{x^2 - a^2}}{2a} \\ &= \frac{x - \sqrt{x^2 - a^2}}{a} \quad \text{Ans.} \end{aligned}$$

Ex. 2. Divide $\frac{2\sqrt{3} + \sqrt{5}}{3 + 2\sqrt{3}}$ by $\frac{2 - \sqrt{3}}{\sqrt{5} - \sqrt{3}}$, and express the quotient with a rational denominator.

$$\begin{aligned}
 \text{The quotient} &= \frac{2\sqrt{3} + \sqrt{5}}{3 + 2\sqrt{3}} \times \frac{\sqrt{5} - \sqrt{3}}{2 - \sqrt{3}} \\
 &= \frac{(2\sqrt{3} + \sqrt{5})(\sqrt{5} - (2\sqrt{3} + \sqrt{5})\sqrt{3})}{(3 + 2\sqrt{3})2 - \sqrt{3}(3 + 2\sqrt{3})} \\
 &= \frac{2\sqrt{15} + 5 - 6 - \sqrt{15}}{6 + 4\sqrt{3} - 3\sqrt{3} - 6} \\
 &= \frac{\sqrt{15} - 1}{\sqrt{3}} \\
 &= \frac{\sqrt{15} - 1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\
 &= \frac{\sqrt{45} - \sqrt{3}}{3} \quad \text{Ans.}
 \end{aligned}$$

EXAMPLES 70.

Find a factor that will rationalize

1. $\sqrt{x} + \sqrt{y}$

2. $2 + \sqrt{3}$

3. $\sqrt[3]{x} - \sqrt[3]{y}$

Rationalize the denominator of

4. $\frac{\sqrt{3}+1}{\sqrt{2}}$

5. $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}}$

6. $\sqrt{\left(\frac{x-a}{x+a}\right)}$

7. $\frac{a+b}{\sqrt{(a^2-b^2)}}$

8. $\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{a}}$

9. $\frac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}}$

10. $\frac{\sqrt{(a+b)}+\sqrt{(a-b)}}{\sqrt{(a+b)}-\sqrt{(a-b)}}$

11. $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

12. $\frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$

13. $\frac{5\sqrt{6}-2\sqrt{7}+\sqrt{24}}{3\sqrt{6}+2\sqrt{7}+\sqrt{24}}$

14. $\frac{(3+\sqrt{5})(\sqrt{5}-2)}{5-\sqrt{5}}$

15. $\frac{1}{1+\sqrt{3}+\sqrt{2}}$

16. $\frac{\sqrt{3}+1}{\sqrt{3}+\sqrt{2}-1}$

Find the value of

17. $\frac{(1-x)^{\frac{1}{2}}+(1+x)^{-\frac{1}{2}}}{1+(1-x^2)^{-\frac{1}{2}}}$

18. $\frac{x-4x^{\frac{1}{2}}}{x^{\frac{3}{2}}+x-20x^{\frac{1}{2}}} \div \left(1+\frac{5}{\sqrt{x}}\right)^{-1}$

19. a^2+b^2+6ab , when $a=\frac{\sqrt{5}+1}{\sqrt{5}-1}$, $b=\frac{1-\sqrt{5}}{1+\sqrt{5}}$

20. $4p-q$, when $p=\frac{2+\sqrt{3}}{2-\sqrt{3}}$, $q=\frac{\sqrt{3}+1}{\sqrt{3}-1}$

Given $\sqrt{2}=1.41421$, $\sqrt{3}=1.73205$, $\sqrt{5}=2.23607$, find correct to four places of decimals the value of,

$$21. \frac{\sqrt{3}+1}{\sqrt{3}-1} \quad 22. \frac{2+\sqrt{2}}{\sqrt{2}-1} \quad 23. \frac{\sqrt{5}+2}{\sqrt{5}-2} \quad 24. \frac{1}{\sqrt{5}-\sqrt{3}} - \frac{1}{\sqrt{5}+\sqrt{3}}$$

CHAPTER XXIV.

INVOLUTION.

141. Definition Involution is the operation of finding any power of a given quantity.

142. Squares and cubes We have already given the formulæ for the squares and cubes of algebraical expressions. We simply re-state them here

$$(a+b)^2 = a^2 + 2ab + b^2.$$

$$(a-b)^2 = a^2 - 2ab + b^2.$$

$$(a+b+c+d+\dots)^2 = a^2 + b^2 + c^2 + d^2 + \dots + 2a(b+c+d+\dots) + 2b(-c+d+\dots) - 2c(d+\dots) + \&c.$$

$$(a+b)^3 = a^3 + 3ab(a+b) + b^3.$$

$$(a-b)^3 = a^3 - 3ab(a-b) - b^3.$$

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a+b)(b+c)(c+a).$$

The form in which we have expressed the expansion of $(a+b+c+d+\dots)^2$ suggests a very convenient form of the rule we have given in Art 74, page 82.

Rule: *The square of any multinomial is equal to the sum of the squares of its several terms plus twice the algebraic product of each term by the sum of those that follow it.*

N. B. Compare this rule with the one in page 82. It should be borne in mind that the square quantities are always positive, but that in forming the products each term must be taken with its proper sign.

Ex. 1. Find the value of $(a^3 - 2a^2x + 4ax^2 - 5x^3)^2$.

$$\begin{aligned} \text{The given expn.} &= (a^3)^2 + (-2a^2x)^2 + (4ax^2)^2 + (-5x^3)^2 \\ &\quad + 2a^3(-2a^2x + 4ax^2 - 5x^3) \\ &\quad - 2 \cdot 2a^2x(4ax^2 - 5x^3) + 2 \cdot 4ax^2(-5x^3) \\ &= a^6 + 4a^4x^2 + 16a^2x^4 + 25x^6 - 4a^5x + 8a^4x^2 - 10a^3x^3 \\ &\quad - 16a^3x^3 + 20a^2x^4 - 40ax^5 \\ &= a^6 - 4a^5x + 12a^4x^2 - 26a^3x^3 + 36a^2x^4 - 40ax^5 + 25x^6. \end{aligned}$$

Ex. 2. Find the co-efficient of x^4 in the expansion of $(1+2x-3x^2+4x^3-x^4)^2$.

The student is referred back to the rule in page 82. We shall first examine which of the usual square quantities contains x^4 ; we shall next examine the usual products.

Evidently, of the square quantities, we must take $(-3x^2)^2$ or $9x^4$; of the products, we must take $2.1.(-x^4)$ and $2.2x.4x^2$, i.e., $-2x^4$ and $16x^4$.

∴ the reqd. coef. = $9 - 2 + 16 = 23$. *Ans.*

143. Higher Expansions The expansions of the fourth and fifth powers of binomials are frequently required. We proceed to find out $(a+b)^4$ and $(a+b)^5$.

$$(a+b)^2 = a^2 + 2ab + b^2;$$

$$\therefore \{(a+b)^2\}^2 = (a^2 + 2ab + b^2)^2;$$

$$\begin{aligned} \text{i.e., } (a+b)^4 &= (a^2 + 2ab + b^2)^2 \\ &= (a^2)^2 + (2ab)^2 + (b^2)^2 + 2a^2(2ab + b^2) + 2ab \cdot b^2 \\ &= a^4 + 4a^2b + 6a^2b^2 + 4ab^3 + b^4, \text{ simplifying.} \end{aligned}$$

$$\begin{aligned} \text{Again, } (a+b)^5 &= (a+b)^4 \times (a+b) \\ &= (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)(a+b) \\ &= (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)a \\ &\quad + (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)b \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5, \\ &\quad \text{simplifying.} \end{aligned}$$

We can similarly find $(a-b)^4$ and $(a-b)^5$, or at once write down their expansions from the above results by changing b into $-b$. In the latter work we must bear in mind that $(-b)^2 = b^2$, $(-b)^4 = b^4$, but $(-b)^3 = -b^3$, $(-b)^5 = -b^5$, &c. Thus we get

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4,$$

$$\text{and } (a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

We have now got the following formulæ :

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4. \text{ (Signs all +).}$$

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4. \text{ (Alternately +, -).}$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5. \text{ (Signs all +).}$$

$$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5. \text{ (Alternately +, -).}$$

N. B. It will be a good exercise for the student to obtain the higher powers of binomials than the third independently of formulæ. Thus he may easily find $(a-b)^4$ from $\{(a-b)^2\}^2$ or $(a-b)^5(a-b)$, $(a+b)^5$ from $\{(a+b)^2\}^2(a+b)$, &c.

144. Signs. Any integral power of a positive quantity is evidently positive. This is true not only in the case of a positive

index, but also in that of a negative index. Thus $a^{-b} = +\frac{1}{a^b}$. In the case of negative quantities we lay down the following rule :

The sign of a power of a negative quantity is positive when the index is an even number (positive or negative), but negative when the index is odd.

$$\text{Thus } (-a)^4 = \{(-a)^2\}^2 = (a^2)^2 = +a^4,$$

$$(-a)^{12} = \{(-a)^3\}^4 = (a^3)^4 = +a^{12},$$

$$(-a)^{-6} = \frac{1}{(-a)^6} = +\frac{1}{a^6} = +a^{-6}, \text{ \&c}$$

In short, when m is any positive or negative integer,

$$(-a)^{2m} = +a^{2m}, \quad 2m \text{ being of course even.}$$

$$\text{Again, } (-a)^3 = (-a) \times (-a) \times (-a) = a^2 \times (-a) = -a^3;$$

$$(-a)^7 = (-a)^6 \times (-a) = a^6 \times (-a) = -a^7;$$

$$(-a)^{-13} = \frac{1}{(-a)^{13}} = \frac{-1}{a^{13}} = -\frac{1}{a^{13}} = -a^{-13}, \text{ \&c.}$$

In short, if m be any positive or negative integer,

$$(-a)^{2m+1} = -a^{2m+1}, \quad 2m+1 \text{ being of course odd}$$

Hence we infer that any odd power of $-1 = -1$, and any even power of $-1 = +1$, the index being positive or negative.

Ex. 1. Expand $(2x-a)^6$.

$$\begin{aligned} (2x-1)^6 &= \{(2x-1)^3\}^2 \\ &= \{2x^3 - 3 \cdot 2x(2x-1) + 1^3\}^2 \\ &= (8x^3 - 12x^2 + 6x - 1)^2 \\ &= (8x^3)^2 + (-12x^2)^2 + (6x)^2 + (-1)^2 \\ &\quad + 16x^3(-12x^2 + 6x - 1) - 24x^2(6x - 1) + 12x(-1) \\ &= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1, \text{ Ans.} \end{aligned}$$

Ex. 2 Find $\left(-\frac{a^2b^3}{c^3d^4}\right)^3$ and $(x-2x^{-1}-2)^3$.

$$\left(-\frac{a^2b^3}{c^3d^4}\right)^3 = -\frac{(a^2b^3)^3}{(c^3d^4)^3} = -\frac{a^{2 \times 3} b^{3 \times 3}}{c^{3 \times 3} d^{4 \times 3}} = -\frac{a^6 b^9}{c^9 d^{12}}. \text{ Ans.}$$

$$\begin{aligned} (x-2x^{-1}-2)^3 &= (x-2x^{-1})^2 - 4(x-2x^{-1}) + 2^3 \\ &= (x^2 - 4x \times x^{-1} + 4x^{-2}) - 4(x-2x^{-1}) + 4 \\ &= x^2 - 4x^0 + 4x^{-2} - 4x + 8x^{-1} + 4, \quad (4x^0 = 4. \text{ Art. 130}) \\ &= x^2 - 4x + 8x^{-1} + 4x^{-2}. \text{ Ans.} \end{aligned}$$

Ex. 3. Find the coefficient of x^3 in the product $(x+2)^3(x-1)^2$.

By formula, $(x+2)^3(x-1)^2 = (x^3 + 6x^2 + 12x + 8)(x^2 - 2x + 1)$.

We now select in all possible ways a pair of terms, one out of each of the factors on the right side, such that their product may contain x^3 . We thus get $x^3.1 + 6x^2.(-2x) + 12x.x^2$, i.e., x^3 , simplifying.

\therefore the reqd. coef. = 1. *Ans.*

EXAMPLES 71.

Find the squares of

1. $2a+b-3c$; a^6+b^6 .
2. $2x^2-5y^2-6z^2$; $a^{10}-a^5b^5$.
3. $a^m+b^m+c^m$.
4. $a^{2m}+a^mb^m+b^{2m}$.
5. $\sqrt[3]{x}-3\sqrt{bx}+\sqrt[3]{b}$.
6. $a^{\frac{3}{2}}-2a^{\frac{1}{2}}b^{\frac{1}{2}}-3a^{\frac{1}{2}}b^{\frac{3}{2}}+b^{\frac{3}{2}}$.
7. x^2-x^{-2} ; $x^{\frac{2}{3}}-2x^{\frac{1}{3}}-\frac{1}{x^{\frac{1}{3}}}$.
8. $\left(x^2+\frac{1}{x^2}\right)-2\left(x+\frac{1}{x}\right)+3$.

Find the cubes of

9. $2x+y-3z$; a^5-b^5 .
10. $2x^6+y^6$; $2a^{10}-3b^{10}$.
11. a^4-b^4 ; $\sqrt[3]{x}+\sqrt[3]{y}$.
12. $a^{-\frac{2}{3}}+b^{-\frac{2}{3}}$; $x^{\frac{3}{2}}+y^{-\frac{3}{2}}$.
13. $x^2+\frac{3}{x^2}-1$; x^m-y^m .
14. $x^{\frac{2}{m}}-x^{-\frac{1}{m}}$; $a^{\frac{2}{n}}+2b^{\frac{2}{n}}$.

Find the fourth powers of

15. a^2-b^2 .
16. a^3-2b^3 .
17. $a^{\frac{1}{2}}b^{\frac{1}{2}}+c^{\frac{1}{2}}d^{\frac{1}{2}}$.
18. $a^{\frac{1}{2}}b^{-\frac{1}{2}}-a^{-\frac{1}{2}}b^{\frac{1}{2}}$.
19. a^2-2a+1 .
20. $a^{\frac{2}{n}}-b^{-\frac{2}{n}}$.

Find the 5th and 6th powers of

21. $x+y$; $x-y$.
22. $2x-3y$.
23. a^3-b^3 .

Find the 7th powers of

24. $x-a$; $2a-b$.
25. x^2+a^3 ; $x^{-3}-a^{-3}$.
26. $1-\frac{a}{b}$; $\frac{x}{a}-\frac{a}{x}$.

Find the value of

27. $(\sqrt[3]{-4x^2yz})^6$; $(-a^2b)^3(-ab^2)^4$
28. $(x^2y^{-2}z)^3(-xy^3z^{-2})^2(-x^{-2}yz^3)^{-4}$.
29. $(a^{2n}+b^{2n})^2+(a^{2n}-b^{2n})^2$.
30. $(a^{2n-3}+b^{2n-1})^3+(a^{2n-1}-b^{2n-3})^3$.

Show that

31. $(x+2a-b)^4+(x+b)^4=2(x+a)^4+12(x+a)^2(a-b)^2+2(a-b)^4$.
32. $27(a+b)^3+(a-b)^3=5(a+2b)^3+4(b+2a)^3-9(a^3+2b^3)$.

Simplify

$$33. \frac{(-xy)^8(-yz)^4(-zx)^6}{(-x^2y)^2(y^2z)^2(-z^2x)^2}$$

$$34. \frac{(a^2b)(\sqrt[4]{a^2b^2})(-2ab)^7}{(-a^3b)^5 \cdot (-2b)^4 \cdot \sqrt[3]{(-a^6b^3)}}$$

$$35. \frac{(b+c-a)^3(c+a-b)^3(a+b-c)^3}{(c-a-b)^3(b-c-a)^3(a-b-c)^3}$$

$$36. \frac{(x^2-1)^3(x^2-5x+6)^4}{(2+x-x^2)^4(4x-x^2-3)^8}$$

Find the co-efficient of x^4 in the expansions of :

$$37. (x^2+a^2)^3; (x-2)^3(x+2)^3.$$

$$38. (x^2+ax-b)^2.$$

$$39. (2x^3-x^2-3x+1)^2$$

$$40. (2x^4-3x^2+x-3)^2.$$

$$41. (x^2+2x-3)^3.$$

$$42. (x+2)^2(x-1)^3.$$

$$43. (2x-3)^6.$$

$$44. (ax-b)^7.$$

$$45. \text{ Find the value of } 16x^6 - 20x^3 + 5x, \text{ when } x = \frac{\sqrt{5}-1}{4}.$$

Show that

$$46. (16x^6 - 20x^3 + 5x)^2 + \{16(1-x^2)^{\frac{5}{2}} - 20(1-x^2)^{\frac{3}{2}} + 5(1-x^2)^{\frac{1}{2}}\}^2 = 1.$$

$$47. x^4 + \left\{\frac{1}{2}x + \frac{\sqrt{3}}{2}\sqrt{1-x^2}\right\}^4 + \left\{\frac{1}{2}x - \frac{\sqrt{3}}{2}\sqrt{1-x^2}\right\}^4 = 8.$$

$$48. \left(\frac{\sqrt{1-x^2}}{x} - \sqrt{1-x^2}\right)^2 + (1-x)^2 = \left(\frac{1}{x} - 1\right)^2.$$

$$49. [xy - \sqrt{\{(1-x^2)(1-y^2)\}}]^2 - \left\{\frac{\frac{\sqrt{1-x^2}}{x} + \frac{\sqrt{1-y^2}}{y}}{1 - \frac{\sqrt{1-x^2}}{x} \cdot \frac{\sqrt{1-y^2}}{y}}\right\}^2 = 1$$

CHAPTER XXV.

EVOLUTION.

145. Definition. **Evolution** is the method of finding any root of a given quantity.

146. Sign. Since by the rule of signs $(+a) \times (+a) = a^2$, and also $(-a) \times (-a) = a^2$, it follows that the square root of a^2 is either $+a$ or $-a$. Again, $(+a)(+a)(+a)(+a)(+a)(+a) = a^6$, and also $(-a)(-a)(-a)(-a)(-a)(-a) = a^6$; hence the sixth root of a^6 is either $+a$ or $-a$. Proceeding in this way, we can show that *any even root of a positive quantity may have either the positive or the negative sign.*

Since the continued product of any real quantity, *positive or negative*, taken any *even* number of times, must be always *positive*,

as we have already seen, it is evident that *there is no real even root of a negative quantity*.

Since $a \times a \times a = a^3$, the cube root of $a^3 = a$;

since $(-a)(-a)(-a) = -a^3$, the cube root of $-a^3 = -a$.

$a \times a \times a \times a \times a = a^5$; therefore the fifth root of $a^5 = a$;

$(-a)(-a)(-a)(-a)(-a) = -a^5$; \therefore the fifth root of $-a^5 = -a$.

Proceeding in the same way, we can show that any odd root of a quantity has the same sign as the quantity itself.

147. Simple cases. We have seen that an even root may have either sign. Thus $\sqrt{(4a^2b^2)} = \pm 2ab$, &c. But we will in the present chapter confine ourselves to the positive sign, whenever any even root is possible. We have then the following **Rules** for the extraction of any root of a monomial:

(1) Find the arithmetical root of the co-efficient, if any, and take this root as the co-efficient of the required root.

(2) Divide the index of every factor of the given monomial by the index number denoting the root.

(3) Take the negative sign in the case of an odd root of a negative quantity, and the positive sign in all other cases.

Ex 1 Find $\sqrt[4]{81x^8y^4}$, $\sqrt[3]{-8x^6y^6z^{12}}$ and $\sqrt{\frac{25a^2b^4}{64x^4y^4}}$

$$\sqrt[4]{81x^8y^4} = \sqrt[4]{81} x^{\frac{8}{4}} y^{\frac{4}{4}} = 3x^2y. \text{ Ans}$$

$$\sqrt[3]{-8x^6y^6z^{12}} = \sqrt[3]{-8} x^{\frac{6}{3}} y^{\frac{6}{3}} z^{\frac{12}{3}} = -2x^2y^2z^4 \text{ Ans.}$$

$$\sqrt{\frac{25a^2b^4}{64x^4y^4}} = \frac{\sqrt{25a^2b^4}}{\sqrt{64x^4y^4}} = \frac{5ab^2}{8x^2y^2} \text{ Ans.}$$

EXAMPLES 72.

Find the square root of

- | | | | |
|-------------------------|-----------------------------------|---------------------------|---------------------------------------|
| 1. $9x^4y^8$ | 2. $36a^2b^6c^4$ | 3. $49a^{-2}b^4$ | 4. $121x^{-4}y^4z^8$ |
| 5. $\frac{x^4}{y^2z^2}$ | 6. $\frac{16a^4b^8}{25c^2d^{10}}$ | 7. $\frac{8x^2y^4}{2z^6}$ | 8. $\frac{32a^6b^8c^6}{162a^3b^7d^3}$ |

Find the cube root of

- | | | | |
|--------------------------|-----------------------------------|-----------------------------------|---|
| 9. $27a^3b^6$ | 10. $125a^3b^{-3}$ | 11. $-64a^3b^3a^6$ | 12. $-343x^{-6}y^8z^9$ |
| 13. $\frac{8a^3x^3}{27}$ | 14. $\frac{125a^3b^6}{216x^3y^6}$ | 15. $-\frac{512x^{12}}{27y^3z^9}$ | 16. $-\frac{1458a^3b^{10}}{250a^4b^3c^3}$ |

Find the value of

- | | | |
|-------------------------------|-----------------------------------|------------------------------|
| 17. $\sqrt[5]{-a^5b^5c^{10}}$ | 18. $\sqrt[5]{32a^{10}b^5c^{10}}$ | 19. $\sqrt[3]{4^3a^3b^3c^3}$ |
|-------------------------------|-----------------------------------|------------------------------|

$$20. \sqrt[7]{\left(-\frac{128a^7}{b^7c^7}\right)}.$$

$$21. \sqrt[9]{\left(\frac{3^9 a^9 b^{10}}{bc^9}\right)}.$$

$$22. \sqrt[11]{-\frac{2^3 8^3 a^3 b^{13}}{3 \cdot 9^4 ab}}.$$

$$23. \sqrt[5]{(-1)} \cdot \sqrt[4]{\frac{1}{a^4}} \cdot \sqrt[4]{\frac{1}{b^4}}.$$

$$24. \sqrt[3]{225^{-1}} (\sqrt[3]{27})^3 - 2\sqrt[6]{243} + \sqrt[3]{9^4}. \quad 25. \sqrt[12]{12a} \cdot \sqrt[12]{18a^2} \div \sqrt[30]{\frac{6a^7}{32}}.$$

$$26. \sqrt[n-b]{\left(x^{a^2-b^2}\right)} \cdot \sqrt[2m+1]{-\left(\frac{x^n}{y^n}\right)^{-b}} \cdot \left(\frac{y^b}{x^b}\right)^{n+1}.$$

148. Square Roots of Polynomials by Inspection.

We know that $(a+b)^2 = a^2 + 2ab + b^2$,
and $(a-b)^2 = a^2 - 2ab + b^2$.

Hence to extract the square root of a given expression by inspection, we should put it in either of the forms,

sq. of a quantity + sq. of another \pm twice their product

N. B. Some other artifices may be more useful in special cases, as will appear from some of the following solutions

Ex 1 Find the square root of $4x^3 - 12xy + 9y^3$

$$\begin{aligned} 4x^3 - 12xy + 9y^3 &= (2x)^2 + (3y)^2 - 2 \times 2x \cdot 3y \\ &= (2x - 3y)^2. \end{aligned}$$

\therefore the reqd. sq. root $= 2x - 3y$ *Ans.*

Ex. 2. Find the square root of $x^4 - 4ax^3 + 8a^2x + 4a^4$.

$$\begin{aligned} x^4 - 4ax^3 + 8a^2x + 4a^4 &= (x^2 - 2ax)^2 + 4a^2x^2 + 8a^2x + 4a^4 \\ &= (x^2 - 2ax)^2 + 2 \times 2a^2(x^2 - 2ax) + (2a^2)^2 \\ &= (x^2 - 2ax - 2a^2)^2. \end{aligned}$$

\therefore the reqd. sq. root $= x^2 - 2ax - 2a^2$. *Ans.*

Ex. 3. Find the square root of $\frac{4a^{2m}}{9b^{2m}} + \frac{b^{2m}}{4c^{2m}} - \frac{2a^m}{3c^m} + \frac{4a^m}{3b^m} - \frac{b^m}{c^m} + 1$.

$$\begin{aligned} \text{The given expn.} &= \left(\frac{2a^m}{3b^m}\right)^2 + \left(\frac{b^m}{2c^m}\right)^2 - 2 \times \frac{2a^m}{3b^m} \times \frac{b^m}{2c^m} + \frac{4a^m}{3b^m} - \frac{b^m}{c^m} + 1 \\ &= \left(\frac{2a^m}{3b^m} - \frac{b^m}{2c^m}\right)^2 + 2\left(\frac{2a^m}{3b^m} - \frac{b^m}{2c^m}\right) + 1 \\ &= \left(\frac{2a^m}{3b^m} - \frac{b^m}{2c^m} + 1\right)^2. \end{aligned}$$

\therefore the reqd sq root $= \frac{2a^m}{3b^m} - \frac{b^m}{2c^m} + 1$. *Ans.*

Ex. 4. Find the square root of $\left(a + \frac{1}{a}\right)^2 - 4\left(a - \frac{1}{a}\right)$.

$$\text{Since } \left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2 = \left(a^2 + \frac{1}{a^2} - 2\right) + 4 = \left(a - \frac{1}{a}\right)^2 + 2^2,$$

$$\therefore \text{ the given expn. } = \left(a - \frac{1}{a}\right)^2 + 2^2 - 2 \cdot 2 \left(a - \frac{1}{a}\right)$$

$$= \left\{ \left(a - \frac{1}{a}\right) - 2 \right\}^2.$$

$$\therefore \text{ the reqd. sq. root } = a - \frac{1}{a} - 2. \text{ Ans.}$$

N. B. It is well to remember the results

$$(a+b)^2 = (a-b)^2 + 4ab, \left(a + \frac{1}{a}\right)^2 = \left(a - \frac{1}{a}\right)^2 + 4, \text{ \&c.}$$

Ex. 5. Extract the square root of $\frac{(a^2+b^2)^2}{a^4+b^4-2a^2b^2} + 4 \times \frac{a}{a+b} \times \frac{b}{a-b}$.

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$$\text{Since } a^4 + b^4 - 2a^2b^2 = (a^2 - b^2)^2,$$

$$\text{the given expn. } = \frac{(a^2 + b^2)^2 + 4ab(a^2 - b^2)}{(a^2 - b^2)^2}.$$

$$\text{Since } (a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2, \text{ we have}$$

$$\text{the given expn. } = \frac{(a^2 - b^2)^2 + 4a^2b^2 + 4ab(a^2 - b^2)}{(a^2 - b^2)^2} = \frac{(a^2 - b^2 + 2ab)^2}{(a^2 - b^2)^2}.$$

$$\therefore \text{ reqd. sq. root } = \frac{a^2 + 2ab - b^2}{a^2 - b^2}. \text{ Ans.}$$

Ex. 6. Find the square root of $3(a^2 + 3b^2)(a-b)^2 + a^2(a-3b)^2$.

$$\begin{aligned} 3(a^2 + 3b^2)(a-b)^2 &= (a^2 + 3b^2) \cdot 3(a^2 - 2ab + b^2) \\ &= (a^2 + 3b^2)(3a^2 - 6ab + 3b^2). \end{aligned}$$

Hence by Art. 86, we have

$$\begin{aligned} 3(a^2 + 3b^2)(a-b)^2 &= \{(2a^2 - 3ab + 3b^2) - a(a-3b)\} \{ (2a^2 - 3ab + 3b^2) \\ &\quad + a(a-3b) \} \\ &= (2a^2 - 3ab + 3b^2)^2 - a^2(a-3b)^2. \end{aligned}$$

\therefore transposing $a^2(a-3b)^2$, we have

$$3(a^2 + 3b^2)(a-b)^2 + a^2(a-3b)^2 = (2a^2 - 3ab + 3b^2)^2.$$

$$\therefore \text{ sq. root reqd. } = 2a^2 - 3ab + 3b^2. \text{ Ans.}$$

EXAMPLES 73.

Find the square root of

1. $4a^2 + 20a + 25$.
2. $9a^2 - 24ax + 16x^2$.
3. $4x^2 + 10xy + 25y^2$.
4. $\frac{1}{16}a^2b^2 - \frac{1}{8}ab + \frac{1}{6}c^2$.
5. $100x^4y^2 + 60x^2yz + 9z^2$.
6. $(ab - 1)^2 + 4ab$.
7. $(7x - 3y)^2 + 84xy$.
8. $(9x + 5y)^2 - 180xy$.
9. $\left(x + \frac{1}{x}\right)^2 + 4\left(x - \frac{1}{x}\right)$.
10. $\frac{a^2}{b^2} + \frac{b^2}{a^2} - 6\left(\frac{a}{b} + \frac{b}{a}\right) + 11$.
11. $(x^2 - ax)^2 + 2x^2 - 2ax + 1$.
12. $(x^2 - 3)^2 - 10ax(x^2 - 3) + 25a^2x^2$.
13. $\frac{x^2}{y^2} + \frac{y^2}{x^2} - 8\left(\frac{x}{y} - \frac{y}{x}\right) + 14$.
14. $\left(x + \frac{1}{2x}\right)^2 - 14\left(x - \frac{1}{2x}\right) + 47$.
15. $(x^2 - 4x + 4)(4y^2 - 4y + 1)$.
16. $(a^2 - 1)(a^2 + a - 2)(a^2 + 3a + 7)$.
17. $(6a^3 + 13ax + 6x^2)9a^2 - 4x^2(6a^2 + 5ax - 6x^2)$.
18. $4x^4 + 12x^2 + 1 - x^2 + 6x + 1$.
19. $x^4 - 8ax^2 + 64a^2x + 64a^4$.
20. $x^4 + 10x^2y + 21x^2y^2 - 20xy^3 + 4y^4$.
21. $a^2 + b^2 + 2a(b + 1) + 2b + 1$.
22. $x^{-4} + 4x^{-2} - 2x^{-2} - 12x^{-1} + 9$.
23. $\frac{16a^2}{x^2} + \frac{z^2}{64a^2} + 1$.
24. $x^2 - \frac{2xy}{3} + \frac{xz}{2} + \frac{y^2}{9} - \frac{yz}{6} + \frac{z^2}{16}$.
25. $x^{2m} - 2x^m y^{\frac{1}{2}} + y^2$.
26. $a^{2m} - 2a^{m+\frac{1}{2}} + a^{2m}$.
27. $9x^2 - 6x^{\frac{3}{2}} + 1$.
28. $9 + 6x - 11x - 4x^{\frac{3}{2}} + 4x^2$.
29. $7 - 6(x^m - x^{-m}) + x^{2m} + x^{-2m}$.
30. $25x^{4m+2} - 20x + 4x^{-2m}$.
31. $p^2 + 8p - \frac{64}{p}\left(1 - \frac{1}{p}\right)$.
32. $\left(\frac{a-b}{a}\right)^2 - \frac{2ab}{a^2 - b^2} + \left(\frac{a+b}{b}\right)^{-2}$.
33. $(p^2 + q^2)(l^2 + m^2) + pq(l^2 - m^2) - 2(p^2 - q^2)lm$.
34. $(x - y)^4 - 2(x^2 + y^2)(x - y)^2 + 2(x^4 + y^4)$.
35. $\frac{a^2}{x^2} + \frac{4b^2}{y^2} + \frac{c^2}{4z^2} - \frac{4ab}{xy} + \frac{ac}{xz} - \frac{2bc}{yz}$.
36. $x^2 + (1 + i^2)(1 + i)^2$.
37. $2x - 1)(2x - 3)(2x - 5)(2x - 7) + 16$.
38. $(5a^2 + 2ab + 3b^2)(a^2 - 2ab - b^2) + 4(a^2 + ab + b^2)^2$.
39. $\frac{1}{a-1}\left(a + 3 + \frac{4}{a-1}\right)$.
40. $\frac{(p^2 + 1)^2}{p^4 - 2p^2 + 1} - \frac{4p}{p^2 - 1}$.

149. **Extraction of Square root.** Knowing that $(a+b)^2 = a^2 + 2ab + b^2$, we proceed to discover a method of finding $a+b$ from $a^2 + 2ab + b^2$. The first term, a , of the whole root, $a+b$, is easily found, since it is the square root of the first term, a^2 , of the given expression. The latter term being thus disposed of, the

remainder of the given expression is $2ab + b^2$. The second term, b , of the square root = $\frac{2ab}{2a} = \frac{\text{1st term of the remainder}}{\text{twice 1st root}}$; it is thus easily obtained. We should now subtract not only b^2 , but $b(2a+b)$, from the remainder just referred to, in order to have nothing. The latter part of the work is therefore the same as in dividing $2ab + b^2$ by the complete divisor $2a + b$, i.e., by twice the first term of the root + the 2nd term of the root.

The above process can evidently be extended to the case of the square root consisting of more than two terms. For, after finding the first two terms of the root, we can look upon them as a single term, and then proceed as before to find the next term of the root. We have only to repeat this process, when more terms of the root are possible.

The following rules will now be evident.

(1) Arrange the given expression in descending or ascending powers of some letter.

(2) Find the square root of the first term and put it down as the first term of the required root; subtract its square from the given expression, and bring down the remainder.

(3) Take twice the first term of the root, and make it the first term of a divisor; divide by it the first term of the remainder, put down the quotient as the second term of the root, and also add it to the first divisor to obtain the complete divisor of the above remainder.

(4) Multiply the divisor thus found by the second term of the root, and subtract the product as in division.

(5) Next regard the two terms of the root already obtained as a single term, and proceed as before; repeat this process until there is no remainder.

Ex. 1. Find the square root of

$$x^6 - 2ax^5 + 3a^2x^4 + 2a^3x^3 - 3a^4x^2 + 4a^5x + 4a^6.$$

$$x^6 - 2ax^5 + 3a^2x^4 + 2a^3x^3 - 3a^4x^2 + 4a^5x + 4a^6(x^3 - ax^2 + a^2x + 2a^3)$$

$$\begin{array}{r} x^6 \\ 2x^3 - ax^2 \overline{) x^6 - 2ax^5 + 3a^2x^4} \\ \underline{-2ax^5 + a^2x^4} \\ 2x^3 - 2ax^2 + a^2x^3 \\ \underline{2a^2x^4 - 2a^3x^3 + a^4x^2} \\ 2x^3 - 2ax^2 + 2a^2x + 2a^3 \overline{) 4a^3x^3 - 4a^4x^2 + 4a^5x + 4a^6} \\ \underline{4a^3x^3 - 4a^4x^2 + 4a^5x + 4a^6} \end{array}$$

The reqd root = $x^3 - ax^2 + a^2x + 2a^3$. Ans.

Explanation. In the above work it will be seen that the first term of the root is $\sqrt{x^6}$, i.e., x^3 , and the first remainder is $-2ax^5$ &c. Hence the first term of the first divisor is $2x^3$. The 2nd term of the root $= \frac{-2ax^5}{2x^3} = -ax^2$, the complete first divisor is $2x^3 - ax^2$, and the 2nd remainder is $2a^2x^4$ &c. The first member of the 2nd divisor is $2(x^3 - ax^2)$ or $2x^3 - 2ax^2$. The third term of the root = the quotient $\frac{2a^2x^4 + 2a^3x^3}{2x^3 - 2ax^2}$. But this is more conveniently found from $\frac{2a^2x^4}{2x^3} = a^2x$. The first member of the third divisor $= 2(x^3 - ax^2 + a^2x) = 2x^3 - 2ax^2 + 2a^2x$, the fourth term of the root $= \frac{4a^3x^3}{2x^3} = 2a^3$, and the complete third divisor is $2x^3 - 2ax^2 + 2a^2x + 2a^3$.

Ex. 2. Extract the square root of

$$\frac{x^3}{4y^3} + \frac{4}{x^2} - \frac{1}{3}\left(x + \frac{4y}{x}\right) + \frac{y^2}{9} + \frac{2}{y}.$$

Arrange the expression in descending powers of x ,

$$\text{viz., } x^3, x, x^0, x^{-1}\left(-\frac{1}{x}\right) \text{ and } x^{-2}\left(-\frac{1}{x^2}\right).$$

Observing that $\frac{y^2}{9} + \frac{2}{y} = \left(\frac{y}{3} + \frac{2}{y}\right)x^0$, $\therefore x^0 = 1$,

we proceed as below.

$$\begin{array}{r} \frac{x^3}{4y^3} - \frac{x}{3} + \frac{y^2}{9} + \frac{2}{y} - \frac{4y}{3x} + \frac{4}{x^2} \left(\frac{x}{2y} - \frac{y}{3} + \frac{2}{x} \right) \\ \hline \frac{x^3}{4y^3} \\ \hline \frac{x}{y} - \frac{y}{3} - \frac{x}{3} + \frac{y^2}{9} \\ \hline -\frac{x}{3} + \frac{y^2}{9} \\ \hline \frac{x}{y} - \frac{2y}{3} + \frac{2}{x} \Big)^2 - \frac{4y}{3x} + \frac{4}{x^2} \\ \hline \frac{2}{y} - \frac{4y}{3x} + \frac{4}{x^2} \end{array}$$

$$\text{The reqd. root} = \frac{x}{2y} - \frac{y}{3} + \frac{2}{x}. \quad \text{Ans.}$$

Explanation. The 1st term of the root = $\sqrt{\frac{x^2}{4y^2}} = \frac{x}{2y}$, the 2nd term = $-\frac{x}{3} \div \frac{x}{y} = -\frac{y}{3}$, and the 3rd term = $\frac{2}{y} \div \frac{x}{y} = \frac{2}{x}$. From this work it is evident that the *trial divisor at any stage is practically the same as the first, viz., $\frac{x}{y}$* . This is true in all cases.

Ex. 3. Find the square root of

$$1 - x^{\frac{2}{3}} + \frac{4}{3}x^{\frac{1}{3}}(x^{\frac{1}{3}} - 3) + \frac{2}{3}x^{\frac{2}{3}}(\frac{8}{3}x^{\frac{1}{3}} + 1).$$

Simplify and arrange the expression in ascending powers of x .

$$\begin{array}{r} 1 - \frac{4}{3}x^{\frac{1}{3}} + \frac{2}{3}x^{\frac{1}{3}} + \frac{4}{3}x^{\frac{2}{3}} - x^{\frac{6}{3}} + \frac{9}{16}x(1 - \frac{2}{3}x^{\frac{1}{3}} + \frac{2}{3}x^{\frac{1}{3}} \\ 1 \\ 2 - \frac{2}{3}x^{\frac{1}{3}}) - \frac{4}{3}x^{\frac{1}{3}} + \frac{2}{3}x^{\frac{1}{3}} + \dots \\ - \frac{4}{3}x^{\frac{1}{3}} + \frac{4}{3}x^{\frac{2}{3}} \\ 2 - \frac{4}{3}x^{\frac{1}{3}} + \frac{2}{3}x^{\frac{1}{3}}) \frac{2}{3}x^{\frac{1}{3}} - x^{\frac{6}{3}} + \frac{9}{16}x \\ \frac{\frac{2}{3}x^{\frac{1}{3}} - x^{\frac{5}{3}} + \frac{9}{16}x}{\dots} \end{array}$$

The reqd. root = $1 - \frac{2}{3}x^{\frac{1}{3}} + \frac{2}{3}x^{\frac{1}{3}}$. *Ans.*

Ex. 4. Extract the fourth root of $\frac{x^4}{y^4} + \frac{y^4}{x^4} - 4\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) + 6$.

Arrange the expression in descending powers of x , observing that $6 = 6x^0$. Then extract the square root.

$$\begin{array}{r} \frac{x^4}{y^4} - 4\frac{x^2}{y^2} + 6 - 4\frac{y^2}{x^2} + \frac{y^4}{x^4} \left(\frac{x^2}{y^2} - 2 + \frac{y^2}{x^2} \right) \\ \frac{x^4}{y^4} \\ 2x^2 - 2 \left) - 4\frac{x^2}{y^2} + 6 \right. \\ - 4\frac{x^2}{y^2} + 4 \\ \frac{2x^2}{y^2} - 4 + \frac{y^2}{x^2} \left) 2 - 4\frac{y^2}{x^2} + \frac{y^4}{x^4} \right. \\ 2 - 4\frac{y^2}{x^2} + \frac{y^4}{x^4} \end{array}$$

The square root of the result $\frac{x^3}{y^3} - 2 + \frac{y^2}{x^2}$ will evidently be the fourth root required

$$\frac{x^3}{y^3} - 2 + \frac{y^2}{x^2} = \frac{x^2}{y^2} - 2 \times \frac{x}{y} \times \frac{y}{x} + \frac{y^2}{x^2} = \left(\frac{x}{y} - \frac{y}{x}\right)^2.$$

\therefore the reqd. fourth root $= \frac{x}{y} - \frac{y}{x}$. *Ans.*

N.B. We might have employed the method of inspection from the beginning; thus the given expr. $= \left(\frac{x^4}{y^4} + \frac{y^4}{x^4} + 2\right) - 4\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) + 4$

$$= \left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right)^2 - 4\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) + 4$$

$$= \left(\frac{x^2}{y^2} + \frac{y^2}{x^2} - 2\right)^2.$$

Ex. 5. Find the condition that $ax^2 + bx + c$ may be a perfect square

Proceed to extract the root

$$\begin{array}{r} ax^2 + bx + c \left(\sqrt{ax^2} + \frac{b}{2\sqrt{a}} \right. \\ \underline{ax^2} \\ 2\sqrt{ax^2} + \frac{b}{2\sqrt{a}} \left. \right) \\ \phantom{2\sqrt{ax^2} +} \frac{bx + c}{bx + \frac{b^2}{4a}} \\ \phantom{2\sqrt{ax^2} +} \underline{- \frac{4a}{4a}} \\ \phantom{2\sqrt{ax^2} +} c - \frac{b^2}{4a} \end{array}$$

Therefore $ax^2 + bx + c$ will be a perfect square, if $c - \frac{b^2}{4a} = 0$, i.e., $b^2 = 4ac$, or in words,

(coeff. of 2nd term)² = four times coeff. of 1st term \times last term.

Otherwise thus: If $ax^2 + bx + c$ be a perfect square, it will be of the form $(lx + m)^2$, i.e., $l^2x^2 + 2lmx + m^2$.

\therefore Comparing coefficients, $a = l^2$, $b = 2lm$, and $c = m^2$.

$$\therefore b^2 = 4l^2m^2 = 4ac.$$

Ex. 6. Find the square root of $1 + x$ to four terms in ascending powers of x , and thence find the nearest value of the square root of $\frac{10}{3}$ to six places of decimals.

$$\begin{array}{r}
 1 + x \left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \right) \\
 2 + \frac{x}{2} \left(x + \frac{x^2}{4} \right) \\
 2 + x - \frac{x^2}{8} \left(\frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{64} \right) \\
 2 + x - \dots \left(\frac{x^3}{8} - \frac{x^4}{64} \right) \\
 \frac{x^2}{8} - \frac{x^4}{16} \dots \\
 - \frac{5x^4}{64} \dots
 \end{array}$$

The reqd. root $= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$ to four terms.

Since $\frac{3}{2} = 1 + \frac{1}{2}$, we will now take $x = \frac{1}{2}$.

$$\begin{array}{rcl}
 1 & & = 1 \\
 \frac{x}{2} = \frac{1}{64} & = \frac{1}{8} \div 8 = \frac{.125}{8} & = +.015625 \\
 -\frac{x^2}{8} = -\left(\frac{1}{4} \cdot \frac{1}{2}\right)x & = -\frac{.00390625}{32} & = -.0001220\dots \\
 \frac{x^3}{16} = \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right)x & = \frac{.000610}{32} & = +.000019\dots \\
 \hline
 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} & & = 1.0155049\dots
 \end{array}$$

The term of the root after $\frac{x^3}{16}$ is evidently $-\frac{5x^4}{128}$ or $-\left(\frac{5x^3}{8 \cdot 16}\right)x$; since it will, as is easily seen, have only ciphers in the first seven places, it is neglected.

Since we reject 9 in the 7th decimal place, the nearest value required $= 1.015505$. *Ans.*

EXAMPLES 74.

Find the square root of

1. $9x^4 + 12x^3 - 8x^2 - 8x + 4$.
2. $3a^2b^3 + a^4 + 10a^3b - 110ab^3 + 121b^4$.
3. $\frac{1}{9} + \frac{4a}{3} - 24a^3 + 36a^4$.
4. $12a^6 + 14a^5 - 40a + 25 + 36a^2 - 20a^4 + 9a^0$.
5. $18x^3 - \frac{3x}{2} + \frac{1}{4} + 36x^4 - \frac{15}{4}x^2$.
6. $\frac{9a^4}{64} + \frac{4}{9} - \frac{3}{2}a^3 + \frac{7}{2}a^2 + \frac{5}{3}a$.
7. $\frac{9x^4}{4} - ax^3 + \frac{a^2x^2}{9} - \frac{6bx^2}{5} + \frac{4abx}{15} + \frac{4b^2}{25}$.
8. $\frac{x^2y^2}{16} - \frac{x^3y}{2} + x^4 - \frac{3x^2}{2} + \frac{3xy}{8} + \frac{9}{16}$.
9. $1 + x^6 - \frac{1}{2}x^2(1 + x^2) + x^2(2 + \frac{9}{16})x$.
10. $\frac{2b}{15} - \frac{3b^2}{5a} - ab^3 + \frac{9b^4}{4} + \frac{a^2b^2}{9} + \frac{1}{25a^2}$.
11. $\frac{4}{9} + \frac{16x}{3y} + 14\frac{x^2}{y^2} - 12\frac{x^3}{y^3} + \frac{9x^4}{4y^4}$.
12. $4p^3 + 9q^2 + 16r^2 - 12pq + 24qr - 16rp$.
13. $a^4x^2 - 4a^3xy + (4y^2 - 6zx)a^2 + 12ayz + 9z^2$.
14. $\frac{a^2}{b^2} - a - \frac{ab}{3} + \frac{2a^2}{3b} + \frac{b^2}{4} + \frac{a^3}{9}$.
15. $\frac{8}{3}a^2b^2(a^2 + \frac{2}{9}ab + \frac{1}{36}b^2) + 4ab(16a^4 + \frac{1}{81}b^4) + 64a^6 + \frac{b^6}{729}$
16. $\frac{a^3}{b^2} + 4\frac{a}{c} - 6\frac{a}{b} + 4\frac{b}{c^2} - 12\frac{b}{c} + 9$.
17. $\frac{x^2}{4y^2} + \frac{4y^2}{9x^2} + \frac{9z^2}{16x^2} - \frac{2x}{3z} - \frac{3z}{4y} + \frac{y}{x}$.
18. $16c^4 + a^2(4b^2 - 12bc + 17c^2) + a^4 + 2a(2b - 3c)(a^2 + 4c^2)$.
19. $25x^9 - 40x^7 + 46x^6 - 44x^5 + 35x^4 - 20x^3 + 10x^2 - 4x + 1$.
20. $\frac{x^2}{y^2} + \frac{y^2}{x^2} + 6\frac{x}{y} - 6\frac{y}{x} + 7$.
21. $\frac{5}{4}\left(1 + \frac{5}{9}\frac{a^2b^2}{c^2}\right) - \frac{5}{6}\frac{ab}{c} - \frac{3}{5}\frac{c}{ab}\left(1 - \frac{3}{5}\frac{c}{ab}\right)$.

$$22. \frac{x^4}{y^4} + 4\frac{x^2}{y^2} + \frac{y^2}{x^2} - 2\frac{x}{y} - 4\frac{y}{x} + 4.$$

$$23. \frac{x^6}{y^6} + \frac{y^6}{x^6} + 6\left(\frac{x^4}{y^4} + \frac{y^4}{x^4}\right) + 3\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) - 20.$$

$$24. \frac{4x^4}{y^2} + \frac{1}{4x^2y^2} + \frac{1}{9x^4} - \frac{2x}{y^2} + \frac{4}{3y} + \frac{1}{3x^2y}.$$

$$25. a^{-6} + 4a^{-2}(1 - b^{-4}) + 4b^{-8} + 4(1 - 2b^{-4}).$$

$$26. (a^{\frac{4}{3}} + a^{\frac{2}{3}})b - 6a^{\frac{1}{3}}b^{\frac{1}{3}}(\sqrt[3]{a} + 1) + 2ab + 9.$$

$$27. 49\sqrt[3]{\frac{1}{b^8}} + 9\sqrt[3]{\frac{8}{a^4}} + \sqrt[3]{\frac{4}{b^2}} - 2b^{\frac{2}{3}}(7a^{\frac{2}{3}}b^{-\frac{2}{3}} + 3a^{-\frac{2}{3}}b^{\frac{2}{3}}).$$

$$28. x^{\frac{4}{11}}\left(1 + \frac{a}{b} + \frac{a^2}{4b^2}\right) - a - 2b + b^2x^{-\frac{1}{11}}.$$

$$29. 4a^{2m} + 9a^{2n} + 4(3a^{m+n} - 2a^m - 3a^n + 1).$$

$$30. \frac{a^{2m}b^{-2n}}{2^4} - 2^2 \cdot 3^8 a^mb^{-n} + \frac{a^{2n}b^{-2m}}{3^4} - 2^4 \cdot 3a^nb^{-m} + 6b + \frac{1}{2}3^2(a^{n-1})^{m+n}.$$

31. Find the fourth root of

$$x^8 - 4x^7 + 14x^6 - 28x^5 + 49x^4 - 56x^3 + 56x^2 - 32x + 16.$$

32. Find the eighth root of

$$r^8 - 8r^7 + 28r^6 - 56r^5 + 70r^4 - 56r^3 + 28r^2 - 8r + 1.$$

Find the condition that the following may be perfect squares, and then find the square roots:

$$33. (a+1)x^2 + 6x + 9$$

$$34. (l+21)x^2 - (2l+12)x + l.$$

$$35. (a^2 + a + 1)x^2 + 2(a^2 + 1)xy + (a^2 - a + 1)y^2.$$

$$36. x^4 - 2(a+b)x^3 + (a^2 + 4ab + b^2)x^2 - 2ab(a+b)x + c.$$

150. Arithmetical Square Root. The arithmetical process of extracting the square root of a number may be easily derived from the algebraical process just detailed. For example, take 172225. Break it up as 160000 + 8000 + 4100 + 100 + 25. Now apply the algebraical method.

$$\begin{array}{r} 160000 + 8000 + 4100 + 100 + 25 \left(400 + 10 + 5 \right. \\ \hline 160000 \\ \hline 800 + 10 \left. \right) 8000 + 4100 + 100 \quad = 12200 \\ \hline = 810 \quad 8000 + 100 \quad = 8100 \\ \hline 800 + 20 + 5 \left. \right) 4000 + 100 + 25 \quad = 4100 + 25 = 4125 \\ \hline = 825 \quad 4000 + 100 + 25 \quad = 4125 \end{array}$$

Dropping the ciphers in the above work, and without breaking up the number we have the usual arithmetical work :

$$\begin{array}{r} 1\overline{)72225} \quad (415 \\ \underline{16} \\ 81 \overline{)122} \\ \underline{81} \\ 825 \overline{)4125} \\ \underline{4125} \end{array}$$

151. Cube Root by Inspection. We know that

$$(a+b)^3 = a^3 + 3ab(a+b) + b^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

$$\text{and } (a-b)^3 = a^3 - 3ab(a-b) - b^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

Note that the number of terms in each expansion is four and that the sign in the last is alternately positive and negative.

Hence if we can arrange an expression in descending powers of some letter so as to have it in any of the above forms, its cube root will be the *algebraical sum of the cube roots of the first and last terms*.

For example, take the expression $x^3 + 12x^2 + 48x + 64$. If it be a complete cube, the cube root must be $\sqrt[3]{x^3}/\sqrt[3]{64}$ or $x + 4$. Having thus guessed out the cube root, we adopt the following process :

$$\begin{aligned} x^3 + 12x^2 + 48x + 64 &= x^3 + 12x(x+4) + 4^3 \\ &= x^3 + 3 \cdot 4 \cdot x(x+4) + 4^3 \\ &= (x+4)^3 \end{aligned}$$

\therefore the reqd. cube root $= x + 4$

The above process can be easily extended to the case in which the cube root consists of three terms. Here it is important to bear in mind that the expressions that readily yield to the foregoing treatment are such as contain only three powers of some letter. Let us try to find the cube root of

$$x^3 - 3ax^2 + 6bx^2 + 3a^2x - 12abx + 12b^2x - a^3 + 6a^2b - 12ab^2 + 8b^3.$$

Arranging in powers of x , we have the given expression

$$\begin{aligned} &= x^3 - 3x^2(a-2b) + 3x(a^3 - 4ab + 4b^3) - (a^3 - 6a^2b + 12ab^2 - 8b^3) \\ &= x^3 - 3x^2(a-2b) + 3x(a-2b)^2 - (a-2b)^3 \\ &= \{x - (a-2b)\}^3 = (x-a+2b)^3. \end{aligned}$$

\therefore the cube root $= x - a + 2b$. *Ans.*

N. B. We first guess out the root here as the algebraical sum of the cube roots of the terms that are perfect cubes,

$$\text{i.e., } \sqrt[3]{x^3} + \sqrt[3]{-a^3} + \sqrt[3]{8b^3} \text{ or } x - a + 2b.$$

(5) Next regard the two terms of the root already found as a single term, and proceed as before; repeat this process until there is no remainder.

Ex. 1. Extract the cube root of

$$\begin{aligned} & 8a^6 - 18a^5 + \frac{5}{2}a^4 - \frac{1}{8}a^3 + \frac{5}{4}a^2 - \frac{3}{2}a + 1 \\ & 8a^6 - 18a^5 + \frac{5}{2}a^4 - \frac{1}{8}a^3 + \frac{5}{4}a^2 - \frac{3}{2}a + 1 \left(2a^2 - \frac{3}{2}a + 1 \right. \\ & \quad \left. 8a^6 \right. \\ & \quad \left. 12a^4 - 9a^3 + \frac{1}{4}a^2 \right) - 18a^5 + \frac{5}{2}a^4 - \frac{1}{8}a^3 \\ & \quad - 18a^5 + \frac{5}{2}a^4 - \frac{2}{8}a^3 \\ & 3(2a^2 - \frac{3}{2}a)^2 + 3(2a^2 - \frac{3}{2}a + 1) \left(12a^4 - 18a^3 + \frac{5}{4}a^2 - \frac{3}{2}a + 1 \right. \\ & = 12a^4 - 18a^3 + \frac{5}{4}a^2 - \frac{3}{2}a + 1 \left. \right) 12a^4 - 18a^3 + \frac{5}{4}a^2 - \frac{3}{2}a + 1. \\ & \therefore \text{the reqd. cube root} = 2a^2 - \frac{3}{2}a + 1. \text{ Ans.} \end{aligned}$$

N.B. The last trial divisor is $3(2a^2 - \frac{3}{2}a)$ or $12a^4 - 18a^3 + \frac{5}{4}a^2$. It is practically only $12a^4$, i.e., the same as at beginning, and the last term of the root = $\frac{12a^4}{12a^4}$ = 1. This practical hint will be found very useful.

Ex. 2. Extract the sixth root of

$$\frac{x^6}{y^6} + \frac{64}{x^6} + \frac{60x^2}{y^4} + \frac{240}{x^2y^2} - \frac{192}{x^4y} - \frac{160}{y^3} - \frac{12x^4}{y^6}.$$

First extract the cube root, and then the square root of the result. Arranging in descending powers of x :

$$\begin{aligned} & \frac{x^6}{y^6} - \frac{12x^4}{y^6} + \frac{60x^2}{y^4} - \frac{160}{y^3} + \frac{240}{x^2y^2} - \frac{192}{x^4y} + \frac{64}{x^6} \left(\frac{x^2}{y^2} - \frac{4}{y} + \frac{4}{x^2} \right. \\ & \quad \left. \frac{x^6}{y^6} \right. \\ & \quad \left. \frac{3x^4}{y^4} - \frac{12x^2}{y^3} + \frac{16}{y^2} \right) - \frac{12x^4}{y^6} + \frac{60x^2}{y^4} - \frac{160}{y^3} \\ & \quad - \frac{12x^4}{y^6} + \frac{48x^2}{y^4} - \frac{64}{y^3} \\ & 3 \left(\frac{x^4}{y^4} - \frac{8x^2}{y^3} + \frac{16}{y^2} \right) + \frac{17(x^2 - 4)}{x^2(y^2 - y)} + \frac{16}{x^4} \left(\frac{12x^2}{y^4} - \frac{96}{y^3} + \frac{240}{x^2y^2} - \frac{192}{x^4y} + \frac{64}{x^6} \right. \\ & = \frac{2x^4}{y^4} - \frac{24x^2}{y^3} + \frac{60}{y^2} - \frac{48}{x^2y} + \frac{16}{x^4} \left. \frac{12x^2}{y^4} - \frac{96}{y^3} + \frac{240}{x^2y^2} - \frac{192}{x^4y} + \frac{64}{x^6} \right. \\ & \text{The cube root} = \frac{x^2}{y^2} - \frac{4}{y} + \frac{4}{x^2} = \frac{x^2}{y^2} - 2 \cdot \frac{x}{y} + (x)^2 = \left(\frac{x}{y} - x \right)^2. \end{aligned}$$

\therefore the reqd. sixth root = $\frac{x}{y} - x$. **Ans.**

EXAMPLES 75.

Find by inspection or otherwise the cube root of

1. $x^3 + 9x^2 + 27x + 27$.
2. $8x^3 + 12x^2 + 6x + 1$.
3. $x^3 - 12x^2 + 48x - 64$.
4. $8x^3 - 60x^2 + 150x - 125$.
5. $64x^3 - 144ax^2 + 80a^2x - 27a^3$.
6. $27a^3b^3 - 54a^2b^2c^2 + 36ab^2c^2 - 8c^3$.
7. $\frac{x^3}{y^3} + \frac{8y^3}{x^3} + 6\left(\frac{x}{y} + \frac{2y}{x}\right)$.
8. $\frac{a^3}{27b^3} - \frac{8b^3}{a^3} - \frac{2}{3}\frac{a}{b} + \frac{b}{a}$.
9. $64a^{-3} - 48a^{-2}b^{-1} + 12a^{-1}b^{-2} - b^{-3}$.
10. $6x^{\frac{2}{3}}y^{-\frac{1}{3}} + 12x^{\frac{1}{3}}y^{-\frac{2}{3}} + x + 8y^{-1}$.
11. $a^3(a^3 + 3b^2)^2 - b^2(3a^2 + b^2)^2$.
12. $(3x^2 + 6x + 4)^2 - (x + 1)^2(x^2 + 2x + 4)^2$.
13. $x^3 + 3x^2(a - b) + 3x(a^2 - 2ab + b^2) + (a^3 - 3a^2b + 3ab^2 - b^3)$.
14. $x^6 - 3ax^4 + 6a^2x^4 + 3a^2x^2 - 12a^3x^2 + 12a^4x^2 - a^5 + 6a^4 - 12a^5 + 8a^6$.
15. $x^3y^3 - 15x^2y^3x + 75xy^3x^2 - 125y^3x^3$.
16. $b^6 - 3ab^5 + 9a^2b^4 - 13a^3b^3 + 18a^4b^2 - 12a^5b + 8a^6$.
17. $8 + 60a^2 + 114a^4 - 55a^6 - 171a^8 + 135a^{10} - 27a^{12}$.
18. $a^3 - 6a^2b + 3a^2 + 12ab^2 - 12ab + 3a - 8b^3 + 12b^2 - 6b + 1$.
19. $\frac{2}{8}(x+y)^3 - \frac{2}{27} - \frac{1}{4}(x+y)^2x + \frac{1}{2}(x+y)x^2$.
20. $1728a^6 + 1728a^4b^3 + 576a^2b^6 + 64b^9$.
21. $\frac{x^6}{64} - \frac{ax^5}{32} + \frac{5a^2x^3}{216} - \frac{a^5x}{162} - \frac{a^6}{729}$.
22. $\frac{x^6}{y^6} + \frac{y^6}{x^6} - 6\left(\frac{x^4}{y^4} + \frac{y^4}{x^4}\right) + 15\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) - 20$.
23. $\frac{x^3}{27y^3} - \frac{x^3}{3y^3} + \frac{x^3}{y} - x^3 + \frac{x}{y} - 6x + 9xy + \frac{9y}{x} - \frac{27y^3}{x} + \frac{27y^3}{x^3}$.
24. Find the sixth root of

$$729x^6 + 729x^5y + 12415x^4y^2 + 1225x^3y^3 + 1185x^2y^4 + 108xy^5 + \frac{y^6}{64}$$
25. Find the ninth root of

$$x^9 - \frac{9}{2}x^8y + 9x^7y^2 - \frac{2}{3}x^6y^3 + \frac{6}{8}x^5y^4 - \frac{9}{16}x^4y^5 + \frac{2}{1}x^3y^6 - \frac{9}{32}x^2y^7 + \frac{9}{512}xy^8 - \frac{y^9}{512}$$
26. Find the cube roots of $8+x$ and $27-x$ to 3 terms, and thence find approximate values of the cube roots of 10 and 25.

MISCELLANEOUS EXAMPLES II.

1. Divide $6a^5 - a^5 - 61a^3 + 10a + 2$ by $2a^2 + 3a - 1$.
2. Show that $(x+ay)(y-ax) + (ax+y)(ay-x) + 2a(x+y)(x-y) = 0$.
3. Find the L. C. M. of $x^4 + 4x^3 - 2x^2 - 12x + 9$ and $x^4 - 10x^3 + 9$, and the H.C.F. of $8 + 12x^{-1} + 2x^{-2} + 2x^{-4}$, $15 + 21x^{-1} - 3x^{-3} + 3 - x^{-4}$.
4. Factorize : (1) $x^6 - 5x^3 - 24$; (2) $x^6 + 5x^3 - 24$;
(3) $x^2 - 4y^2 - 6x + 9$; (4) $xx(a^2 + b^2) + ab(y^2 - z^2 - x^2) + xy(a^2 - b^2)$
5. Simplify $\frac{(x+2)^2 - 9x^2}{(x-3)^2 - 4x^2} \times \frac{x^2 + x - 12}{x^2 + 7x - 8} \times \frac{x^2 + 9x + 8}{x^2 + 2x - 8}$.
6. Find the product
 $(x^{2^{n-1}} + a^{2^{n-2}} \cdot x^{2^{n-2}} + a^{2^{n-1}})(x^{2^{n-1}} - a^{2^{n-2}} x^{2^{n-2}} + a^{2^{n-1}})$.
7. Extract the square root of
 $4x^6 - 20x^4 + 16x^2 + 25x^2 - 40x + 16$.
8. Simplify $\frac{1 + \sqrt{1+x^2}}{1 - \sqrt{1+x^2}} + \frac{1 - \sqrt{1+x^2}}{1 + \sqrt{1+x^2}}$.
9. Solve the following equations :
(1) $[x - \{2x - 3(x-2)\}] = 1$; (2) $2x + 7 - 2(5x + 3) = x - 8$.
10. A bag contains the same number of pounds, shillings and pence ; if their total value be £ a , and the number of each kind of coins be b , shew that $240a = 253b$.
11. Divide $6a^6 - 19a^5 + 6a^3 - 3a + 2$ by $3a^2 - 2a + 1$.
12. Find the coefficient of x^3 in the product
 $(ax^3 + bx^2 + cx + d)(a^1x^3 + b^1x^2 + c^1x + d^1)$.
13. Find the H.C.F. of $2a^2 - ab - b^2$, $6a^2 + ab - b^2$, $16a^3b + 2b^4$.
14. Factorize : (1) $4x^2 + 12ax + 6ay - y^2$; (2) $60x^2 - 52x + 8$;
(3) $a^3 + 3ab + 2b^2 - bc - 3c^2 - 2ac$.
15. Simplify $\frac{2x-3}{2x^2-x-3} - \frac{2x+5}{2x^2+x-3} + \frac{2x^2}{x^2-1}$.
16. Evaluate $\frac{\sqrt{3+1}}{\sqrt{3-1}} \times \frac{3+\sqrt{6}}{\sqrt{3}+\sqrt{2}}$ to three places of decimals
17. Extract the square root of $4(x+x^{-1}) - 4(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) + 9$.
18. What value of a will make the expression $x^3 - ax^2 + 11x - a$ exactly divisible by $x - 2$?
19. For what value of y is $20(3y-5) = 12(4y-3)$?
20. Find two numbers, whose sum is 42, such that three times the greater increased by their difference may be equal to six times the less diminished by 8.

21. Find without actual division the quotient of $a^2(a-2b)-b^2(b-2a)$ by $a-b$; $(a+b)^2+(a-2b)^2$ by $3a-b$.
22. Find the H. C. F. of $x^4-(a-b)x^3+(a-b)b^2x-b^4$ and $x^4-2a(a-b)x^3+(a^2+b^2)(a-b)x-a^2b^2$.
23. Simplify $(5x+2y)^2+3(5x+2y)(x+y)-10(x+y)^2$.
24. Resolve into elementary factors : (1) $12-28x-5x^2$; (2) $(a+1)^3+(a-1)^3-3a^3+4$; (3) $2x^2-11xy-6y^2-x+19y-3$.
25. Simplify $\frac{a+b}{(a^2-bc)(b^2-ca)}+\frac{b+c}{(b^2-ca)(c^2-ab)}+\frac{c+a}{(c^2-ab)(a^2-bc)}$.
26. Divide $x^2y^2+2+x^{-2}y^2$ by $x^{\frac{2}{3}}y^{-\frac{2}{3}}-1+x^{-\frac{2}{3}}y^{\frac{2}{3}}$.
27. $(a-cx)^2+(x^2-1)(b^2-a^2)$ is a complete square, if $x=\frac{a+b}{c-d}$.
28. Find the value of $(x^{\frac{1}{m-n}})^{\frac{m}{n}} \times (x^{\frac{m}{n-1}})^{\frac{n}{m-1}} (x^{\frac{n}{1-m}})^{\frac{1}{n-m}}$.
29. Find the value of $\sqrt{12}-\sqrt{5}+\sqrt{27}+\sqrt{80}-\sqrt{48}-\sqrt{45}$.
30. One of two bills exceeded the other by one third of the latter, and in paying them off the change out of a ten-rupee note was half their difference; what was the amount of each bill?
31. Find the value of the expression $\sqrt{s(s-a)(s-b)(s-c)}$, when $a=13$, $b=14$, $c=15$, and $2s=a+b+c$.
32. Simplify $(x-a)(x+b)(x+c)-(x+a)(x-b)(x+c)-2x(x-c)(a-b)$.
33. Find the condition that the expressions x^2+ax+b and x^2+cx+d may have a common linear factor in x .
34. Resolve into factors : (1) $a^2-4(b^2-a-1)$; (2) $a^2-56ab-44b^2$; (3) $x^3-13x+12$.
35. If $2s=a+b+c$, shew that $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}$.
36. Divide $x^{\frac{3}{2}}y^{-\frac{3}{2}}+2+x^{-\frac{3}{2}}y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}y^{-\frac{1}{2}}-1+x^{-\frac{1}{2}}y^{\frac{1}{2}}$.
37. Find the square root of $(a+b)(a+3b)(a-5b)(a-7b)+64b^4$.
38. If $y-z=ax$, $z-x=by$, $x-y=cz$, prove that $abc+a+b+c=0$.
39. If $2s=a+b+c$, prove that $(2as+bc)(2bs+ca)(2cs+ab)$ is a perfect square.
40. I ride $(b-c)$ miles at the rate of a miles per hour, then walk $(a-b)$ miles at the rate of c miles per hour, next rest for a/c hours, and lastly get back to my starting place by train; if I have been out altogether for c/a hours, find the rate of the train.

41. Add together $3x^{\frac{4}{3}} - 2x^{\frac{2}{3}}y^{\frac{2}{3}} + 4y^{\frac{4}{3}}$, $3x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} - 3y^{\frac{4}{3}}$, $4x^{\frac{2}{3}}y^{\frac{4}{3}} - 3x^{\frac{4}{3}}$, $4x^{\frac{2}{3}}y^{\frac{2}{3}} - x^{\frac{4}{3}} + y^{\frac{4}{3}}$, and $5(x^{\frac{4}{3}} + y^{\frac{4}{3}})$
42. Multiply $x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 2$ by $x^{\frac{3}{2}} - x$
43. Simplify $(5a+2b)^2 - 7(5a+2b)(a+b) + 10(a+b)^2$.
44. Find the L. C. M. of the following : $3a^3 - 7a^2b + 5ab^2 - b^3$, $a^3b + 3ab^2 - 3a^2b - b^3$, and $3a^3 + 5a^2b + ab^2 - b^3$.
45. Simplify $\frac{\left(\frac{3a+a^3}{1+3a^2}\right)^3 - 1}{3^3a - 1} \div \frac{\frac{3^3 - a^3}{3a^2 + 1} - 9}{\frac{2(a^2+3)}{(a^3-a)^2} - \frac{3}{a^2}}$
46. If $a = \frac{2xy+y^3}{x^2+xy+y^2}$, $b = \frac{x^2-y^2}{x^2+xy+y^2}$, then $a^3 + b^3 = a$.
47. Expand $\left(\frac{x}{y} - \frac{2y}{x} - 7\right)^2$ and $\left(\frac{2x}{y} - \frac{3y}{x}\right)^4$.
48. Simplify $\left(\frac{a-b}{c-d}\right)^{2m} \times \left(\frac{a-b}{d-c}\right)^{-m}$, and $\frac{(a^{a+b}-x^a)(x^a-x^{a-b})}{(a^{a+b}-x^a)-(x^a-x^{a-b})}$.
49. Divide $a(b-c)^2 - b(c-a)^2 + c(a-b)^2$ by $(a-b)(b-c)$.
50. I bought a number of plums at 8 a pice ; I kept one-fifth of them, and sold the rest at 5 a pice, gaining 7 pice ; how many did I buy ?
51. Find the value of $(1+2a+3a^2+4a^3)^2$.
52. If $2s = a+b+c$, shew that $(s-a)^3 + (s-b)^3 + (s-c)^3 = s^3 - 3abc$.
53. Shew that $a(b+c)(b^2+c^2-a^2) + b(c+a)(c^2+a^2-b^2) + c(a+b)(a^2+b^2-c^2) = 2abc(a+b+c)$
54. Resolve into factors :
(1) $x^2 + \frac{8}{3}xy - \frac{7}{3}y^2$; (2) $18 - 3x - 28x^2$; (3) $x^4 + 64a^4$.
55. Simplify the fractions :

$$(1) \frac{\frac{a^2}{b^2}\left(1+\frac{b}{a}\right)^2 - \left(1+\frac{a}{b}\right)}{\frac{a^2}{b^2}\left(1+\frac{b}{a}\right)^2 - \frac{a}{b}\left(1+\frac{a}{b}\right)} ;$$

$$(2) \frac{\left\{1 - \frac{x}{x+y} + \frac{x^2}{(x+y)^2}\right\} \left\{1 - \frac{x^2}{(x+y)^2}\right\}}{\left\{1 + \frac{x^3}{(x+y)^3}\right\} \left(1 - \frac{x}{x+y}\right)}.$$

56. Simplify $\frac{1}{1+x^{a-b}+x^{a-c}} + \frac{1}{1+x^{b-a}+x^{b-c}} + \frac{1}{1+x^{c-a}+x^{c-b}}$.

57. Extract the square root of

$$\frac{x^8}{4} + x^{\frac{5}{2}} + \frac{5x^3}{2} + 5x^{\frac{3}{2}} + \frac{25x}{4} + 6x^{\frac{1}{2}} + 4.$$

58. Multiply $x^n - 1 + x^{-n}$ by $(x^{2n} + x^{-2n}) + (x^n + x^{-n})$.

59. Express as the difference of two squares, giving three different results : $x(x+3a)(x+5a)(x+7a)$.

60. At an election the majority was 100, which was one-eighth of the whole number of voters : find the number of votes on each side.

61. If $x^2 - x - 2$ and $x^2 + 4x + a$ have a common linear factor in x , shew that a must be either 3 or -12.

62. Find the value of $(1-x)^3(1+x)^3(1+x+x^2)^3(1-x+x^2)^3$.

Find the L. C. M. of $1+x^{\frac{1}{2}}+x+x^{\frac{3}{2}}$ and $2x+2x^{\frac{3}{2}}+3x^2+3x^{\frac{5}{2}}$.

63. Divide $m(bx^2 - cx) + a(mx^3 - nx^2) - n(bx - c)$ by $mx - n$.

64. Factorize : (1) $(xy + yz + zx)^2 - (x^2y^2 + y^2z^2 + z^2x^2)$;

(2) $3a^2 + 16a - 2$; (3) $x^2 + \left(a + \frac{1}{a}\right)xy + y^2$.

65. Find the value of the expression $(x-a)^2 - (y-b)^2$,

when $4x = 4(a+b) + \frac{(a-b)^2}{a+b}$, and $4y = a+b + \frac{4ab}{a+b}$.

66. Simplify

$$\frac{a^2(m-b)(m-c)}{(a-b)(a-c)} + \frac{b^2(m-c)(m-a)}{(b-c)(b-a)} + \frac{c^2(m-a)(m-b)}{(c-a)(c-b)}.$$

67. Write down at once the following quotients :

(1) $(a^7 - b^7) \div (a - b)$; (2) $(a^{10} - x^{10}) \div (a^2 - x^2)$;

(3) $32 + 243x^5$ by $2 + 3x$.

68. If $x^2 - yz = a^2$, $y^2 - zx = b^2$, and $z^2 - xy = c^2$, find the value of

$$\frac{a^2x + b^2y + c^2z}{x + y + z}$$
 in terms of a , b and c .

69. If $s = \frac{1}{2}(a^2 + b^2 + c^2 + d^2)$, shew that

$$(ad + bc)^2 - s^2 = \frac{1}{4}(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a).$$

70. What is the total outlay of a man whose income is £396, when one-third of it brings in 2 per cent, one-fourth 3 per cent., one-fifth 4 per cent., and the remainder 5 per cent. ?

71. Simplify $(x - 2y + z)^2 + (x - 2y - z)(3x - 6y + z)$.

72. Prove that

$$(b-a)(c-b-a) + (c-b)(a-c-b) + (a-c)(b-a-c) = 0.$$

73. Find the H. C. F. of the following : $2a^5 - 6a^4b + 8a^3b^2$, $4a^5 - 8a^4b - 4a^3b^2 + 8a^2b^3$, and $6a^5 + 12a^4b - 30a^3b^2 - 36a^2b^3$.

74. Resolve into factors : (1) $(x^2 + y^2 + z^2)^2 - 4(xy + yz + zx)^2$;

$$(2) x^2(x+a)^2 - 4a^4 ; (3) x^{12} - y^{12} ; (4) 48 + \frac{3a^8}{4}.$$

75. Reduce to the lowest terms :

$$(1) \frac{(x+2a)^3 + (x+2b)^3 - 8(x+a+b)^3}{(x+2a)(x+2b)(x+a+b)} ;$$

$$(2) \frac{c^3 + 2(a+b)c^2 + (1+4ab)c + 2b}{1 + 2(a+b)c + (1+4ab)c^2 + 2bc^3}.$$

76. Simplify

$$(a-b)\frac{a^2}{(a-c)(x-a)} + (b-a)\frac{b^2}{(b-c)(x-b)} + (c-a)\frac{c^2}{(c-b)(x-c)}.$$

77. Simplify $\frac{x^{2m} - x^{2n}}{x^m + x^n} \times \frac{x^{2m} - x^{2n}}{x^{2m} - 2x^{m+n} + x^{2n}} \times \frac{1}{x^{2m} + x^{m+n} + x^{2n}}.$

78. Simplify $(x+a)^5 - (x-b)^5 - (a+b)^5.$

79. Shew that $b(c-a)^n + c(a-b)^n = (b-c)(c-a)^n + c(c-a)^n + (a-b)^n$, and thence shew that $a(b-c)^n + b(c-a)^n + c(a-b)^n$ is exactly divisible by $(b-c)(c-a)(a-b)$ when n is any odd whole number ; write down the quotient in the latter case, when $n=3$.

80. If 57 be added to a number, the sum will be six times the number diminished by 8 ; what is the number ?

81. Divide $a^8 + a^4x^4 + x^8$ by $a^2 + ax + x^2$.

82. If L denote the L. C. M. and G the H. C. F. of any n quantities, then LG^{n-1} = the continued product of those quantities.

83. Find the L. C. M. of

$$3x^4 - 8x^5 + x^2 - 1, 3x^4 - 2x^5 - 4x^2 + 2x + 1.$$

84. Resolve into factors :

$$(1) (a+1)^2(b-c) + (b+1)^2(c-a) + (c+1)^2(a-b) ;$$

$$(2) 252 - 332x + 96x^2 ; (3) (a^2 + b^2)^3 - 8a^2b^3 ; (4) x^8 - y^8.$$

85. Simplify $\frac{(a+1)^2}{(a-b)(a-c)} + \frac{(b+1)^2}{(b-a)(b-c)} + \frac{(c+1)^2}{(c-b)(c-a)}.$

86. Shew that

$$a(b-c)^3 + b(c-a)^3 + c(a-b)^3 + a^2(b-c) + b^2(c-a) + c^2(a-b) = 0.$$

87. When n is any odd integer, prove that $(x+y+z)^n - x^n - y^n - z^n$

is exactly divisible by $(x+y)(y+z)(z+x)$; find the quotient when $n=5$.

88. Find the term independent of x in the expansion of

$$\left(\frac{x}{2y} - y + \frac{2}{x}\right)^3.$$

89. Find by inspection the square roots of

$$\left(\frac{x}{a} + \frac{2a}{x}\right)^2 - 4\left(\frac{x}{a} - \frac{2a}{x}\right) - 4, \text{ and } a^2\left(1 + \frac{1}{b} + \frac{1}{4b^2}\right) - c\left(2 + \frac{1}{b}\right) + \frac{c^2}{a^2}.$$

90. A father's age is a years, and the son's age b years; when will the father be twice as old as the son?

91. Shew that $(a+c)^3 - b^3 - (a-b)^3 - c^3 = 3a(a-b+c)(b+c)$.

92. Simplify

$$\frac{a+b}{ab}(a^2+b^2-c^2) + \frac{b+c}{bc}(b^2+c^2-a^2) + \frac{c+a}{ca}(c^2+a^2-b^2).$$

93. Find the L. C. M. of

$$a^2b^3 - a^2b^3, 2(a^4b^2 - a^2b^4), 4(a^5b^2 - a^2b^5), 6(a^7c^3 - a^2b^6).$$

Find the H. C. F. of

$$e^{2x}.x^3 + e^{3x}.x^3 - x^3 - 1 \text{ and } e^{2x}.x^2 + 2e^x.x^3 - e^{2x} - 2e^x + x^3 - 1.$$

94. Factorize: (1) $a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2) + 3abc$;

$$(2) \frac{9x^4}{16} + \frac{x^2y^2}{4} + \frac{y^4}{9}; (3) \frac{x^{16}}{4} - 16; (4) 3a^3 - \frac{11}{2}a^2b + \frac{3}{2}ab^2.$$

95. Shew that

$$\frac{(a+1)^3(b-c)^2}{(c-a)(a-b)} + \frac{(b+1)^3(c-a)^2}{(a-b)(b-c)} + \frac{(c+1)^3(a-b)^2}{(b-c)(c-a)} = 3(a+1)(b+1)(c+1).$$

96. Find the value of

$$.1 + \sqrt[3]{3} + \sqrt{2} + 1 + \sqrt[3]{3} - \sqrt{2} + 1 + \sqrt{2} - \sqrt[3]{3} + 1 - \sqrt{2} - \sqrt[3]{3}.$$

97. Shew that $(x+y)^n - x^n - y^n$ is always divisible by $xy(x+y)$ when n is an odd integer, and find the quotient when $n=5$; hence factorize $(a-b)^5 + (b-c)^5 + (c-a)^5$.

98. Find the sixth root of

$$\frac{x^6}{y^6} + \frac{y^6}{x^6} + 6\left(\frac{x^4}{y^4} + \frac{y^4}{x^4}\right) + 15\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) + 20.$$

99. Find the coefficient of x^3 and of x^4 in the expansion of

$$(a^3 - 2a^2x + 3ax^2 - 4x^3)^2.$$

100. A man at a party at cards betted 4s. to three upon every deal, and won 12s. after 25 deals; how many deals did he win, supposing no deal to have resulted in a drawn game?

CHAPTER XXVI.

• SIMPLE EQUATIONS IN ONE VARIABLE.

153. Definitions. (An **Equation** is a statement of the equality of two expressions.

An **Identical Equation** or simply an **Identity** is a statement of the equality of two expressions for all values of the symbols involved

Thus, $(x+y)^2 = x^2 + 2xy + y^2$ is an Identity.

An **Equation of Condition**, more commonly called an **Equation** only, is a statement of the equality of two expressions for some particular value or values of one or more of the symbols involved.

Thus, $x+3=5$, $2x+3y=6$, are Equations of Condition.

Ex. Distinguish between an Equation and an Identity, and give an example of each. What value of c makes $(x-2)^2 - (x-1) \times (x-3) = c$ an Identity? Can any value of c make it an Equation?

B. U. 1876.

When two expressions are connected by the sign of equality, the relation of equality may hold good for all values of the symbols used, or for some value or values of one or more of the symbols used. In the former case we have what is called an Equation of Identity, or more shortly an Identity, and in the latter case we have an Equation of condition, or more shortly an Equation. An Equation restricts some of the symbols therein to some particular value or values, whereas an Identity imposes no such condition on the symbols used. An Identity gives only different forms of the same expression connected by the sign of equality, and one form results from the other on performing some operations involved in the latter, whereas in an Equation no such operations will convert any of the expressions connected by the sign of equality into the other. In short an Equation is *conditionally* true, and an Identity is *unconditionally* true. For example, $x+1=2$ is an Equation, for $x+1$ is equal to 2 only when $x=1$, and $x+1$ is not equal to 2 if x be given any other value. Again, $(x+a)(x+b) = x(x+a+b) + ab$ is an Identity, as the equality holds good, whatever may be the values of x, a, b . For, let $x=1, a=2, b=3$. Then $(x+a)(x+b) = x(x+a+b) + ab$ becomes $(1+2)(1+3) = 1(1+2+3) + 2 \times 3$, which is evidently true, each side being = 12. In fact $(x+a)(x+b)$ and $x(x+a+b) + ab$ are both different forms of one and the same expression, viz., $x^2 + ax + bx + ab$.

$$(x-2)^2 - (x-1)(x-3) = (x^2 - 4x + 4) - (x^2 - 4x + 3) = 1, \text{ always}$$

$$\therefore (x-2)^2 - (x-1)(x-3) = 1, \text{ is an identity.}$$

$$\therefore \text{if } c=1, (x-2)^2 - (x-1)(x-3) = c, \text{ is an identity.}$$

No value of c free from x can make it an Equation in x . It may be said to be an Equation in c .

154. Definitions. The symbol on whose value an equation depends is called the Unknown Quantity, or the Variable. There may be more than one variable in an equation.

When any value of the variable is found, which when substituted for it in the two sides of an equation, makes those sides identically equal, the equation is said to be **satisfied** by that value of the variable, and this value of the variable is called a **root** of the equation.

The **solution** of an equation is the method of finding its root or roots.

Thus, $2x+3=x+9$ is an equation in x (variable).

It is *satisfied* by $x=6$, for on substituting 6 for x we have $2 \times 6+3=6+9$, which is evidently true, for each side = 15. 6 is therefore a *root* of the equation.

A **Simple Equation** is one which depends on the first power only of the unknown quantity or quantities involved. It is also called an **Equation of the First Degree**, or sometimes a **Linear Equation**.

Thus $ax+b=c$, and $ax+by=c$ are Simple Equations, the first in x , and the second in x and y .

N.B. The last letters x, y, z , &c. are *generally* used to denote unknown quantities, and the first letters, a, b, c , &c. are used to denote known quantities.

155. General Method. The student is referred back to the axioms given in Art. 66, which easily lead to the following general rule for the solution of Simple Equations involving one unknown quantity.

Simplify the two sides. Transpose the known terms to the right-hand side of the equation, and the terms involving the variable to the left-hand side.

Simplify again the two sides.

Lastly divide each side by the coefficient of the variable.

Ex. 1. Solve $ax-c^2=cx-a^2$.

Transposing, $ax-cx=c^2-a^2=-(a^2-c^2)$;

i.e., $x(a-c)=-(a-c)(a+c)$;

dividing by $a-c$, $x=-(a+c)$. *Ans.*

Ex. 2. Solve $2(x-3)+5(2x-4)=4(x-6)-(x+2)$.

Removing brackets, $2x-6+10x-20=4x-24-x-2$;

simplifying, $12x-26=3x-26$;

transposing, $12x-3x=26-26$;

i.e., $9x=0$;

dividing by 9, $x=0$. *Ans*

Ex. 3. Solve $(bx+a)(ax+b)-a^4=(bx-a)(ax-b)-b^4$.

Perform the multiplications; then

$$abx^2+(a^2+b^2)x+ab-a^4=abx^2-(a^2+b^2)x+ab-b^4.$$

Subtract abx^2+ab from both sides (*i.e.*, cancel abx^2+ab).

Thus, $(a^2+b^2)x-a^4=-(a^2+b^2)x-b^4$;

transposing, $2(a^2+b^2)x=a^4-b^4$
 $=(a^2-b^2)(a^2+b^2)$;

dividing by $2(a^2+b^2)$, $x=\frac{1}{2}(a^2-b^2)$. *Ans.*

Ex. 4. Solve $(6x+9)^2+(8x-7)^2=(10x+3)^2-71$. C.U.1882.

Expanding, $(36x^2+108x+81)+(64x^2-112x+49)$
 $=100x^2+60x+9-71$.

Simplify each side; then

$$100x^2-41+130=100x^2+60x-62$$

cancelling $100x^2$, $-41+130=60x-62$;

transposing, $-41+60x=-62-130$;

$$-64x=-192$$

changing signs, $64x=192$;

dividing by 64, $x=3$. *Ans.*

EXAMPLES 76.

Solve the following equations :

1. $x+4=8$
2. $x-4=8$
3. $2x+7=15$
4. $2x-9=16$
5. $5x+17=7$
6. $10x+11=36$
7. $51x+71=173$
8. $5x+8=3x+12$
9. $20+3x=97-4x$
10. $11x+12-17x=6-3x$
11. $12x-(x-4)=16x-(2x-21)$
12. $6x+2(3-x)=17(x-3)+5$
13. $7x-4(x-1)=3x+2(x+2)$
14. $51x-4(x+9)=18(2x+1)-x$
15. $23(x+2)-10(2x+3)=5(x+1)-25$
16. $4(x-15)+3(x-13)+6(3x-46)=0$

17. $3(2x-11)+4(2x^2-23)+7(3x-38)=0$.
 18. $5(x+2)+11(3x+1)-4(4-2(x+7))^2=2x+3$.
 19. $5(x+1)-17(2x+1)=13(x+3)-11(3x+5)$.
 20. $2(x+2)+3(2x+3)+4(3x+4)=9(4x+7)$.
 21. $13(x-13)+12(11x-13)+11(7x-1)-17(x-3)=2380$
 22. $ax-b=ab-x$. 23. $x(a+1)+b=c+x+(a+1)(b-c)$.
 24. $a(x+b)+c(x+d)=x(a+c)+(x+1)(ab+cd)$.
 25. $a(x+b-c)+b(x+c-a)+c(x+a-b)=a+b+c$.
 26. $a(x-b)-b(x-a)=a^2-b^2$. 27. $a(x-a)+b(x-b)=2ab$.
 28. $(x+1)(a+b)+(x+3)(b+c)-x(c+a)=a+3c$.
 29. $(x+1)(x+5)=(x+2)(x+3)$
 30. $(x+2)(x-3)+(x+5)(x-7)=(2x+1)(x+8)$.
 31. $(ax+b)(bx+a)=(ax-b)(bx-a)$.
 32. $(x+3)(x+5)=(x-7)(x+3)$. 33. $a(x+1)^2=(ax+1)(x+a)$.
 34. $(mx+n)^2+(nx+m)^2=(m^2+n^2)x^2+2(m+n)^2x$.

156. Convenient reduction. The following examples are intended to show how an equation may sometimes be most conveniently reduced to a simpler one.

Ex. 1. Solve $(2x+1)^2+(3x+4)^2=(2x-11)^2+(3x+7)^2$.

By transposition, we get

$$(2x+1)^2-(2x-11)^2=(3x+7)^2-(3x+4)^2.$$

$$\therefore \{(2x+1)+(2x-11)\}\{(2x+1)-(2x-11)\} = \{(3x+7)+(3x+4)\}\{(3x+7)-(3x+4)\};$$

Art. 75,

$$\text{or } (4x-10) \times 12 = (6x+11) \times 3;$$

dividing by 3, $(4x-10) \times 4 = 6x+11$;

$$\therefore 16x-40=6x+11;$$

$$\therefore 16x-6x=40+11;$$

$$\therefore 10x=51;$$

$$\therefore x=\frac{51}{10}=5\frac{1}{10}. \quad \text{Ans.}$$

N.B. The student is recommended to solve the above equation by expanding each side,

Ex. 2. Solve $(x-a)^2+(x-b)^2+(x-c)^2=3(x-a)(x-b)(x-c)$.

Transposing, $(x-a)^2+(x-b)^2+(x-c)^2-3(x-a)(x-b)(x-c)=0$.

We know that

$$l^2+m^2+n^2-3lmn=\frac{1}{2}(l+m+n)\{(l-m)^2+(m-n)^2+(n-l)^2\}.$$

For l put $x-a$, for m put $x-b$, and for n put $x-c$;
thus $(x-a)^3 + (x-b)^3 + (x-c)^3 - 3(x-a)(x-b)(x-c)$

$$= \frac{1}{2} \{ (x-a) + (x-b) + (x-c) \} \{ (x-a-x-b)^2 + (x-b-x-c)^2 + (x-c-x-a)^2 \} \\ = \frac{1}{2} (3x-a-b-c) \{ (b-a)^2 + (c-b)^2 + (a-c)^2 \}.$$

But we have seen that from the given equation

$$(x-a)^3 + (x-b)^3 + (x-c)^3 - 3(x-a)(x-b)(x-c) = 0$$

$$\therefore \frac{1}{2} (3x-a-b-c) \{ (b-a)^2 + (c-b)^2 + (a-c)^2 \} = 0 ;$$

dividing each side by $\frac{1}{2} \{ (b-a)^2 + (c-b)^2 + (a-c)^2 \}$, we have

$$3x - a - b - c = 0 ;$$

$$\therefore 3x = a + b + c ;$$

$$\therefore x = \frac{1}{3}(a+b+c). \text{ Ans.}$$

EXAMPLES 77.

Solve

1. $(x-7)^3 + (x-9)^3 + (x-11)^3 = 3x^3 + 305.$

2. $(x+2)^3 + 3(x+7)^3 = 4(x+8)^3$

3. $5(x+1)^3 + 7(x+3)^3 = 12(x+2)^3$

4. $(2x+5)^3 - (2x-7)^3 = (2x+9)^3 - (2x-11)^3.$

5. $(x-a)^3 + (x-b)^3 + (x-c)^3 = (x+a)^3 + (x+b)^3 + (x+c)^3.$

6. $(7x+41)^3 + (5x+29)^3 = (5x+31)^3 + (7x+39)^3.$

7. $(x+1)^3 = x(x+1)(x+2).$

8. $(x-a)^3 + (x-b)^3 + (x+a+b)^3 = 3(x^3 - b^3).$

9. $8(x-1)^3 + (2x-3)^3 + (2x-7)^3 = 6(x-1)(2x-3)(2x-7).$

10. $(x+1)^4 = (x^2-1)^2 + 4x^2(x+2) + 7.$

157. Number of roots. Every Simple Equation has only one root. C. U. 1881, 1902.

For, every simple equation can by transposition and simplification be reduced to the form $ax+b=0$.

If possible, let a and β be two different roots of the equation $ax+b=0$.

Then we must have identically

$$\left. \begin{aligned} aa+b &= 0, \\ a\beta+b &= 0. \end{aligned} \right\}$$

Dividing each of these by a , we get

$$a + \frac{b}{a} = 0, \quad (1)$$

$$\text{and } \beta + \frac{b}{a} = 0. \quad (2)$$

Subtract (2) from (1) ; thus $a - \beta = 0$;

$$\therefore a = \beta.$$

$\therefore a$ and β are not different from each other, which is contrary to the hypothesis.

Therefore a simple equation cannot have two *different* roots.

Hence it cannot have three, four, five, &c., different roots ; for if it has so many as three, four, &c., different roots, it certainly has two different roots, which has been proved to be absurd.

Therefore a simple equation cannot have two or more different roots, but has only one root ; for example in the typical equation $ax + b = 0$, $x = -b/a$, so that $-b/a$ is the only root.

158. The following example illustrates the case in which the root of a given equation is an algebraical fraction.

Ex. 1. Solve $2(x+a)(x+b) = x^2 + (x-a)(x-b)$.

Perform the multiplications ; then

$$2\{x^2 + x(a+b) + ab\} = x^2 + \{x^2 - x(a+b) + ab\} ;$$

$$\text{or } 2x^2 + 2x(a+b) + 2ab = x^2 - x(a+b) + ab ;$$

$$\text{cancelling } 2x^2, \quad 2x(a+b) + 2ab = -x(a+b) + ab ;$$

$$\text{transposing, } 2x(a+b) + x(a+b) = ab - 2ab ;$$

$$\therefore 3x(a+b) = -ab ;$$

$$\therefore x = -\frac{ab}{3(a+b)}.$$

159. Equations with Fractional Terms.

An Equation involving fractional terms may be reduced to an equivalent Equation without fractions by the following rule : *Multiply each side of the given Equation, or, which is the same thing, each term of the given Equation, by the L. C. M. of the denominators of the fractional terms on both sides.*

Ex. 1. Solve $\frac{2x-3}{6} + \frac{3x-8}{11} = \frac{4x+15}{33} + \frac{1}{2}$. C. U. 1877.

The L. C. M. of the denominators (6, 11, 33, 2) is 66.

Multiply both sides of the given equation by 66.

$$\text{Then, } 66\left(\frac{2x-3}{6} + \frac{3x-8}{11}\right) = 66\left(\frac{4x+15}{33} + \frac{1}{2}\right)$$

$$\text{i. e., } 11(2x-3) + 6(3x-8) = 2(4x+15) + 33;$$

$$\therefore 22x - 33 + 18x - 48 = 8x + 30 + 33;$$

$$\text{simplifying, } 40x - 81 = 8x + 63;$$

$$\text{transposing, } 40x - 8x = 81 + 63.$$

$$\therefore 32x = 144.$$

$$\therefore x = 4\frac{1}{2}.$$

$$\text{Ex. 2. Solve } \frac{a(x-1)}{b} - \left\{x - \frac{x-a}{m}\right\} = c - \frac{x+\frac{c}{b}}{d}.$$

$$\text{We have } \frac{a(x-1)}{b} - x + \frac{x-a}{m} = c - \frac{bx+c}{bd};$$

$$\text{multiplying by } mbd, \quad mbd \times a(x-1) - mbdx^2 + bd(x-a) = mbd.c - m(bx+c);$$

$$\text{multiplying out, } madx - mad - mbdx + bdx - abd = mbc - mbx - mc;$$

$$\text{transposing, } (mad - mbd + bd + mb)x = mad + abd + mbc - mc.$$

$$\therefore x = \frac{mad + abd + mbc - mc}{mad - mbd + bd + mb}. \text{ Ans.}$$

EXAMPLES 78.

$$1. \quad ax+b=d-x. \quad *2. \quad a(x+b)+c(x+d)=b(x-a)+c(x-d).$$

$$3. \quad 2mx+(n-m)(2x-m)=0.$$

$$4. \quad (ux+c)^2 - (bx+d)^2 = (a^2-b^2)(x^2+1).$$

$$5. \quad (x+l)^2 + (x+m)^2 + (x+n)^2 = 3x^2.$$

$$6. \quad (x+c+d)^2 + (x+c-d)^2 = 2x^2.$$

$$7. \quad (x+a)^2 + (x+b)^2 + (x+c)^2 = 3x^2(x+a+b+c).$$

$$8. \quad \frac{x}{2} + \frac{x}{3} = 6 - \frac{x}{6}.$$

$$9. \quad \frac{2x+1}{3} - \frac{3x-2}{4} = \frac{x-2}{6}.$$

$$10. \quad \frac{3x-1}{3} + \frac{x}{4} = \frac{x}{4} + \frac{2x+1}{5}.$$

$$11. \quad \frac{x+2}{3} + 2 = \frac{x+4}{5} + \frac{x+6}{7}.$$

$$12. \quad \frac{3x-13}{8} - \frac{4x+6}{9} = 1 - \frac{x-1}{10}.$$

$$13. \quad \frac{x-1}{5} - \frac{x-2}{4} = \frac{x-6}{10}.$$

$$14. \quad \frac{3x-1}{11} - \frac{2(x-1)}{5} = \frac{2-x}{10}.$$

$$15. \quad \frac{y-5}{2} - \frac{y-3}{4} + \frac{y+3}{5} - \frac{y+5}{6} = 0.$$

$$10. \sqrt{\frac{x+1}{8}} - \frac{2x+1}{3} = \frac{4x-1}{6} - \frac{5x-1}{4}. \quad 17. 7(x-2) - 5(2x-9) = \frac{x+31}{2}.$$

$$18. \frac{1}{3}(2x-3) - \frac{2}{3}(x+6) = \frac{x}{3} - 5. \quad 19. \frac{x+1}{3} + \frac{5-2x}{4} - \frac{2+5x}{2} + \frac{5-x}{3} = 0.$$

$$20. \frac{1}{3}(2x+5) + \frac{1}{3}(2x-5) = \frac{1}{3}(3x+1) + \frac{1}{3}(3x-1).$$

$$21. 7\frac{1}{2} - \frac{2x-3}{7} = 3\frac{1}{2} - \frac{4-5x}{7}.$$

$$22. \frac{4x-11}{15} - \frac{2x-7}{6} + \frac{1}{10} = \frac{5x-14}{18} - \frac{3x-11}{9}.$$

$$23. \frac{x+\frac{3}{2}}{9} + \frac{2x-1}{12} = \frac{x}{5} + \frac{2x+2}{25}.$$

$$24. \frac{2x+3}{8} + \frac{3x-2\frac{1}{2}}{5} = \frac{6x+7}{11} - \frac{5x-12\frac{1}{2}}{9}.$$

$$25. \frac{7x-10}{30} + \frac{\frac{x}{2}-\frac{7}{6}}{6} - \frac{\frac{2x}{3}-5}{30} = \frac{1}{2}. \quad 26. \frac{x+7\frac{1}{2}}{15} + \frac{x+3\frac{1}{2}}{25} = \frac{x^2+4\frac{1}{2}}{55}.$$

$$27. \frac{2x+\frac{1}{2}}{5} + \frac{\frac{x}{2}+\frac{x}{5}}{3} + \frac{2-\frac{x-8}{3}}{4} = \frac{x}{3}.$$

$$28. \frac{1}{2} \left[x - \frac{1}{2} \left\{ x - \frac{1}{2} \left(x - \frac{x-1}{5} \right) \right\} \right] = 53.$$

$$29. \frac{1}{3}(x-\frac{2}{3}) + \frac{2}{3}(x-\frac{1}{3}) - \frac{1}{3}(x+\frac{1}{3}) + \frac{1}{3}(x-\frac{1}{3}) = 0.$$

$$30. \frac{x}{ab} + \frac{x}{bc} + \frac{x}{ca} = a+b+c. \quad 31. \frac{ax}{b} + \frac{bx}{c} + \frac{cx}{a} = \frac{(ab^2+bc^2+ca^2)^2}{a^2b^2c^2}.$$

$$32. \frac{x-a}{a} + \frac{x-b}{b} + \frac{x-c}{c} = 1. \quad 33. \frac{a-x^2}{bx} - \frac{b-x}{c} = \frac{-x}{b} - \frac{b-x^2}{cx}.$$

$$34. \frac{a}{x} + \frac{b}{x} + \frac{c}{x} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}. \quad 35. \frac{a}{x+a} + \frac{b}{x+b} = \frac{3ab}{(x+a)(x+b)}.$$

$$36. \frac{3\{ab-x(a+b)\}}{a+b} + \frac{(2a+b)b^2x}{a(a+b)^2} = \frac{bx}{a} - \frac{a^2b^2}{(a+b)^2}.$$

$$37. \frac{3x+2}{3x-1} - \frac{7}{9x^2-1} = \frac{3x-1}{3x+1}. \quad 38. \frac{1}{3x-5} + \frac{1}{2x-2} = \frac{1}{3(x-5)} + \frac{1}{3(x-5)}.$$

$$39. \frac{1}{1+x} - \frac{b}{2(a+bx)} = \frac{a}{2(b+ax)}. \quad 40. \frac{m}{n} \left(1 - \frac{m}{x} \right) + \frac{m}{n} \left(1 - \frac{m}{x} \right) = 1.$$

$$41. \frac{1}{2x} + \frac{2}{3x} + \frac{3}{4x} - \frac{4}{5x} = 1. \quad 42. \frac{1}{ax+a^2} + \frac{m}{bx+ab} = 2.$$

160. Useful Principle. If $\frac{a}{b} = \frac{c}{d}$ and then will $ad = bc$:

$$\text{Given } \frac{a}{b} = \frac{c}{d}$$

Multiplying each side by bd , we get

$$\frac{a}{b} \times bd = \frac{c}{d} \times bd,$$

$$\text{i.e., } ad = bc.$$

The above principle may be expressed thus :

*When two fractions are equal, numr. of 1st \times denr. of second
= numr. of 2nd \times denr. of 1st.*

We exhibit to the eye the quantities to be multiplied together by cross lines, each connecting the proper factors :

$$\begin{array}{cc} a & c \\ b & d \end{array}$$

Ex. 1 Solve $\frac{a-b}{x-c} = \frac{a+b}{x+2c}$ C U. 1862.

Multiply both sides by $(x-c)(x+2c)$; then

$$(a-b)(x+2c) = (a+b)(x-c) ; \quad (\text{A})$$

multiplying out, $(a-b)x + 2c(a-b) = (a+b)x - c(a+b)$;

transposing, $(a-b)x - (a+b)x = -2c(a-b) - c(a+b)$.

simplifying, $-2bx = (b-3a)c$;

changing signs, $2bx = (3a-b)c$;

$$\therefore x = \frac{(3a-b)c}{2b}. \text{ Ans}$$

N. B. The equation, (A), follows at once from Art. 160.

Ex. 2 Solve $\frac{x+3}{x+5} = \frac{2x-1}{2x-3}$.

$$(x+3)(2x-3) = (x+5)(2x-1) ; \quad \text{Art. 160.}$$

multiplying out,

$$2x^2 + 3x - 9 = 2x^2 + 9x - 5 ;$$

cancelling $2x^2$,

$$3x - 9 = 9x - 5 ;$$

transposing,

$$3x - 9x = 9 - 5 ;$$

$$\therefore -6x = 4 ;$$

$$\therefore x = \frac{4}{-6}$$

$$= -\frac{2}{3}. \text{ Ans.}$$

EXAMPLES 70.

Solve

1. $\frac{x+5}{x-2} = \frac{7}{5}$

2. $\frac{2x+1}{2x-3} = \frac{3}{5}$

3. $\frac{9x+11}{7x+3} = \frac{5}{7}$

4. $\frac{x+\frac{3}{4}}{x+\frac{2}{5}} = \frac{2}{5}$

5. $\frac{ax+b}{bx+a} = \frac{c}{d}$

6. $\frac{x-a}{x-b} = \frac{2a}{3b}$

7. $\frac{x-\frac{1}{2}(x-4)}{2x-\frac{1}{3}(x+2)} = \frac{10}{9}$

8. $\frac{x-1}{x-3} = \frac{x+2}{x+3}$

9. $\frac{2x+3}{3x+2} = \frac{2x-5}{3x-4}$

10. $\frac{x}{2} = \frac{2x^2+x+1}{4x+5}$

11. $\frac{(l-m)x+n}{(l-m)x+r} = \frac{(l+m)x+k}{(l+m)x-n}$

12. $\frac{ax^2+bx+c}{px+q} = \frac{ax+d}{p}$

13. $\frac{x^2-x+1}{x^2+1} = \frac{x}{x+1}$

161. Judicious combination of terms. In Equations of which the following are typical, *combine by transposition of terms, if necessary, the simplest parts or fractions with like denominators.*

Ex. 1. Solve $\frac{4x+3}{9} + \frac{29-7x}{12-5x} = \frac{8x+19}{18}$. C. U. 1868.

Transposing,

$$\frac{29-7x}{12-5x} = \frac{8x+19}{18} - \frac{4x+3}{9}$$

$$= \frac{8x+19-8x-6}{18}$$

$$= \frac{13}{18}$$

$$18(29-7x) = 13(12-5x); \quad \text{Art. 160.}$$

multiplying out, $522 - 126x = 156 - 65x;$

transposing, $65x - 126x = 156 - 522;$

i.e., $-61x = -366;$

$\therefore x = 6.$ *Ans.*

Ex. 2. Solve $\frac{6-5x}{5} - \frac{3}{14} \times \frac{7-2x^2}{x-1} - \frac{1}{10} = \frac{1+3x}{7} - x + \frac{1}{35}$.

M. U. 1867.

$$\begin{aligned} \text{Transposing, } -\frac{3}{14} \times \frac{7-2x^2}{x-1} &= \frac{1+3x}{7} - x + \frac{1}{35} - \frac{6-5x}{5} + 1\frac{1}{10} \\ &= \frac{10+30x-70x+2-84+70x+77}{70} \\ &= \frac{30x+5}{70} \\ &= \frac{6x+1}{14}; \end{aligned}$$

$$\text{multiplying by 14, } -\frac{3(7-2x^2)}{x-1} = 6x+1.$$

$$\therefore -3(7-2x^2) = (6x+1)(x-1); \quad \text{Art 160}$$

$$\text{multiplying out, } -21+6x^2 = 6x^2-5x-1;$$

$$5x = 21-1 = 20;$$

$$x=4 \quad \text{Ans}$$

EXAMPLES 80

Solve

$$1. \frac{2x}{5} + \frac{3(x-6)}{x-4} = 3 - \frac{1-4x}{10}, \quad 2. \frac{12x+19}{18} - \frac{7x-2}{3x-10} = \frac{8x-25}{12} + \frac{5}{36}.$$

$$3. \frac{x+1}{3} + \frac{7\frac{1}{2}-x}{2} = \frac{2\frac{1}{2}-x}{6} + \frac{4x-1}{2x-3}, \quad 4. \frac{2x}{3} - \frac{3-x}{8(x-1)} = \frac{x-2}{2} + \frac{x-1}{6} + 1\frac{1}{2}.$$

$$5. \frac{x+1}{18} - \frac{5-\frac{x}{3}}{4} - \frac{7+\frac{x}{4}}{6} = \frac{x}{6} + \frac{2-\frac{x}{4}}{3-\frac{1}{5}}, \quad 6. \frac{x}{8} - \frac{x-1}{6} + \frac{\frac{1}{3}-\frac{x}{4}}{\frac{1}{4}-x} = \frac{1+\frac{x}{6}}{4} + \frac{1-\frac{x}{4}}{3}.$$

$$7. \frac{21(x+2)}{2x+7} - \frac{11(x-1)}{2x+1} = 5 + \frac{4}{x+1}, \quad 8. \frac{2-3x}{2\frac{1}{2}-x} + \frac{2\frac{1}{2}}{x+1} = \frac{x+\frac{1}{2}}{x+1} + 2.$$

$$9. \frac{4+17x}{36x} + \frac{1+2x}{6x} + \frac{1-20x}{72x} = \frac{7}{24} - \frac{x-\frac{1}{3}}{17-32x}.$$

$$10. \frac{5x+14}{18} - \frac{5x-9}{11x-8} + \frac{1}{3} = \frac{10x+29}{36}.$$

$$11. \frac{6x-7\frac{1}{2}}{13-2x} + 2x + \frac{1+16x}{24} = 4\frac{5}{12} - \frac{12\frac{5}{6}-8x}{3}.$$

$$12. \frac{bc-ad}{c(ax+d)} + \frac{c}{a} - \frac{c}{a} \cdot \frac{x^2+x+1}{x(x+1)} = \frac{cx^2+x-1}{a \cdot x(x+1)} - \frac{a}{c}.$$

162. **Reduction by Division.** When the numerator of a fractional term is not of a lower degree than the denominator, it

is often useful to begin by reducing it by division to a form partly integral and partly fractional.

Ex. 1. Solve $\frac{(x-a)(x+b)}{x-a+b} = \frac{x(x-c)-b(x+c)}{x-b-c}$, B. U. 1886.

Perform the multiplications in the numerators ; then

$$\frac{x^2 - ax + bx - ab}{x - a + b} = \frac{x^2 - bx - cx - bc}{x - b - c}.$$

Divide $x^2 - ax + bx - ab$ by $x - a + b$, and
 $x^2 - bx - cx - bc$ by $x - b - c$.

Thus,
$$x - \frac{ab}{x - a + b} = x - \frac{bc}{x - b - c};$$

cancelling x ,
$$\frac{-ab}{x - a + b} = \frac{-bc}{x - b - c};$$

dividing by $-b$,
$$\frac{a}{x - a + b} = \frac{c}{x - b - c};$$

$$\therefore a(x - b - c) = c(x - a + b); \quad \text{Art. 160};$$

simplifying and transposing, $ax - cx = a(b + c) + c(-a + b)$
 $= b(a + c);$

$$\therefore x(a - c) = b(a + c);$$

$$\therefore x = \frac{b(a + c)}{a - c}. \quad \text{Ans.}$$

Ex. 2. Solve $\frac{2x+11}{x+5} - \frac{9x-9}{3x-4} = \frac{4x+13}{x+3} - \frac{15x-47}{3x-10}$, C. U. 1866.

By division the given equation reduces to

$$\left(2 + \frac{1}{x+5}\right) - \left(3 + \frac{3}{3x-4}\right) = \left(4 + \frac{1}{x+3}\right) - \left(5 + \frac{3}{3x-10}\right)$$

$$\therefore \frac{1}{x+5} - \frac{3}{3x-4} - 1 = \frac{1}{x+3} - \frac{3}{3x-10} - 1;$$

cancelling -1 ,
$$\frac{1}{x+5} - \frac{3}{3x-4} = \frac{1}{x+3} - \frac{3}{3x-10};$$

simplifying,
$$\frac{-19}{(x+5)(3x-4)} = \frac{-19}{(x+3)(3x-10)};$$

$$\therefore (x+5)(3x-4) = (x+3)(3x-10);$$

multiplying out,
$$3x^2 + 11x - 20 = 3x^2 - x - 30;$$

$$\therefore 11x - 20 = -x - 30;$$

whence
$$x = -\frac{5}{6}. \quad \text{Ans.}$$

163. Principle of Alternando. If $\frac{a}{b} = \frac{c}{d}$, then will $\frac{a}{c} = \frac{b}{d}$; that is, if two fractions be equal, then

$$\frac{\text{1st numerator}}{\text{2nd numerator}} = \frac{\text{1st denominator}}{\text{2nd denominator}}$$

This is known as the **Principle of Alternando**.

Proof: Given $\frac{a}{b} = \frac{c}{d}$

Multiply each side by $\frac{b}{c}$; then

$$\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c};$$

$$\text{i.e., } \frac{a}{c} = \frac{b}{d}$$

Ex 1. Solve $\frac{x+n}{x+m} = \frac{(2x+n+r)^2}{(2x+m+r)^2}$. P.U. 1883.

We have $\frac{(2x+m+r)^2}{x+m} = \frac{(2x+n+r)^2}{x+n}$; Alternando.

expanding, $\frac{4x^2 + 4(m+r)x + (m+r)^2}{x+m} = \frac{4x^2 + 4(n+r)x + (n+r)^2}{x+n}$;

by division, $4x + 4r + \frac{(r-m)^2}{x+m} = 4x + 4r + \frac{(r-n)^2}{x+n}$;

$$\therefore \frac{(r-m)^2}{x+m} = \frac{(r-n)^2}{x+n};$$

$$\therefore (r-m)^2(x+n) = (x+m)(r-n)^2;$$

$$\therefore \text{by transposition, } x\{(r-m)^2 - (r-n)^2\} = m(r-n)^2 - n(r-m)^2;$$

$$\text{or } x\{(r-m) - (r-n)\}\{(r-m) + (r-n)\} = m(r^2 - 2nr + n^2) - n(r^2 - 2mr + m^2)$$

$$\therefore x(n-m)(2r-m-n) = (m-n)r^2 + mn(n-m) = (n-m)(mn-r^2)$$

$$\therefore x = \frac{mn-r^2}{2r-m-n}. \text{ Ans}$$

Otherwise thus: Subtract 1 from both sides;

$$\text{thus } \frac{x+n}{x+m} - 1 = \frac{(2x+n+r)^2}{(2x+m+r)^2} - 1;$$

$$\begin{aligned} \text{or } \frac{n-m}{x+m} &= \frac{(2x+n+r)^2 - (2x+m+r)^2}{(2x+m+r)^2} \\ &= \frac{\{(2x+n+r) - (2x+m+r)\} \{(2x+n+r) + (2x+m+r)\}}{(2x+m+r)^2} \\ &= \frac{(n-m)(4x+m+n+2r)}{(2x+m+r)^2}; \end{aligned}$$

dividing by $n-m$, $\frac{1}{x+m} = \frac{4x+m+n+2r}{(2x+m+r)^2};$

multiplying by $2x+m+r$, $\frac{2x+m+r}{x+m} = \frac{4x+m+n+2r}{2x+m+r};$

\therefore by division, $2 + \frac{r-m}{x+m} = 2 + \frac{n-m}{2x+m+r};$

$$\therefore \frac{r-m}{x+m} = \frac{n-m}{2x+m+r};$$

$$\therefore (r-m)(2x+m+r) = (n-m)(x+m); \dots \text{Art 260.}$$

multiplying out, $(2r-2m)x + r^2 - m^2 = (n-m)x + mn - m^2;$

\therefore by transposition, $x(2r-2m-n+m) = mn - r^2;$

simplifying, $x(2r-m-n) = mn - r^2;$

$$\therefore x = \frac{mn - r^2}{2r - m - n}.$$

N. B. The first method is specially useful in dealing with numbers instead of symbols, (m, n, r , &c.).

Ex 2. Solve $\frac{x+7}{x+9} = \left(\frac{x+5}{x+6}\right)^2.$

We have $\frac{(x+6)^2}{x+9} = \frac{(x+5)^2}{x+7}, \dots \text{Alternando.}$

expanding, $\frac{x^2+12x+36}{x+9} = \frac{x^2+10x+25}{x+7};$

by division, $x+3 + \frac{9}{x+9} = x+3 + \frac{4}{x+7};$

cancelling $x+3$, $\frac{9}{x+9} = \frac{4}{x+7};$

$$\therefore 9(x+7) = 4(x+9),$$

whence $5x = -27,$

and $x = -\frac{27}{5} = -5\frac{2}{5}. \text{ Ans.}$

Or thus : Subtract 1 from both sides ; then

$$\begin{aligned}\frac{x+7}{x+9} - 1 &= \frac{(x+5)^2}{(x+6)^2} - 1 \\ \therefore \frac{-2}{x+9} &= \frac{(x+5)^2 - (x+6)^2}{(x+6)^2} \\ &= \frac{\{(x+5) - (x+6)\} \{(x+5) + (x+6)\}}{(x+6)^2} \\ &= \frac{-1(2x+11)}{(x+6)^2}\end{aligned}$$

$$\therefore 2(x+6)^2 = (x+9)(2x+11) ; \dots\dots\dots \text{Art. 160 ;}$$

multiplying out, $2x^2 + 24x + 72 = 2x^2 + 20x + 99$;

$$\therefore -5x = 27, \text{ whence } x = -5\frac{2}{5}. \text{ Ans.}$$

EXAMPLES 8'

1. $\frac{x+2}{x-3} + \frac{x-2}{x-6} = 2.$
2. $\frac{ax}{x-b} + \frac{bx}{x-a} = a+b.$
3. $\frac{9x^2-8}{x-3} = \frac{9x-13}{x-4}.$
4. $\frac{4(x+1)}{x+2} - \frac{x+1}{x-1} = 3.$
5. $\frac{2x-3}{7x-11} = \frac{4x-5}{14x-23}.$
6. $\frac{2x-1}{x-1} - \frac{2x-14}{x-5} = \frac{2x-5}{x-3} - \frac{3x-20}{x-7}.$
7. $\frac{x^2-24}{x-5} - \frac{x^2-8x-8}{x-9} = \frac{6x-41}{x-7} - \frac{2x-21}{x-11}.$
8. $\frac{8x^2+23x+32}{8x+11} = \frac{4x^2+22x+45}{4x+16}.$
9. $\frac{x^2-10x-20}{3x-25} = \frac{3x^2-20x-60}{9x-45}$ [Multiply by 3 and divide.]
10. $\frac{23x^2-30x-12}{24x-18-23x^2} = \frac{23x-30}{24-23x}.$
11. $\frac{x^2+3x-9}{x+5} + \frac{x^2+5x-23}{x+8} = \frac{2x^2+7x-28}{x+6}.$
12. $\frac{(x+a)(x+b)}{x+a+b} = \frac{(x+c)(x+d)}{x+c+d}.$
13. $\frac{16x-29}{4x-7} - 2 \times \frac{16x-31}{8x-15} = 4 \times \frac{5x-11}{4x-9} - \frac{40x-83}{8x-17}.$
14. $4\left(\frac{9x-1\frac{1}{2}}{6x-1}\right) + 5\left(\frac{28x-17}{12x-7}\right) = 6\left(\frac{16x-1}{12x-1}\right) + \frac{2}{3}\left(\frac{15x-8}{2x-1}\right).$

$$\begin{array}{ll}
 15. \frac{(x+1)(x+2)}{(x+3)(x+4)} = \frac{x+3}{x+7} & 16. \frac{lx+m+n}{lx+p+q} = \frac{(lx+m)(lx+n)}{(lx+p)(lx+q)} \\
 17. \frac{x+8}{x+6} = \left(\frac{x+4}{x+3}\right)^2 & 18. \frac{x+2a}{x+2b} = \frac{(x+a)^2}{(x+b)^2} \\
 19. \frac{x+11}{x+9} = \left(\frac{2x+19}{2x+17}\right)^2 & 20. \frac{mx+b}{mx+a} = \left(\frac{2mx+b+c}{2mx+a+c}\right)^2
 \end{array}$$

164. **Judicious transposition.** The following example shows the importance of suitable transposition of terms.

Ex. 1. Solve $\frac{a}{x+a} + \frac{b}{x+b} = \frac{a-c}{x+a-c} + \frac{b+c}{x+b+c}$. B U. 1882-83

Transposing, $\frac{a}{x+a} - \frac{a-c}{x+a-c} = \frac{b+c}{x+b+c} - \frac{b}{x+b}$.

$$\therefore \frac{a(x+a-c) - (a-c)(x+a)}{(x+a)(x+a-c)} = \frac{(b+c)(x+b) - b(x+b+c)}{(x+b+c)(x+b)};$$

simplifying, $\frac{cx}{(x+a)(x+a-c)} = \frac{cx}{(x+b)(x+b+c)};$

dividing by cx , $\frac{1}{(\bar{x}+a)(\bar{x}+a-c)} = \frac{1}{(x+b)(x+b+c)};$
 $\therefore (x+a)(x+a-c) = (x+b)(x+b+c);$

multiplying out, $x^2 + (2a-c)x + a(a-c) = x^2 + x(2b+c) + b(b+c);$

$$\therefore x\{(2a-c) - (2b+c)\} = b(b+c) - a(a-c)$$

$$= b^2 - a^2 + (b+a)c;$$

$$\therefore 2x(a-b-c) = (b+a)(b-a+c)$$

$$= -(a+b)(a-b-c);$$

$$\therefore x = -\frac{a+b}{2}. \text{ Ans.}$$

N. B. We will give another method of solving the same equation later on.

EXAMPLES 82.

Solve :

1. $\frac{2}{x+2} + \frac{3}{x+3} = \frac{1}{x+1} + \frac{4}{x+4}.$

2. $\frac{5}{x+5} + \frac{6}{x+6} = \frac{3}{x+3} + \frac{8}{x+8}.$

3. $\frac{1}{x+1} + \frac{5}{x+5} = \frac{1-c}{x+1-c} + \frac{5+c}{x+5+c}.$

4. $\frac{3}{x+3} + \frac{4}{x+4} = \frac{9}{x+9} + \frac{2}{x-2}$

$$5. \frac{11}{2x+11} - \frac{9}{2x-9} = \frac{9}{2x+9} - \frac{7}{2x-7}.$$

$$6. \frac{2(x+6)}{x+4} + \frac{3(x+8)}{x+6} = \frac{x+4}{x+2} + \frac{4(x+10)}{x+8}. \quad (\text{First divide}),$$

$$7. \frac{1}{(a-b)(x-a)} - \frac{1}{(c-d)(x-c)} = \frac{1}{(a-b)(x-b)} - \frac{1}{(c-d)(x-d)}.$$

$$8. \frac{1}{3x-8} - \frac{1}{4x-6} = \frac{1}{3x-5} - \frac{1}{4x-2}.$$

165. Breaking up of Terms. The artifice illustrated in the following examples deserves close attention.

$$\begin{aligned} \text{Ex. 1. Solve } \frac{3}{x+1} + \frac{5}{x+9} &= \frac{8}{x+8}; \\ \frac{3}{x+1} + \frac{5}{x+9} &= \frac{8}{x+8} \\ &= \frac{3}{x+8} + \frac{5}{x+8}; \end{aligned}$$

$$\text{transposing, } \frac{3}{x+1} - \frac{3}{x+8} = \frac{5}{x+8} - \frac{5}{x+9};$$

$$\text{simplifying, } \frac{21}{(x+1)(x+8)} = \frac{5}{(x+8)(x+9)};$$

$$\text{multiplying by } x+8, \quad \frac{21}{x+1} = \frac{5}{x+9};$$

$$\therefore 21(x+9) = 5(x+1);$$

$$\therefore 21x - 5x = 5 - 189;$$

$$\therefore 16x = -184;$$

$$\therefore x = -\frac{23}{2}. \quad \text{Ans.}$$

N. B. The student should notice that the sum of the numerators on one side (3, 5) = the numerator on the other side (8). We have therefore broken up 8 into 3 + 5, and then by transposition we have combined fractions having the same numerator.

$$\text{Ex. 2. Solve } \frac{x-4}{(x-1)(x-3)} + \frac{x-7}{(x-1)(x-6)} + \frac{x-9}{(x-3)(x-6)} = \frac{3}{x}.$$

M. U. 1874.

Breaking up $\frac{3}{x}$ into $\frac{1}{x} + \frac{1}{x} + \frac{1}{x}$, we have by transposition,

$$\left\{ \frac{x-4}{(x-1)(x-3)} - \frac{1}{x} \right\} + \left\{ \frac{x-7}{(x-1)(x-6)} - \frac{1}{x} \right\} + \left\{ \frac{x-9}{(x-3)(x-6)} - \frac{1}{x} \right\} = 0,$$

simplifying,
$$\frac{-3}{x(x-1)(x-3)} + \frac{-6}{x(x-1)(x-6)} + \frac{-18}{x(x-3)(x-6)} = 0;$$

dividing by -3,
$$\frac{1}{x(x-1)(x-3)} + \frac{2}{x(x-1)(x-6)} + \frac{6}{x(x-3)(x-6)} = 0,$$

multiplying by $x(x-1)(x-3)(x-6)$, $1(x-6) + 2(x-3) + 6(x-1) = 0,$

$\therefore 9x - 18 = 0$; whence $x = 2$. *Ans.*

EXAMPLES 83.

Solve

1. $\frac{3}{x+3} + \frac{4}{x+5} = \frac{7}{x+1}$

2. $\frac{4}{2x+1} + \frac{5}{2x+3} = \frac{9}{2x-1}$

✓ 3. $\frac{5}{x-13} + \frac{8}{x-10} = \frac{13}{x-7}$

✓ 4. $\frac{2}{3x+7} + \frac{1}{x+3} - \frac{5}{3x+5} = 0$

✓ 5. $\frac{3}{10x+9} + \frac{4}{45x+2} = \frac{7}{18x+5}$

✓ 6. $\frac{l}{ax+b} + \frac{m}{ax+c} = \frac{l+m}{ax+d}$

✓ 7. $\frac{a-b}{x+c} + \frac{b-d}{x+d} = \frac{a-d}{x+c+d}$

8. $\frac{a+b}{mx+n} + \frac{a-b}{mx-n} = \frac{2a}{mx+p}$ ✓

✓ 9. $\frac{x+a}{x-a} - \frac{x-b}{x+b} = \frac{2(a+b)}{x+c}$ [First divide].

10. $\frac{3(2x-1)}{2x-3} - \frac{6x+5}{2x+5} = \frac{16}{2x+7}$

11. $\frac{a}{ax+b} + \frac{c}{cx+d} = \frac{2l}{lx+m}$

12. $\frac{x-5}{(x-2)(x-3)} + \frac{x-7}{(x-3)(x-4)} + \frac{x-6}{(x-4)(x-2)} = \frac{3}{x}$

13. $\frac{x-5}{(x-2)(x-4)} + \frac{x-8}{(x-2)(x-7)} + \frac{x-10}{(x-4)(x-7)} = \frac{3}{x-1}$

14. $\frac{4(x-2)}{(2x-1)(2x-3)} + \frac{2x-7}{(2x-1)(x-3)} + \frac{2x-9}{(2x-3)(x-3)} = \frac{3}{x}$

15. $\frac{x-a+b}{(x-a)(x+b)} + \frac{x+b-c}{(x+b)(x-c)} = \frac{3}{x} + \frac{a+c-x}{(x-a)(x-c)}$

16. $\frac{b+c}{x+3a} + \frac{c+a}{x+3b} + \frac{a+b}{x+3c} = \frac{2(a+b+c)}{x} - \frac{6(ab+bc+ca)x+27abc}{(x+3a)(x+3b)(x+3c)}$

✓ **166. Useful Theorem.** If $\frac{a}{b} = \frac{c}{d}$, then will each fraction

$$= \frac{a+c}{b+d} = \frac{a-c}{b-d};$$

i.e., if two fractions be equal to one another, then will each

$$= \frac{\text{sum of numerators}}{\text{sum of denominators}}, \text{ also } = \frac{\text{difference of numerators}}{\text{difference of denominators}}.$$

Proof : For, let $\frac{a}{b} = \frac{c}{d} = k$;

then $a = bk$, and $c = dk$;

∴ by addition, $a + c = bk + dk = k(b + d)$,(1)

and by subtraction, $a - c = bk - dk = k(b - d)$;(2)

∴ from (1) and (2), k (i.e., each fraction) $= \frac{a+c}{b+d}$, also $= \frac{a-c}{b-d}$

Ex. 1. Solve $\frac{(x+5)(x+9)}{(x+3)(x+7)} = \frac{(x+4)(x+10)}{(x+2)(x+8)}$.

Multiplying out, $\frac{x^2 + 14x + 45}{x^2 + 10x + 21} = \frac{x^2 + 14x + 40}{x^2 + 10x + 16}$;

∴ each = $\frac{\text{dif. of numerators}}{\text{dif. of denominators}} = \frac{45 - 40}{21 - 16} = \frac{5}{5} = 1$,

$$\therefore x^2 + 14x + 45 = x^2 + 10x + 21$$

whence $4x = -24$, and $x = -6$. Ans.

Ex. 2. Solve $\left(\frac{x+c}{x+d}\right)^2 = \frac{x+2c-d}{x+2d-c}$

Multiplying by $\frac{x+d}{x+c}$, $\frac{(x+c)^2}{(x+d)^2} = \frac{(x+2c-d)(x+d)}{(x+2d-c)(x+c)}$;

expanding, $\frac{x^2 + 2cx + c^2}{x^2 + 2dx + d^2} = \frac{x^2 + 2cx + 2cd - d^2}{x^2 + 2dx + 2cd - c^2}$;

∴ each = $\frac{\text{dif. of numerators}}{\text{dif. of denominators}} = \frac{c^2 - 2cd + d^2}{d^2 - 2cd + c^2} = 1$;

$$\therefore x^2 + 2cx + c^2 = x^2 + 2dx + d^2;$$

whence, cancelling x^2 , $2(c-d)x = d^2 - c^2 = -(c-d)(c+d)$;

$$\therefore x = -\frac{1}{2}(c+d). \text{ Ans.}$$

N. B. We shall give another mode of solving the above equation later on.

Ex. 3. Solve $\frac{a}{x+a} + \frac{b}{x+b} = \frac{a-c}{x+a-c} + \frac{b+c}{x+b+c}$. B. U. 1882-83.

$$\text{Adding up, } \frac{(a+b)x+2ab}{(x+a)(x+b)} = \frac{(a-c)(x+b+c) + (b+c)(x+a-c)}{(x+a-c)(x+b+c)};$$

$$i. e., \frac{(a+b)x+2ab}{x^2+(a+b)x+ab} = \frac{(a+b)x+2(b-c)(b+c)}{x^2+(a+b)x+(a-c)(b+c)}$$

$$\therefore \text{ each fraction} = \frac{\text{dif. of numerators}}{\text{dif. of denominators}}$$

$$= \frac{2ab - 2(a-c)(b+c)}{ab - (a-c)(b+c)}$$

$$= 2,$$

$$\text{since } \frac{(a+b)x+2ab}{x^2+(a+b)x+ab} = 2,$$

$$\therefore (a+b)x + 2ab = 2x^2 + 2(a+b)x + 2ab;$$

$$\therefore -2x^2 = (a+b)x;$$

$$\therefore -2x = a+b;$$

$$\therefore x = -\frac{1}{2}(a+b). \text{ Ans. See Ex. 1, Art. 164.}$$

Ex. 4. Solve $\frac{ax+b}{cx+d} = \frac{m-bpx-afx^2}{n-dpx-cpx^2}$.

$$\text{Since } \frac{ax+b}{cx+d} = \frac{px}{px} \times \frac{ax+b}{cx+d}, \text{ we have}$$

$$\frac{apx^2+bp}{cp^2+dp} = \frac{m-bpx-afx^2}{n-dpx-cpx^2}$$

$$= \frac{\text{sum of numerators}}{\text{sum of denominators}}$$

$$= \frac{m}{n};$$

$$i. e., \frac{ax+b}{cx+d} = \frac{m}{n};$$

$$\therefore nax+nb = mcx+md; \dots\dots\dots \text{Art. 160.}$$

$$\therefore x(na-mc) = md-nb;$$

$$\therefore x = \frac{md-nb}{na-mc} \text{ Ans.}$$

EXAMPLES 84.

Solve

$$1. (x+2)(x+5)(x+10)(x+11) = (x+3)(x+9)(x+4)(x+12).$$

$$\left[\text{Put it as } \frac{(x+5)(x+11)}{(x+3)(x+9)} = \frac{(x+4)(x+12)}{(x+2)(x+10)} \right].$$

$$2. (x+13)(x+11)(x+8)^2 = (x+7)(x+9)(x+12)^2.$$

$$3. \frac{(x+1)^3}{(x+3)} = \frac{x-1}{x+5} \quad 4. \frac{(x+2)^3}{(x+4)} = \frac{x}{x+6} \quad 5. \frac{(x-7)^3}{(x-9)} = \frac{x-5}{x-11}.$$

$$6. (x-3)(x-5)(x-9)(x-15) = (x-1)(x-7)(x-11)(x-13).$$

$$7. (2x-7)^2(2x-11) = (2x-9)^2(2x-5).$$

$$8. \frac{ax+b}{cx+d} = \frac{ax^2+bx+c}{cx^2+dx+e} \quad 9. \frac{x+1}{x+2} = \frac{x^2+x+2}{x^2+2x+3}.$$

$$10. \frac{x^2+x+1}{x^3-x-1} = \frac{(x^2+1)(x+1)+1}{(x^2-1)(x-1)+1}.$$

167. Common Factor of the two sides of an Equation.

If $ax+b$ occurs as a factor in each side of an equation, then by transposition of terms we can put the equation in the form $(ax+b) \times \text{some expression} = 0$. This equation is evidently satisfied, if $ax+b=0$, or $x=-b/a$. Therefore the *common factor of the two sides equated to zero gives a solution*. And since a simple equation has only one root, the solution thus found in the case of a simple equation is the only one possible.

$$\text{Ex. 1. Solve } \frac{x+b}{a-b} = \frac{x-b}{a+b} \dots\dots\dots \text{C. U. 1893.}$$

$$\text{Add 1; then } \frac{x+b}{a-b} + 1 = \frac{x-b}{a+b} + 1;$$

$$\therefore \frac{x+a}{a-b} = \frac{x+a}{a+b};$$

$$\therefore (x+a) \left(\frac{1}{a-b} - \frac{1}{a+b} \right) = 0, \text{ by transposition.}$$

$$\therefore x+a (= \text{common factor}) = 0,$$

$$\therefore x = -a. \text{ Ans.}$$

N. B. It should be noticed that if $AB=AC$, then $AB-AC$ or $A(B-C)=0$; the last equation is satisfied if $A=0$, or $B-C=0$. Hence, if $AB=AC$, either A must be zero, or $B=C$. Let us apply this principle to Ex. 1, Art. 164.

Subtracting each side from 2, we may put

$$\left(1 - \frac{a}{x+a}\right) + \left(1 - \frac{b}{x+b}\right) = \left(1 - \frac{a+c}{x+a-c}\right) + \left(1 - \frac{b+c}{x+b+c}\right);$$

$$\therefore \frac{x}{x+a} + \frac{x}{x+b} = \frac{x}{x+a-c} + \frac{x}{x+b+c};$$

$$\therefore \text{either } x=0, \text{ or } \frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{x+a-c} + \frac{1}{x+b+c},$$

$$\text{i.e., } \frac{2x+a+b}{(x+a)(x+b)} = \frac{2x+a+b}{(x+a-c)(x+b+c)},$$

$$\text{whence either } 2x+a+b=0, \text{ i.e., } x = -\frac{1}{2}(a+b),$$

$$\text{or } (x+a)(x+b) = (x+a-c)(x+b+c),$$

i.e., $x^2 + (a+b)x + ab = x^2 + (a+b)x + (a-c)(b+c)$, which gives no solution

Thus x may have either of the values, 0 and $-\frac{1}{2}(a+b)$. But as such equations are really of the Second Degree in x , and therefore to some extent beyond the Entrance Course, we have contented ourselves with treating them as Linear Equations, and with giving only one solution.

Ex 2. Solve $\frac{3}{x-3} - \frac{4}{x+9} - \frac{5}{x-27} + \frac{6}{x-15} = 0$. M U. 1873.

By transposition, $\frac{3}{x-3} + \frac{6}{x-15} = \frac{4}{x+9} + \frac{5}{x-27}$

$$\therefore \frac{3(x-15) + 6(x-3)}{(x-3)(x-15)} = \frac{4(x-27) + 5(x+9)}{(x+9)(x-27)};$$

$$\text{i.e., } \frac{9x-63}{x^2-18x+45} = \frac{9x-63}{x^2-18x-243};$$

$$\therefore \text{putting } 9x-63 \text{ (i.e., common factor)} = 0,$$

we have

$$x-7. \text{ Ans}$$

Ex. 3. Solve $\frac{(x+c)^3}{(x+d)^3} = \frac{x+2c-d}{x+2d-c}$.

Adding 1, $\frac{(x+c)^3}{(x+d)^3} + 1 = \frac{x+2c-d}{x+2d-c} + 1;$

$$\therefore \frac{(x+c)^3 + (x+d)^3}{(x+d)^3} = \frac{(x+2c-d) + (x+2d-c)}{x+2d-c} = \frac{2x+c+d}{x+2d-c};$$

factorizing, $\frac{(x+c+x+d)\{(x+c)^2 - (x+c)(x+d) + (x+d)^2\}}{(x+d)^3} = \frac{2x+c+d}{x+2d-c}.$

Now, $x+c+x+d=2x+c+d$;

$\therefore 2x+c+d$ is a factor of both sides ;

\therefore putting $2x+c+d=0$, $x=-\frac{c+d}{2}$ Ans.

N. B. For another mode of solution see Ex. 2, Art. 166.

EXAMPLES 85.

Solve

$$1. \quad \frac{1}{x+2} + \frac{1}{x+10} = \frac{1}{x+4} + \frac{1}{x+8}.$$

$$2. \quad \frac{2x-1}{3x-1} = \frac{2x-3}{3x+1}.$$

$$3. \quad \frac{1}{x+5} + \frac{1}{x+3} = \frac{1}{x+2} + \frac{1}{x+6}$$

$$4. \quad \frac{1}{x-5} - \frac{1}{x-7} = \frac{1}{x-6} - \frac{1}{x-8}$$

$$5. \quad \frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{x+a-c} + \frac{1}{x+b+c}.$$

$$6. \quad \frac{x-a}{b+c} + \frac{x-b}{c+a} + \frac{x-c}{a+b} = 3.$$

$$7. \quad \frac{3}{3x+1} + \frac{1}{x+1} = \frac{6}{6x+1} + \frac{6}{6x+7}.$$

$$8. \quad \left(\frac{x+a+b}{x+a-b} \right)^3 = \frac{x+a+3b}{x+a-3b}.$$

$$9. \quad \left(\frac{x-7}{x-9} \right)^3 = \frac{x-5}{x-11}.$$

$$10. \quad \left(\frac{x-13}{x-17} \right)^3 = \frac{x-9}{x-21}.$$

166. Terms with Decimal Coefficients The difficulty here is only arithmetical

Ex. 1 Solve $65x + \frac{585x-975}{6} = \frac{156}{2} - \frac{39x-78}{9}$. C U. 1882.

Divide both sides by '13 ; then we have $5x + \frac{45x-75}{6} = \frac{12}{2} - \frac{3x-6}{9}$.

Now, $\frac{45x-75}{6} = \frac{45x-75}{6} \times \frac{10}{10} = \frac{45x-75}{6} = \frac{15x-25}{2}$;

similarly, $\frac{3x-6}{9} = \frac{30x-60}{9} = \frac{10x-20}{3}$, and $\frac{12}{2} = 6$.

\therefore the above equation reduces to $5x + \frac{15x-25}{2} = 6 - \frac{10x-20}{3}$;

multiplying by 6, $30x + 45x - 75 = 36 - 20x + 40$.

$\therefore 95x = 475$, whence $x = 5$. Ans.

Ex. 2. Solve $\frac{4.05}{9x} - \frac{.3}{.8-2x} = \frac{1.8}{x} - \frac{3.6}{2.4-6x}$. C. U. 1881.

Now, $\frac{4.05}{9x} = \frac{.45}{x}$, and $\frac{3.6}{2.4-6x} = \frac{3 \times 1.2}{3(.8-2x)} = \frac{1.2}{.8-2x}$.

\therefore the given eqn. reduces to $\frac{.45}{x} - \frac{.3}{.8-2x} = \frac{1.8}{x} - \frac{1.2}{.8-2x}$.

\therefore by transposition $\frac{1.2}{.8-2x} - \frac{.3}{.8-2x} = \frac{1.8}{x} - \frac{.45}{x}$;

$\therefore \frac{.9}{.8-2x} = \frac{1.35}{x}$;

dividing by '9,

$\frac{1}{.8-2x} = \frac{1.5}{x}$.

$x = 1.5(.8-2x) = 1.2-3x$;

transposing,

$4x = 1.2$, whence $x = .3$. *Ans.*

EXAMPLES 86.

1. $2.5(x-2) - 4.5(x-1) = 7.5x - 6.25(x-2)$

2. $\frac{.9x-.8}{2.5} - \frac{6x-1.1}{1.2} = \frac{x-6.8}{5} - \frac{1}{12}$.

3. $\frac{4.5}{2.09}(3x-1) - \frac{3}{.95}(x-1) = 3 \times \frac{1-.5x}{1.9}$.

4. $\frac{7.5-5x}{3} = \frac{15-12x}{7.2} + \frac{5-2.5x}{1+.5x}$.

5. $\frac{2.25}{.5x+2} - \frac{1.35}{.5x+1} = \frac{.9}{.5x+3}$.

6. $\frac{2}{x-1.6} - \frac{2}{2x-1} = \frac{1}{x-2.5}$.

7. $\frac{3}{.3} - \frac{1.4x-2}{1.3-6x} = 2.6 \times \frac{.5x-.25}{.5x-1}$. 8. $\frac{1+.2x}{1+x} + \frac{2+.26x}{3+x} = .46 + \frac{1.6}{1+x}$.

CHAPTER XXVII.

HARDER EQUATIONS IN ONE UNKNOWN.

169. Special forms of Equations of Higher Degree.
Some equations of higher degree than the first are easily solved by evolution, i.e., extraction of roots.

Ex. 1 Solve $x^3(x-3a)=a^3(a-3x)$

We have $x^3-3ax^2=a^3-3a^2x$;

transposing, $x^3-3ax^2+3a^2x-a^3=0$,

i e., $(x-a)^3=0$;

extracting cubic root, $x-a=0$; whence $x=a$. *Ans.*

Ex. 2. Solve $16\left(\frac{a-x}{a+x}\right)^8 = \frac{a+x}{a-x}$ C. U. 1886

Multiply by $\frac{a-x}{a+x}$; $16\left(\frac{a-x}{a+x}\right)^8 \times \frac{a-x}{a+x} = \frac{a+x}{a-x} \times \frac{a-x}{a+x}$;

i e., $16 \times \left(\frac{a-x}{a+x}\right)^4 = 1$

Extracting the 4th root,

$$2 \times \frac{a-x}{a+x} = \sqrt[4]{1}$$

$= 1$ or -1 Art 146.

First, let

$$2 \times \frac{a-x}{a+x} = 1;$$

then

$$2(a-x) = a+x;$$

$$-3x = -a,$$

$$\therefore x = \frac{a}{3} \quad \text{Ans.}$$

Secondly, let

$$2 \times \frac{a-x}{a+x} = -1,$$

then

$$2(a-x) = -a-x,$$

$$\therefore -x = -3a.$$

$$\therefore x = 3a \quad \text{Ans.}$$

\therefore there are *two roots*, viz, $\frac{a}{3}$ and $3a$.

N. B. The student may easily see that each of these values of x will satisfy the given equation. For, take $x=3a$. Then $a-x=a-3a=-2a$, and $a+x=4a$. On substitution the given equation reduces to $16\left(\frac{-2a}{4a}\right)^8 = \frac{4a}{-2a}$, which is evidently true, each side being equal to -2 .

Ex 3. Solve $\frac{(x+1)^6+(x-1)^6}{(x+1)^8+(x-1)^8} = 10$. M. U. 1877

By Art. 143, $(x+1)^5 + (x-1)^5 = (x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1) + (x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1)$
 $= 2x^5 + 20x^3 + 10x$;
 and $(x+1)^3 + (x-1)^3 = (x^3 + 3x^2 + 3x + 1) + (x^3 - 3x^2 + 3x - 1)$
 $= 2x^3 + 6x$.

∴ the given equation reduces to

$$\frac{2x^5 + 20x^3 + 10x}{2x^3 + 6x} = 10;$$

or $\frac{x^4 + 10x^2 + 5}{x^2 + 3} = 10;$

$$\therefore x^4 + 10x^2 + 5 = 10(x^2 + 3)$$

$$= 10x^2 + 30;$$

$$\therefore x^4 = 25;$$

$$\therefore x^2 = 5 \text{ or } -5.$$

Now, for a real value of x , x^2 cannot be -5 . Art. 146

∴ rejecting -5 , we have $x^2 = 5$;

$$\therefore x = \sqrt{5} \text{ or } -\sqrt{5}. \text{ Ans.}$$

Hence, there are two roots, viz., $\sqrt{5}$ and $-\sqrt{5}$, usually written $\pm \sqrt{5}$. (Verify).

EXAMPLES 87. †

Solve

1. $(x+a)^2 - 4ax.$

2. $x + \frac{1}{x} = 2.$

3. $\frac{x}{a} + \frac{a}{x} = 2.$

4. $\frac{(x+1)^3 + (x-1)^3}{(x+1)^2 + (x-1)^2} = 2x.$

5. $\left(\frac{x+a}{x-a}\right)^2 + \left(\frac{x-a}{x+a}\right)^2 = 2.$

6. $\left(x + \frac{4}{x}\right)\left(x + \frac{1}{x}\right) = 4.$

7. $125\left(\frac{x+2}{x-2}\right)^3 = \frac{x-2}{x+2}.$

8. $\frac{(2x+3)^6 + (2x-3)^6}{(2x+3)^3 + (2x-3)^3} = 90.$

170. Irrational Equations :

Irrational Equations are those which contain one or more terms involving the variable in the form of surds. The general mode of solution depends upon the following :

(1) *Repeated transposition of terms so as to keep as far as convenient the irrational terms on one side and the rational terms on the other.*

(2) *Repeated process of involution.*

Ex. 1. Solve $\sqrt{4x^2+20x+17} - \sqrt{16x^2+11x+10} = 2(x+2)$.

C. U. 1878.

Squaring, $4x^2+20x+17 - \sqrt{16x^2+11x+10} = 4(x^2+4x+4)$;
transposing,

$$-\sqrt{16x^2+11x+10} = 4(x^2+4x+4) - (4x^2+20x+17) = -4x-1;$$

$$\text{squaring again,} \quad 16x^2+11x+10 = 16x^2+8x+1;$$

$$\text{cancelling } 16x^2, \text{ and transposing,} \quad 3x = -9;$$

$$\therefore x = -3. \text{ Ans.}$$

N. B. Putting -3 for x , the given equation stands as $\sqrt{-7} - \sqrt{121} = -2$. If we take $\sqrt{121} = -11$ instead of 11 , we have $\sqrt{-7} + 11$ or $\sqrt{4} = -2$, so that the square roots must be taken negatively in order that the equation may be satisfied. The proper form should therefore be $\sqrt{4x^2+20x+17} + \sqrt{16x^2+11x+10} + 2(x+2) = 0$.

Ex. 2. Solve $\sqrt{x} + \sqrt{4+x} = \frac{2}{\sqrt{x}}$ C. U. 1873.

Multiplying by \sqrt{x} , $x + \sqrt{x(4+x)} = 2$;

transposing, $\sqrt{x(4+x)} = 2-x$;

squaring, $x(4+x) = 4-4x+x^2$,

cancelling x^2 , and transposing, $8x = 4$;

$$\therefore x = \frac{1}{2}. \text{ Ans.}$$

Ex. 3. Solve $\sqrt[n]{x^2+2bx+c^2} = \sqrt[n]{x+d}$.

Raise both sides to the $2n$ th power, then

$$\begin{aligned} x^2+2bx+c^2 &= \{(x+d)^{\frac{1}{n}}\}^{2n} \\ &= (x+d)^2 \\ &= x^2+2dx+d^2; \end{aligned}$$

transposing, $2(b-d)x = d^2-c^2$;

$$\therefore x = \frac{d^2-c^2}{2(b-d)}. \text{ Ans.}$$

Ex. 4. Solve $\sqrt{x^2+11x+20} - \sqrt{x^2+5x-1} = 3$. C. U. 1881.

By transposition, $\sqrt{x^2+11x+20} = 3 + \sqrt{x^2+5x-1}$;

squaring, $x^2+11x+20 = 9+x^2+5x-1+6\sqrt{x^2+5x-1}$
 $= x^2+5x+8+6\sqrt{x^2+5x-1}$;

transposing, $6x+12 = 6\sqrt{x^2+5x-1}$;

dividing by 6, $x+2 = \sqrt{x^2+5x-1}$;

squaring again, $x^2+4x+4 = x^2+5x-1$;

whence $x = 5$. *Ans.*

•EXAMPLES 88.

1. $\sqrt{2x+3} = \sqrt{x+6}$.
2. $\sqrt{3x+2} = \sqrt{x+7}$.
3. $\sqrt[3]{x+4} = \sqrt[3]{(x+2)(x+3)}$.
4. $\sqrt{a+bx} = \sqrt[4]{c^2+2dx+b^2x^2}$.
5. $\sqrt{x^2-2} = \sqrt[3]{x(x^2-3)}$.
6. $\frac{1}{6}(\sqrt{x+20}+1) = \frac{1}{2}$.
7. $\frac{\sqrt[3]{11x+5}}{3} + 6 = 7$.
8. $\sqrt{5x-1} = 1 + \sqrt{5x-2}$.
9. $\sqrt{x+a} = \sqrt{b} + \sqrt{x}$.
10. $\sqrt{x+48} + \sqrt{x} = 12$.
11. $\sqrt{x+a} + \sqrt{x+b} = \sqrt{a+b}$.
12. $\sqrt{2x+5} - \sqrt{2x+1} = \sqrt{2(4x-3)}$.
13. $\sqrt{x+a-b} + \sqrt{x+b-a} = 2\sqrt{x+a+b}$.
14. $\sqrt{ax-(b-1)^2} + \sqrt{ax-(b+1)^2} = 2\sqrt{b}$.
15. $x+1 = \sqrt{x^2-3} + \sqrt[3]{x^2+12}$.
16. $x+a = \sqrt{x^2+a^2-2ab+2a\sqrt{3x^2+b^2}}$.
17. $\frac{x}{2} + a = \sqrt{x} \sqrt{\frac{x}{4} + a} + \sqrt{x+4a}$.
18. $na+m = \sqrt{n^2x^2-2an+m^2+2n\sqrt{m^2x^2+2mb+a^2}}$.
19. $\sqrt{x} + \sqrt{x-l\sqrt{l^2-x}} = l$.
20. $\sqrt{a+bx} + \sqrt{a-bx} = 2\sqrt{a^2-b^2x^2}$.
21. $\sqrt{13x-1} - \sqrt{13x-10} = 1$.
22. $\sqrt{8x+1} - \sqrt{2x+3} = \sqrt{2(x-1)}$.
23. $2\sqrt{2x+1} + \sqrt{2x+8} = \sqrt{12x+52}$.
24. $\sqrt{9x+4} + \sqrt{x+4} = 2x$.
25. $\sqrt{3}\sqrt{x+10} + 2\sqrt{\sqrt{9x-2}} = \sqrt{27}\sqrt{x+10}$.
26. $\sqrt{\sqrt{x+1}+5} + \sqrt{\sqrt{x+1}+12} = 7$.
27. $\sqrt[3]{x+2} + \sqrt[3]{x+7} = \sqrt[3]{4x+17}$.
28. $\sqrt{ax^{\frac{1}{n}}+b} + \sqrt{4ax^{\frac{1}{n}}+c} = \sqrt{9ax^{\frac{1}{n}}+m}$.
29. $\sqrt{2x+1} + \sqrt{2x+7} = 3 + \sqrt{3}$.
30. $\sqrt{x+1} - \frac{\sqrt{x+1}}{5} = \sqrt{x-8}$.
31. $a(a+x)^n + c = d - x(a+x)^n$.
32. $\frac{1}{3}\sqrt[3]{x+3} + \frac{1}{x}\sqrt[3]{x+3} = \sqrt[3]{3x}$.
33. $\sqrt{x+1} + \sqrt{x+5} = \frac{2}{\sqrt{x+1}}$.
34. $\sqrt{2x+3} + \sqrt{2x+7} = \frac{5}{\sqrt{2x+3}}$.

$$35. \sqrt{x+1} - \sqrt{x-2} = \frac{3}{\sqrt{4x-3}}.$$

$$36. \sqrt{\frac{x^2}{4} - 5x + 7} - \sqrt{\frac{x^2}{4} - 3x + 1} = 2.$$

$$37. \sqrt{\frac{x-a}{x-b}} - \frac{a}{x} = \sqrt{\frac{x-b}{x-a}} - \frac{b}{x}. \quad 38. x-3 = \sqrt{2x-7}.$$

$$39. x-b = \sqrt{a-b}. \sqrt{2x-a-b}. \quad 40. ax+c = \sqrt{(b+c)(2ax-b+c)}.$$

171. Method of Identity. This method is best illustrated by examples.

Ex. 1. Solve the equation $\sqrt{5x+6} - \sqrt{5x-6} = 2$.

By simple subtraction, we have

$$(5x+6) - (5x-6) = 5x+6 - 5x+6 = 12 \dots\dots\dots (1)$$

\therefore (1) is true identically, i.e., for all values of x ;

$$i.e., \quad (\sqrt{5x+6})^2 - (\sqrt{5x-6})^2 = 12 \dots\dots\dots (2)$$

Now, the given equation is

$$\sqrt{5x+6} - \sqrt{5x-6} = 2 \dots\dots\dots (3)$$

Dividing (2) by (3), we have

$$\frac{\{\sqrt{5x+6}\}^2 - \{\sqrt{5x-6}\}^2}{\sqrt{5x+6} - \sqrt{5x-6}} = \frac{12}{2};$$

$$\therefore \text{reducing, } \sqrt{5x+6} + \sqrt{5x-6} = 6 \dots\dots\dots (4)$$

$$\text{Adding (4) to (3),} \quad 2\sqrt{5x+6} = 6+2=8;$$

$$\text{dividing by 2,} \quad \sqrt{5x+6} = 4;$$

$$\text{squaring,} \quad 5x+6=16;$$

$$\text{transposing,} \quad 5x=10;$$

$$\therefore x=2. \text{ Ans.}$$

N. B. It will be observed that (1) has been deduced by simple subtraction, the result of which is always the same, whatever values be assigned to x . Hence also (1) is true for the particular value that x has in the given equation, which fact may be readily verified. The next step is the division of the identity (1) by the given equation. This gives

us the fraction $\frac{(\sqrt{5x+6})^2 - (\sqrt{5x-6})^2}{\sqrt{5x+6} - \sqrt{5x-6}}$, which is of the form $\frac{a^2 - b^2}{a - b}$

($=a+b$). The last step of the artifice is the operation of addition (or subtraction, if convenient), applied to the given equation (3) and the deduced equation (4); the rest of the work is easy. The student is advised to attempt the solution by the ordinary method from the beginning.

Ex. 2. Solve $\sqrt{5x+7} + \sqrt{4x+9} = \sqrt{5x+6} + \sqrt{4x+10}$

By transposition, $\sqrt{5x+7} - \sqrt{5x+6} = \sqrt{4x+10} - \sqrt{4x+9}$(1)

Now, $(5x+7) - (5x+6) = 5x+7-5x-6 = 1$, } ...By subtraction.

and $(4x+10) - (4x+9) = 4x+10-4x-9 = 1$; }

\therefore whatever may be the value of x (*i.e.*, *identically*),

we have $(5x+7) - (5x+6) = (4x+10) - (4x+9)$;

i.e., $(\sqrt{5x+7})^2 - (\sqrt{5x+6})^2 = (\sqrt{4x+10})^2 - (\sqrt{4x+9})^2$(2)

Divide (2) by (1) ; thus we have

$$\sqrt{5x+7} + \sqrt{5x+6} - \sqrt{4x+10} + \sqrt{4x+9} \dots \dots (3)$$

Add (1) and (3) ; then $2\sqrt{5x+7} = 2\sqrt{4x+10}$: (3)

$$\therefore \sqrt{5x+7} = \sqrt{4x+10} ;$$

squaring,

$$5x+7 = 4x+10 ;$$

whence

$$x-3 \text{ Ans}$$

EXAMPLES 89.

1. $\sqrt{2x+3} - \sqrt{2x-5} = 2$

2. $\sqrt{7x+8} + \sqrt{7x-3} = 11.$

3. $\sqrt{4x+5} = 2 + \sqrt{4x-11}$

4. $\sqrt{ax+b} + \sqrt{a(x+1)} = c.$

5. $\sqrt{2x+3} + 2\sqrt{x+1} = \sqrt{4x+13} + \sqrt{2x-6}.$

6. $\sqrt{3x+7} + \sqrt{5x+3} - \sqrt{5x-1} + \sqrt{3x+11}.$

7. $\sqrt{7x-4} + \sqrt{13x+6} = \sqrt{13x+9} + \sqrt{7(x-1)}.$

8. $\sqrt{11}\sqrt{x+5} + \sqrt{10}\sqrt{x-1} = \sqrt{10}\sqrt{x+6} + \sqrt{11}\sqrt{x-2}$

9. $2\sqrt{x+2} + 3\sqrt{x+3} = \sqrt{4x+5} + \sqrt{3\sqrt{x+10}}.$

10. $\sqrt{x^2+5x+7} - \sqrt{x^2+7x+5} = 1.$

11. $\sqrt{x+1}(\sqrt{9x+1} + \sqrt{9x-5}) = 1.$

12. $\sqrt{x-1} + \sqrt{x+1} = \sqrt{4x-3}.$

172. Reduction and Rationalization of Denominator. To prepare for the methods illustrated below, the student should revise the chapter on Surds.

Ex. 1. Solve $\frac{5(x-1)}{\sqrt{5x+4}+3} + \frac{5x+2}{\sqrt{5x+11}-3} = \frac{1}{\sqrt{5x+4}}$

$$\begin{aligned} \text{Now,} \quad & 5(x-1) = 5x+4-9 = (\sqrt{5x+4})^2 - 3^2, \\ \text{and} \quad & 5x+2 = 5x+11-9 = (\sqrt{5x+11})^2 - 3^2; \end{aligned}$$

$$\therefore \text{ we have } \frac{(\sqrt{5x+4})^2 - 3^2}{\sqrt{5x+4} + 3} + \frac{(\sqrt{5x+11})^2 - 3^2}{\sqrt{5x+11} - 3} = \frac{1}{\sqrt{5x+4}},$$

$$\therefore \text{ reducing, } \sqrt{5x+4} - 3 + \sqrt{5x+11} - 3 = \frac{1}{\sqrt{5x+4}};$$

$$\text{i. e., } \sqrt{5x+4} + \sqrt{5x+11} = \frac{1}{\sqrt{5x+4}},$$

$$\text{multiplying by } \sqrt{5x+4}, \quad 5x+4 + \sqrt{(5x+4)(5x+11)} = 1.$$

$$\therefore \text{ by transposition, } \sqrt{(5x+4)(5x+11)} = -5x-3;$$

$$\text{squaring, } (5x+4)(5x+11) = 25x^2 + 30x + 9;$$

$$\text{simplifying, } 25x^2 + 75x + 44 = 25x^2 + 30x + 9;$$

$$\text{transposing and simplifying, } 45x = -35; \\ x = -\frac{7}{9} \text{ Ans}$$

$$\text{Ex. 2. Solve } \frac{2}{x + \sqrt{x^2-1}} - \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} = \frac{1}{2}.$$

Rationalize the denominators [See Art 140]

$$\begin{aligned} \frac{2}{x + \sqrt{x^2-1}} &= \frac{2}{x + \sqrt{x^2-1}} \times \frac{x - \sqrt{x^2-1}}{x - \sqrt{x^2-1}} = \frac{2(x - \sqrt{x^2-1})}{x^2 - (x^2-1)} \\ &= \frac{2(x - \sqrt{x^2-1})}{1} \end{aligned}$$

$$\begin{aligned} \text{and } \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} &= \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} \times \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} \\ &= \frac{\{\sqrt{x+1} - \sqrt{x-1}\}^2}{(x+1) - (x-1)} \\ &= \frac{(x+1) + (x-1) - 2\sqrt{(x^2-1)}}{2} \\ &= \frac{2x - 2\sqrt{(x^2-1)}}{2} \\ &= x - \sqrt{(x^2-1)} \end{aligned}$$

\therefore the given equation reduces to

$$2x - 2\sqrt{(x^2-1)} - \{x - \sqrt{(x^2-1)}\} = \frac{1}{2};$$

$$\text{simplifying, } x - \sqrt{(x^2-1)} = \frac{1}{2};$$

$$\text{transposing, } x - \frac{1}{2} = \sqrt{(x^2-1)};$$

$$\text{squaring, } x^2 - x + \frac{1}{4} = x^2 - 1;$$

$$\therefore x = \frac{5}{4} \text{ Ans.}$$

EXAMPLES 90.

Solve

$$1. \frac{4x^2 - a}{2x - \sqrt{a}} + \frac{4x^2 - a}{2x + \sqrt{a}} = 2x - \frac{2x - \sqrt{a}}{2}.$$

$$2. \frac{lx - m^2}{\sqrt{lx - m}} = \sqrt{nx + l}.$$

$$3. \frac{ax + b - c}{\sqrt{ax + b} + \sqrt{c}} = \sqrt{ax - b}.$$

$$4. \frac{2x - 1}{\sqrt{2x + 3} + 2} = \sqrt{2x - 3}.$$

$$5. \frac{x}{\sqrt{3x + 16} + 4} = \frac{\sqrt{3x - 16}}{3}.$$

$$6. \frac{8 - x}{\sqrt{x + 1} + 3} + \frac{4x - 13}{\sqrt{4x + 3} + 4} = \frac{9x}{\sqrt{9x + 1} + 1}.$$

$$7. \frac{3x + 1}{\sqrt{9x + 7} + 2} + \frac{3(x - 1)}{\sqrt{9x + 16} + 5} = \frac{2}{3}.$$

$$8. \frac{x^2}{1 + \sqrt{1 - x^2}} + \sqrt{1 - x^2} = x \quad 9. \frac{x}{a - \sqrt{(a^2 - x^2)}} = \frac{\sqrt{(a^2 + x^2)}}{x}.$$

$$10. \frac{1}{\sqrt{2x + 2} + \sqrt{(2x + 1)}} + \frac{1}{\sqrt{(x + 6)} - \sqrt{(x + 5)}} = \sqrt{(x + 5)} + \sqrt{(2x + 2)}.$$

173. **Special form.** The method in the following example deserves special attention.

Ex. Solve $(1+x)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}} = 2^{\frac{1}{3}}$ C U 1885.

Cubing both sides, we get

$$(1+x) + (1-x) + 3(1-x^2)^{\frac{1}{3}}\{(1+x)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}}\} = 2;$$

$$\text{i. e., } 2 + 3(1-x^2)^{\frac{1}{3}}\{(1+x)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}}\} = 2;$$

$$\text{cancelling } 2, 3(1-x^2)^{\frac{1}{3}}\{(1+x)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}}\} = 0 \dots \dots (1).$$

But by the given equation $(1+x)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}} = 2^{\frac{1}{3}}$.

\therefore substitute $2^{\frac{1}{3}}$ for $(1+x)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}}$ in (1); thus

$$3(1-x^2)^{\frac{1}{3}} 2^{\frac{1}{3}} = 0;$$

$$\text{cubing again, } 27(1-x^2).2 = 0;$$

$$\therefore 1 - x^2 = 0;$$

$$\therefore x^2 = 1;$$

$$\therefore x = 1 \text{ or } -1. \text{ Ans.}$$

EXAMPLES 91.

Solve

$$1. (4+3x)^{\frac{1}{3}} + (4-3x)^{\frac{1}{3}} = 2.$$

$$2. (a+x)^{\frac{1}{3}} + (a-x)^{\frac{1}{3}} = (2a+3)^{\frac{1}{3}}.$$

3. $(m+nx)^{\frac{1}{3}} + (m-nx)^{\frac{1}{3}} = r^{\frac{1}{3}}$ 4. $(x+2)^{\frac{1}{3}} + (x-4)^{\frac{1}{3}} = (2x-2)^{\frac{1}{3}}$.
 5. $\sqrt[3]{2}\sqrt{x+3a} - \sqrt[3]{2}\sqrt{x-3a} = a$. 6. $\sqrt[3]{ax+b^3} + \sqrt[3]{ax-b^3} = \sqrt[3]{2ax}$.
 7. $\sqrt[3]{2x+3} - \sqrt[3]{2x-5} = 2$. 8. $\sqrt[3]{ax+b^3} - \sqrt[3]{ax-b^3} = \sqrt[3]{c^3}$.

174. Componendo and Dividendo *If two fractions be equal, the sum of the numerator and denominator of one of them divided by their difference is equal to a similar fraction formed from the other*

That is, if $\frac{a}{b} = \frac{c}{d}$, then will $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Proof Let $\frac{a}{b} = \frac{c}{d} \dots\dots\dots (1)$

Add 1 to both sides ; $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

$$\therefore \frac{a+b}{b} = \frac{c+d}{d} \dots\dots\dots (2)$$

Again from (1), $\frac{a}{b} - 1 = \frac{c}{d} - 1$, taking 1 from each side ;

$$\therefore \frac{a-b}{b} = \frac{c-d}{d} \dots\dots\dots (3)$$

Divide (2) by (3) ; thus, by Art 113, we get

$$\frac{a+b}{b} \times \frac{b}{a-b} = \frac{c+d}{d} \times \frac{d}{c-d},$$

simplifying, $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Otherwise thus : Let $\frac{a}{b} = \frac{c}{d} = k$.

Then $\therefore \frac{a}{b} = k, a = bk$;

$$\therefore \frac{a+b}{a-b} = \frac{bk+b}{bk-b} = \frac{b(k+1)}{b(k-1)} = \frac{k+1}{k-1} \dots\dots\dots (a)$$

Again, $\therefore \frac{c}{d} = k, c = dk$,

$$\therefore \frac{c+d}{c-d} = \frac{dk+d}{dk-d} = \frac{d(k+1)}{d(k-1)} = \frac{k+1}{k-1} \dots\dots\dots (b)$$

\therefore from (a) and (b), $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

175. Useful surd identities. By expansion, we get

$$[\sqrt{1+x} + \sqrt{1-x}]^2 = 2 + 2\sqrt{1-x^2};$$

$$[\sqrt{a+x} + \sqrt{a-x}]^2 = 2a + 2\sqrt{a^2-x^2};$$

$$[\sqrt{a+x} - \sqrt{a-x}]^2 = 2a - 2\sqrt{a^2-x^2}.$$

Ex. 1. Solve $\frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}} = \frac{1}{2}$. B. U. 1863.

By Componendo and Dividendo,

$$\frac{(\sqrt{x+1} - \sqrt{x-1}) + (\sqrt{x+1} + \sqrt{x-1})}{(\sqrt{x+1} - \sqrt{x-1}) - (\sqrt{x+1} + \sqrt{x-1})} = \frac{1+2}{1-2};$$

$$\text{i.e.,} \quad \frac{2\sqrt{x+1}}{-2\sqrt{x-1}} = \frac{3}{-1};$$

$$\text{reducing and squaring,} \quad \frac{x+1}{x-1} = 9 = \frac{9}{1};$$

Again by Componendo and Dividendo,

$$\frac{(x+1) + (x-1)}{(x+1) - (x-1)} = \frac{9+1}{9-1};$$

$$\text{simplifying,} \quad \frac{2x}{2} = \frac{10}{8} = \frac{5}{4};$$

$$\text{i.e.,} \quad x = \frac{5}{4}. \quad \text{Ans.}$$

Ex. 2. Solve $\frac{a + \sqrt{a^2 - x^2}}{\sqrt{a+x} - \sqrt{a-x}} = 27$. $\frac{a - \sqrt{a^2 - x^2}}{\sqrt{a+x} + \sqrt{a-x}}$.

$$\left. \begin{aligned} \therefore a + \sqrt{a^2 - x^2} &= \frac{1}{2}(2a + 2\sqrt{a^2 - x^2}) \\ &= \frac{1}{2}(\sqrt{a+x} + \sqrt{a-x})^2, \\ \text{and } a - \sqrt{a^2 - x^2} &= \frac{1}{2}(2a - 2\sqrt{a^2 - x^2}) \\ &= \frac{1}{2}(\sqrt{a+x} - \sqrt{a-x})^2, \end{aligned} \right\} \text{ See Art. 175.}$$

$$\text{we have } \frac{1}{2} \frac{\{\sqrt{a+x} + \sqrt{a-x}\}^2}{\sqrt{a+x} - \sqrt{a-x}} = 27 \times \frac{1}{2} \frac{\{\sqrt{a+x} - \sqrt{a-x}\}^2}{\sqrt{a+x} + \sqrt{a-x}};$$

multiply both sides by $\frac{2\{\sqrt{a+x} + \sqrt{a-x}\}}{\{\sqrt{a+x} - \sqrt{a-x}\}^2}$; then

$$\frac{\{\sqrt{a+x} + \sqrt{a-x}\}^3}{\{\sqrt{a+x} - \sqrt{a-x}\}^3} = 27;$$

extracting the cube root, $\frac{\sqrt[3]{(a+x)} + \sqrt[3]{(a-x)}}{\sqrt[3]{(a+x)} - \sqrt[3]{(a-x)}} = \frac{3}{1}$;

$$\therefore \frac{\{\sqrt[3]{(a+x)} + \sqrt[3]{(a-x)}\} + \{\sqrt[3]{(a+x)} - \sqrt[3]{(a-x)}\}}{\{\sqrt[3]{(a+x)} + \sqrt[3]{(a-x)}\} - \{\sqrt[3]{(a+x)} - \sqrt[3]{(a-x)}\}} = \frac{3+1}{3-1}; \text{Comp., Div.}$$

$$\text{i.e.,} \quad \frac{\sqrt[3]{(a+x)}}{\sqrt[3]{(a-x)}} = \frac{4}{2} = \frac{2}{1};$$

$$\text{squaring,} \quad \frac{a+x}{a-x} = \frac{4}{1};$$

$$\therefore \frac{(a+x) + (a-x)}{(a+x) - (a-x)} = \frac{4+1}{4-1}; \text{Comp, Div.}$$

$$\therefore \text{reducing,} \quad \frac{a}{x} = \frac{5}{3};$$

$$\text{whence } 5x = 3a, \text{ and } x = \frac{3}{5}a. \text{ Ans}$$

Ex. 3. Solve

$$\sqrt[4]{\left(\frac{m}{m+x}\right)} + \sqrt[4]{\left(\frac{n}{n+x}\right)} = 2 \sqrt[4]{\left\{\frac{mn}{(m+x)(n+x)}\right\}}.$$

By transposition, we can put the equation as

$$\left\{\sqrt[4]{\left(\frac{m}{m+x}\right)}\right\}^2 + \left\{\sqrt[4]{\left(\frac{n}{n+x}\right)}\right\}^2 - 2 \times \sqrt[4]{\left(\frac{m}{m+x}\right)} \times \sqrt[4]{\left(\frac{n}{n+x}\right)} = 0,$$

$$\text{i.e.,} \quad \left\{\sqrt[4]{\left(\frac{m}{m+x}\right)} - \sqrt[4]{\left(\frac{n}{n+x}\right)}\right\}^2 = 0;$$

$$\therefore \text{extracting sq. root,} \quad \sqrt[4]{\left(\frac{m}{m+x}\right)} - \sqrt[4]{\left(\frac{n}{n+x}\right)} = 0;$$

$$\text{transposing,} \quad \sqrt[4]{\left(\frac{m}{m+x}\right)} = \sqrt[4]{\left(\frac{n}{n+x}\right)};$$

raising to the 4th power,

$$\frac{m}{m+x} = \frac{n}{n+x},$$

\therefore by Art. 160,

$$m(n+x) = n(m+x);$$

$$\text{i.e.,} \quad mn + mx = mn + nx;$$

transposing,

$$(m-n)x = 0;$$

$$\therefore x = 0. \text{ Ans.}$$

Ex. 4. Solve $\frac{1}{a} \sqrt[4]{a+x} + \frac{1}{x} \sqrt[4]{a+x} = \frac{1}{b} \sqrt[4]{x}$.

The left side $= \left(\frac{1}{a} + \frac{1}{x}\right) \sqrt[4]{a+x} = \frac{a+x}{ax} \sqrt[4]{a+x}$

\therefore the given eqn. reduces to $\frac{(a+x)(a+x)^{\frac{1}{4}}}{ax} = \frac{x^{\frac{1}{4}}}{b}$;

$$\text{i.e.,} \quad \frac{(a+x)^{\frac{5}{4}}}{ax} = \frac{x^{\frac{1}{4}}}{b};$$

multiplying by ax , $(a+x)^{\frac{5}{4}} = \frac{a}{b} x^{\frac{5}{4}};$

raising to the fourth power, $(a+x)^5 = \left(\frac{a}{b}\right)^4 x^5;$

taking the 5th root, $a+x = \left(\frac{a}{b}\right)^{\frac{4}{5}} x;$

transposing, $a = x \left\{ \left(\frac{a}{b}\right)^{\frac{4}{5}} - 1 \right\}$
 $= x \cdot \frac{a^{\frac{4}{5}} - b^{\frac{4}{5}}}{b^{\frac{4}{5}}};$

$$x = \frac{ab^{\frac{4}{5}}}{a^{\frac{4}{5}} - b^{\frac{4}{5}}}. \quad \text{Ans.}$$

EXAMPLES 92.

Solve

$$1. \quad \frac{\sqrt{(2x+1)} + \sqrt{x}}{\sqrt{(2x+1)} - \sqrt{x}} = 5.$$

$$2. \quad \frac{\sqrt{(ax+b)} + \sqrt{(ax-b)}}{\sqrt{(ax+b)} - \sqrt{(ax-b)}} = c.$$

$$3. \quad \frac{\sqrt{(mx)} + \sqrt{(nx+b)}}{\sqrt{(mx)} - \sqrt{(nx+b)}} = \frac{\sqrt{r+c}}{\sqrt{r-c}}.$$

$$4. \quad \frac{\sqrt{(2x+3)} + \sqrt{(5-2x)}}{\sqrt{(x+5)} + \sqrt{(5-x)}} = \frac{\sqrt{(x+3)} - \sqrt{(5-2x)}}{\sqrt{(x+5)} - \sqrt{(5-x)}}.$$

$$5. \quad \frac{a + \sqrt{(a^2 - x^2)}}{a - \sqrt{(a^2 - x^2)}} = b^2 \times \frac{\sqrt{(a+x)} - \sqrt{(a-x)}}{\sqrt{(a+x)} + \sqrt{(a-x)}}.$$

$$6. \quad \frac{1 + \sqrt{(1-x^2)}}{1 - \sqrt{(1-x^2)}} = 32 \left\{ \frac{\sqrt{(1+x)} - \sqrt{(1-x)}}{\sqrt{(1+x)} + \sqrt{(1-x)}} \right\}^8.$$

$$7. \quad \frac{x+a - \sqrt{(x^2+2ax)}}{x+a + \sqrt{(x^2+2ax)}} = b. \quad \frac{\sqrt{(x+2a)} - \sqrt{x}}{\sqrt{(x+2a)} + \sqrt{x}}.$$

8. $\frac{x+3-\sqrt{(x^2+6x)}}{x+3+\sqrt{(x^2+6x)}} = \frac{1}{27} \frac{\sqrt{(x+6)}+\sqrt{x}}{\sqrt{(x+6)}-\sqrt{x}}$
9. $\left\{ \frac{\sqrt{(ax+b)}+\sqrt{(ax-b)}}{\sqrt{(ax+b)}-\sqrt{(ax-b)}} \right\}^m = \left(\frac{n+1}{n} \right)^m \left\{ \frac{ax-\sqrt{(a^2x^2-b^2)}}{ax+\sqrt{(a^2x^2-b^2)}} \right\}^{\frac{m}{2n}}$
10. $\frac{1-mx}{1+mx} \sqrt{\left(\frac{1+nx}{1-nx} \right)} = 1$ 11. $\frac{x+a+\sqrt{(x^2-a^2)}}{x+\sqrt{(x^2-a^2)}} = \frac{x}{b}$
12. $\sqrt[3]{\frac{9}{7} \frac{a-x}{a+x}} + \sqrt[3]{\frac{a^3-ax+x^2}{a^2+ax+x^2}} = 0$
13. $\frac{1}{\sqrt{(1+x)}} + \frac{1}{\sqrt{(1-x)}} = \frac{2}{\sqrt{(1-x^2)}}$ 14. $\frac{x-\sqrt{(1-x^2)}}{\sqrt{x}+\sqrt{(x-x^2)}} = a$
15. $\sqrt[3]{\left(\frac{2}{1+x} \right)} + 2 \sqrt[3]{\left(\frac{4}{1-x} \right)} = \sqrt[4]{\frac{4}{(1-x^2)}}$
16. $\sqrt{\frac{b}{a+x}} + \sqrt{\frac{c}{a-x}} = 2 \sqrt{\frac{bc}{a^2-1}}$
17. $\sqrt{\frac{2+3x}{1+x}} + \sqrt{\frac{1+x}{2+3x}} = 2$ 18. $\frac{\sqrt{p}+\sqrt{x}}{\sqrt[3]{2}} + \frac{\sqrt{p}-\sqrt{x}}{\sqrt[3]{r}} = \sqrt[6]{x}$
19. $(ax+b)^{\frac{1}{3}} + 4(ax-b)^{\frac{1}{3}} = 4(a^2x^2-b^2)^{\frac{1}{3}}$
20. $a^2 \left(\frac{1+x}{1-x} \right)^{\frac{2}{3}} + b^2 \left(\frac{1-x}{1+x} \right)^{\frac{2}{3}} = 2ab$
21. $\frac{1}{a} \sqrt{(a+x)} + \frac{1}{x} \sqrt{(a+x)} = \frac{1}{b} \sqrt{x}$ 22. $\sqrt[3]{x^2+3} = 3 \sqrt[3]{x} + \sqrt[3]{2}$
23. $\frac{1}{2} \sqrt[4]{(2+x)} + \frac{1}{x} \sqrt[4]{(2+x)} = \sqrt[4]{(2x)}$
24. $\frac{1+\sqrt{(x-1)}}{4 \sqrt{(x-1)}} - \frac{\sqrt{(x-1)}-1}{x-2}$ 25. $\sqrt{\left(\frac{x^2+2a^2}{2x^2+a^2} \right)} - \sqrt{\left(\frac{a}{x} \right)} = 0$

176 Exponential Equations. These are equations in which the variable occurs as some power of a quantity; for example, $a^x=c$. In an elementary treatise only those forms of such equations can be treated which may after a little reduction be made to depend upon the following:—

(1) If we have $a^x=a^m$, then $x=m$.

(2) If we have $a^{mx-n}=1$, then, knowing that $a^0=1$ from Art. 130, we have $mx-n=0$

The student is referred back to the chapter on the Theory of Indices.

Ex. 1. Solve $2 \times 4^{x-1} = \frac{8^{x-3}}{2}$.

Observe that we can express all the terms of the given equation in powers of 2; for, by Arts 135 and 134,

$$4^{x-1} = (2^2)^{x-1} = 2^{2x-2},$$

$$\frac{8^{x-3}}{2} = \frac{(2^3)^{x-3}}{2} = \frac{2^{3x-9}}{2^1} = 2^{3x-10}.$$

\therefore the given equation reduced to

$$2^1 \times 2^{2x-2} = 2^{3x-10};$$

$$\text{i.e., } 2^{2x-1} = 2^{3x-10};$$

$$\therefore 2x-1 = 3x-10;$$

$$\therefore x = 9. \text{ Ans}$$

Ex 2. Solve $2^{x+9} \cdot 3^{x+1} = 144$

We shall express each term in powers of 2 and 3; $144 = 16 \times 9 = 2^4 \cdot 3^2$.

\therefore the given equation reduces to

$$2^{x+9} \cdot 3^{x+1} = 2^4 \cdot 3^2.$$

Divide both sides by the right-hand member;

$$\text{then } \frac{2^{x+9}}{2^4} \cdot \frac{3^{x+1}}{3^2} = 1;$$

$$\therefore 2^{(x+9)-4} \cdot 3^{(x+1)-2} = 1;$$

$$\text{i.e., } 2^{x+5} \cdot 3^{x-1} = 1;$$

$$\therefore (2 \cdot 3)^{x-1} = 1; \dots \text{Art. 136.}$$

$$\therefore x-1 = 0; \dots \text{Art. 130.}$$

$$\therefore x = 1. \text{ Ans.}$$

Ex. 3. Solve $3^{x+2} = 3^{x+1} + 18$.

By transposition, $3^{x+2} - 3^{x+1} = 18$;

$$\therefore 3^{x+1}(3-1) = 18, \quad \therefore 3^{x+1} \cdot 3 = 3^{x+2};$$

$$\therefore 3^{x+1} \times 2 = 18;$$

$$\therefore 3^{x+1} = 9 = 3^2,$$

$$\therefore x+1 = 2;$$

$$\therefore x = 1. \text{ Ans.}$$

EXAMPLES 93.

Solve

1. $2^{x-8} = 16$

2. $3 \cdot 2^{x-1} = 24.$

3. $a^{bx-c} = 1.$

4. $l^{mx-n} = k^{mx-n}.$

5. $2^{x-4} = 4a^{x-6}.$

6. $\left(\frac{a}{b}\right)^{4x-4} = \left(\frac{b}{a}\right)^{4x-4}$.
 7. $2^{3x-5} a^{x-2} = 2^{x-5} \cdot 2a^{1-x}$
 8. $\frac{3^{3x-4} a^{2x-5}}{3^{x+1}} = b^{2x-5}$
 9. $a^{x^2} = \{(8\sqrt{a})\}^n$.
 10. $a^{x-2}(a^{2x+2} + a^{1-x}) = a^{-3}(a^0 + a^2)$.
 11. $3^{x+1} \cdot 5^{x+2} = 16875$
 12. $2^{x+1} - 2^x - 8 = 0$.
 13. $a^{x+c} - a^x = a^{b+c} - a^b$.
 14. $3^{x+5} = 3^{x+8} + \frac{8}{3}$.
 15. $4^{x+2} = 2^{2x+1} + 14$.

177'. In the following examples, the student is expected to find out the most convenient method applicable to each.

EXAMPLES 94.

Solve

- $5(x+2) + 11(3x+1) - 4\left(1 - \frac{x+7}{2}\right) = 21 + 3$.
- $r(a+4) + (r+4)a = (b+4)(r+a) + c(x+1)$.
- $2(x-2)(x-1) + 3(x-4)(x-1) = (51-1)1$.
- $e^{2x}(e^{x+2} + e^{x+1}) = e^{2+6} + e^6$
- $(17x+3)^2 - (112-7)^2 = (13x+10)^2 + (10+x)(14-x)$
- $(x+1)(x+4)(x+6) = (x+2)^2(x+7)$
- $\sqrt{(13x+9)} + \sqrt{(13x-3)} = \sqrt{(52x+81)}$
- $\frac{c}{x-d} - \frac{d}{x+c} = \frac{c-d}{x}$.
 9. $\frac{x-1}{x-1} - \frac{3}{5}\left(\frac{1}{x-1} - \frac{1}{3}\right) = \frac{23}{10(x-1)}$.
- $\frac{ax}{b} - \frac{1}{b}\left(\frac{1}{c} + x\right) + d = \frac{d}{b}\left(bx - \frac{1}{cd}\right) - \frac{x}{b} + \frac{a}{b}$.
- $\sqrt{(x^2+13x+1)} + \sqrt{(x^2+7x+2)} = 3$.
- $\frac{x+5}{2x+1} = \frac{x+6}{2x+3} + \frac{9x}{4x^2+8x+3}$.
- $\frac{2}{2x+1} + \frac{1}{x+1} = \frac{2}{2x+1+2x} + \frac{1}{x+1-c}$.
- $\frac{15x-14}{3x-4} + \frac{47-28x}{10-7x} - \frac{9x+37}{x+4} = 0$.
 $\frac{x}{2} - \frac{1}{6} - \frac{x}{2}\left\{\frac{4x^2}{3} - \frac{4}{3}\left(\frac{x-1}{3}\right)^2\right\} + x+2 = \frac{7}{x-2}$.
- $\frac{2x-1}{x} - \frac{2x+1}{x+1} = \frac{2x+5}{x+3} - \frac{2x+7}{x+4}$.

$$17. \frac{5-x}{2} - 25x + \frac{15x+5}{20} = \frac{17}{17-51x^4}$$

$$18. \sqrt{x+b} + \sqrt{\frac{a}{x+b}} = \sqrt{x+b+c}.$$

$$19. 27 \times \frac{1-x}{1+x} = 8 \times \frac{1+2x+x^2}{1-2x+x^2}$$

$$20. \frac{x+a}{x+c} = \frac{x^2+ax+b}{x^2+cx+d}, \quad 21. \frac{2x+31}{x+13} + \frac{3x-32}{x-10} = \frac{5x-32}{x-7}.$$

$$22. \frac{(125)^x}{5^{x^2+b}} = (25)^{x+c}, \quad 23. \sqrt{(11x+3)} + \sqrt{(11x-5)} = 6 + 2\sqrt{7}.$$

$$24. 5\sqrt{(2x+1)} + 3\sqrt{(x+2)} = 3\sqrt{(x+3)} + 2\sqrt{(25x+8)}.$$

$$25. x-2 = \sqrt{7} \cdot \sqrt{(2x-11)}, \quad 26. \left(\frac{x-2}{x-10}\right)^3 = \frac{x+2}{x-10}.$$

$$27. 2^{x+1} 3^{x-1} = 16, \quad 28. \frac{x-3}{x-5} = \left(\frac{2x-9}{2x-11}\right)^2.$$

$$29. \frac{1 + \sqrt{(x^2-4a^2)}}{x - \sqrt{(x^2-4a^2)}} = 32 \left\{ \frac{\sqrt{(x+2a)} - \sqrt{(x-2a)}}{\sqrt{(x+2a)} + \sqrt{(x-2a)}} \right\}^8.$$

$$30. \frac{a+b}{x+1} + \frac{a-b}{x+3} = \frac{2a}{x+2}.$$

$$31. a \sqrt{\frac{1+x}{1-x}} + (a+2) \sqrt{\frac{1-x}{1+x}} = 2 \sqrt{a(a+2)}.$$

$$\left[i. e., \left\{ \sqrt{a} \sqrt{\frac{1+x}{1-x}} - \sqrt{(a+2)} \sqrt{\frac{1-x}{1+x}} \right\}^2 = 0 \right].$$

$$32. \frac{2}{x-2} - \frac{3}{x-7} - \frac{4}{x+8} + \frac{5}{x+3} = 0.$$

$$33. \frac{\sqrt{(x-a)}}{\sqrt{(x-b)} \sqrt{(x-c)}} + \frac{\sqrt{(x-b)}}{\sqrt{(x-c)} \sqrt{(x-a)}} + \frac{\sqrt{(x-c)}}{\sqrt{(x-b)} \sqrt{(x-a)}} = 0.$$

$$34. \sqrt{(1+a)} \sqrt{\left(\frac{1+x}{1-x}\right)} + \sqrt{(1-a)} \sqrt{\left(\frac{1-x}{1+x}\right)} = 2 \sqrt{(1-a^2)}.$$

$$35. \frac{\sqrt{(a+x)} + \sqrt{(a-x)}}{\sqrt{(a+x)} - \sqrt{(a-x)}} = \frac{1}{x}.$$

$$36. \frac{x}{\sqrt{(a+x)} - \sqrt{(a-x)}} = \frac{16a^3}{a + \sqrt{(a^3-x^3)}}.$$

$$37. \frac{x}{3} \sqrt[4]{(x+6)} = 10\frac{2}{3} - 2 \sqrt[4]{(x+6)}, \quad 38. \frac{1+x}{1-x} = \frac{14}{13} \frac{1+x+x^2}{1-x+x^2}.$$

$$39. \frac{ax+1}{ax+2} + \frac{ax+9}{ax+10} = \frac{ax+3}{ax+4} + \frac{ax+7}{ax+8}.$$

$$40. (ax+b)^{\frac{1}{3}} - (ax-b)^{\frac{1}{3}} = (a+b)^{\frac{1}{3}}.$$

$$41. \frac{2x-1}{\sqrt{(2x+3)+2}} + \frac{4x-5}{2\sqrt{(x+1)+3}} = \frac{4(x+3)}{\sqrt{(4x+13)+1}} + \frac{2(x-11)}{\sqrt{(2x-6)+4}}.$$

$$42. \sqrt{(5x+9)} + \sqrt{(5x-9)} = 4 + \sqrt{34}.$$

$$43. \sqrt{\frac{1}{a}} + \sqrt{\frac{\{(b-c)(ac-bx)+ac^2\}}{abc}} = 1.$$

$$44. \frac{1}{\sqrt{(a-\sqrt{x})} + \sqrt{a}} + \frac{1}{\sqrt{(a+\sqrt{x})} - \sqrt{a}} = 3\sqrt{\frac{a}{x}}.$$

$$45. \frac{1 + \sqrt{(x-4)}}{1 + 2\sqrt{(x-4)}} = \frac{\sqrt{(x-4)} - 1}{x-5}$$

$$46. [x-3 + \sqrt{\{(x-1)(x+5)\}}]^3 + [(x-3) - \sqrt{\{(x-1)(x+5)\}}]^3 \\ = 2(x-3)^3 + (x+2)(6x^2 - 6x + 1).$$

$$47. \frac{2(1+x) + \sqrt{(12x+3x^2)}}{2(1+x) - \sqrt{(12x+3x^2)}} = a, \frac{\sqrt{(x+4)} - \sqrt{(3x)}}{\sqrt{(x+4)} + \sqrt{(3x)}}.$$

$$48. \frac{l\sqrt{(a^2-x^2)} + m(a+x)}{m(a+x) - l\sqrt{(a^2-x^2)}} = \frac{li+md}{md-li}.$$

$$49. \frac{(x^2+3a^2)(3a^2+b^2)}{(3x^2+a^2)(a^2+3b^2)} = \frac{a^2}{bx}. \quad 50. \sqrt[p]{x^{p+q}} - \frac{p}{2a} \left(\sqrt[p]{x} + \sqrt[p]{x} \right) = 0.$$

CHAPTER XXVIII.

PROBLEMS LEADING TO SIMPLE EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

178. The student is referred back to Chapter XI. We begin by solving a few easy problems.

Ex. 1. Find a number such that if one-fourth of the next lower number be subtracted from one third of the next higher, the remainder will be less than one-sixth of the number by 5.

Let x = the number required.

Then $x+1$ = next higher number,

and $x-1$ = „ lower „ .

By the question, $\frac{1}{3}$ of the next higher number $-\frac{1}{4}$ of the next lower $= \frac{1}{6}$ of the number -5 .

$$\therefore \frac{x+1}{3} - \frac{x-1}{4} = \frac{x}{6} - 5.$$

Multiply both sides by 12 (=L. C. M. of 3, 4, 6.) ;

then, $4(x+1) - 3(x-1) = 2x - 60$;

$$\text{i.e., } x+7 = 2x-60 ;$$

by transposition, $x-2x = -60-7$;

$$\therefore -x = -67 ;$$

$$\therefore x = \underline{67} \text{ Ans.}$$

N. B. In order to test the correctness of the answer, it will be well to verify it. When the required number = 67, $\frac{1}{4}$ of the next higher number $-\frac{1}{3}$ of the next lower = $\frac{67}{4} - \frac{66}{3} = \frac{201}{12} - \frac{264}{12} = -\frac{63}{12} = -\frac{21}{4}$;

$$\frac{1}{6} \text{ of the number } - 5 = \frac{67}{6} - 5 = \frac{67}{6} - \frac{30}{6} = \frac{37}{6}.$$

$$\therefore \frac{1}{4} \text{ of the next higher number } - \frac{1}{3} \text{ of the next lower}$$

$$= \frac{1}{6} \text{ of the number } - 5.$$

Hence the answer is correct.

Ex. 2. A bankrupt paid only 17s. 6d. in the pound to his creditors, and then gave $\frac{1}{8}$ of what he still owed to the lawyers. This left him £20 for his current expenses. What was the amount of his debt ? C. U. 1886.

Let x = the amount of his debt in pounds.

In £1, he paid 17½s. or $\frac{35}{2}$ s. ;

$$\therefore \text{ in } £x, \text{ " } \frac{35x}{2} \text{ s., or } £\frac{35x}{40}, \text{ i.e., } £\frac{7x}{8} ;$$

$$\therefore \text{ he still owed } £\left(x - \frac{7x}{8}\right) \text{ or } £\frac{x}{8} ;$$

and of this amount he gave away $\frac{1}{8}$ ths.

$$\therefore \frac{1}{8} \text{ of } £\frac{x}{8} \text{ or } £\frac{x}{64} \text{ remained with him.}$$

By the question, he had £20 left ;

$$\therefore \frac{x}{64} = 20 ;$$

$$\therefore x = 800 ;$$

$$\therefore \text{ the amount of the debt } = \underline{£800} \text{ Ans. [Verify]}$$

Ex. 3. A farmer bought equal numbers of two kinds of sheep, one at £3 each, the other at £4 each. Had he expended his money equally in the two kinds, he would have had two more sheep than he had. How many did he buy ? A. U. 1891.

Let x = the number of each kind of sheep bought.

Then $2x$ = the total number of sheep bought

The sum spent on one kind of sheep = £31,

„ „ „ the other kind = £42.

the total sum spent = £(31 + 42) = £73.

If he spent his money equally in the two kinds,

i.e., £ $\frac{73}{2}$ on one kind and £ $\frac{73}{2}$ on the other,

then the number he would have had of one kind = £ $\frac{73}{2} \div £3 = \frac{73}{6}$,

and that of the other kind = £ $\frac{73}{2} \div £4 = \frac{73}{8}$;

then would the total number = $\frac{73}{6} + \frac{73}{8}$.

By the question, this total = a total number (= $2x$) + 2;

$$\frac{73}{6} + \frac{73}{8} = 2x + 2;$$

\therefore by transposition, $\left(\frac{73}{6} - x\right) + \left(\frac{73}{8} - x\right) = 2$;

$$\frac{1}{6} - \frac{1}{8} = 2,$$

$$\therefore \frac{4x - 3x}{24}, \text{ or } \frac{x}{24} = 2,$$

$$\therefore x = 48.$$

\therefore the total number of sheep bought = $2x = 96$. *Ans.*

Ex 4 Two boys own together 13 mangoes and 17 apples; wishing to divide, one takes 11 mangoes and 7 apples, and pays the other 3 as $4\frac{1}{2}$ p. If mangoes be an anna dearer per dozen than apples, find the total value of the fruits

Let x denote the price of an apple in annas.

Then, by the question, the price of a dozen mangoes in annas = that of a dozen apples + 1 = $12x + 1$;

\therefore the price of a mango = $\frac{12x + 1}{12} = x + \frac{1}{12}$ in annas;

\therefore the total value of the fruits in annas = $13\left(x + \frac{1}{12}\right) + 17x$
 $= 30x + \frac{13}{4}$; (A)

\therefore the value of the half share in annas = $15x + \frac{13}{8}$.

The value of 11 mangoes and 7 apples

$$\text{in annas} = 11(x + \frac{1}{2}) + 7x = 18x + \frac{11}{2};$$

\therefore the boy taking 11 mangoes and 7 apples takes in money value $\{18x + \frac{11}{2} - (15x + \frac{1}{2})\}$ annas more than his proper share. This sum he ought to pay to the other.

\therefore by the question, $\{18x + \frac{11}{2} - (15x + \frac{1}{2})\}$ annas = 3 as. 4½p = $3\frac{3}{8}$ as.; simplifying, we get $3x + \frac{3}{8} = 3\frac{3}{8}$; whence $x = 1$;

\therefore by (A), the total value sought = $(30 + \frac{1}{2})$ as. = Re. 1 15as 1p. *Ans.*

N.B. The student should notice that instead of denoting the total value sought by v , we have denoted the price of an apple by x , upon which the total value evidently depends. It is often a matter of great convenience to so denote the simplest quantity in a problem, and to express the others in terms of it.

Ex. 5. I spend a pence and b pence upon eggs, when they are dearer per dozen by a pence and b pence respectively, and find that on the average they are only c pence dearer per dozen. Find the ordinary bargain price per dozen.

Let x denote the price sought (in pence);

then the buying prices are $(x + a)$ and $(x + b)$ pence per dozen,

the fraction of a dozen bought for a pence = $\frac{a}{x+a}$.

and " " " " " " " " $\frac{b}{x+b}$;

\therefore the total no. bought = $\left(\frac{a}{x+a} + \frac{b}{x+b}\right)$ of a dozen;

the same total also = $\frac{\text{total sum spent}}{\text{average price per dozen}} = \frac{a+b}{x+c}$.

$$\therefore \frac{a}{x+a} + \frac{b}{x+b} = \frac{a+b}{x+c} = \frac{a}{x+c} + \frac{b}{x+c};$$

transposing, $\frac{a}{x+a} - \frac{a}{x+c} = \frac{b}{x+c} - \frac{b}{x+b}$;

simplifying, $\frac{a(c-a)}{(x+a)(x+c)} = \frac{b(b-c)}{(x+c)(x+b)}$;

multiplying both sides by $(x+a)(x+b)(x+c)$,

$$a(c-a)(x+b) = b(b-c)(x+a);$$

multiplying out and transposing, $\{c(a+b) - a^2 - b^2\}x = ab(a+b-2c)$;

$$\therefore x = \frac{ab(a+b-2c)}{c(a+b) - a^2 - b^2}.$$

\therefore the price sought = $\frac{ab(a+b-2c)}{c(a+b) - a^2 - b^2}$ pence. *Ans.*

EXAMPLES 95.

1. Find a number such that a third and a fourth part of it together exceed one-fifth of the same by 46.
2. 2 is added to $\frac{2}{3}$ of a certain number, and 3 is subtracted from $\frac{2}{3}$ of the same number; then the sum is divided by the remainder, and the result is $\frac{1}{9}$. Find the number.
3. Find the number which increased by 5 is equal to double the next lower number.
4. A fraction is equal to $\frac{5}{7}$, and the denominator is greater than the numerator by 16; find it.
5. The numerator of a certain fraction is two-sevenths of the denominator, and if both the numerator and denominator be increased by 4, the new fraction is equal to $\frac{1}{2}$; find the fraction.
6. 20 is divided into two parts, so that the quotient on dividing one part by the other is equal to what it would be if the parts were increased by 4 and 6 respectively; find the parts.
7. In an examination the number of passes was at first equal to $\frac{2}{3}$ rds. of the number of failures; on a re-examination of papers, 4 more passed, and the number of passes was then found to be $\frac{2}{3}$ ths of the number of failures. How many appeared?
8. Divide 560 rupees between *A*, *B* and *C*, so that *A*'s share will be equal to twice *B*'s and *B*'s share will be equal to thrice *C*'s.
9. Divide £52. 13s between *A*, *B* and *C*, so that twice *A*'s share = thrice *B*'s = four times *C*'s.
10. Divide £318. 5s. amongst *A*, *B*, *C* and *D*, so that *A*'s share will exceed twice *B*'s by £2, thrice *C*'s by £3, and four times *D*'s by £4.
11. *A*, *B*, *C* and *D* have amongst them 726 apples; *B* has as much again as *A* and 3 more, *C* twice as much again as *B* and 3 more, and *D* thrice as much again as *C* and 3 more; find the number that each has.
12. Rs. 49 were distributed amongst a gang of men and women, numbering 150, so that each woman received 4 as., and each man 8 as. How many men and how many women were there?
13. A man bequeathed $\frac{5}{8}$ of his property to one son, $\frac{1}{3}$ of the remainder to another, and the surplus to his widow. The difference of his sons' legacies was £754. How much did the widow receive?
14. A person finds that if he invest a certain sum in railway shares paying £6 per share when the £100 share is at £132, he will obtain £10 16s. a year more for his money than if he invest it in 3 per cent. consols at 93. What sum has he to invest?

15. A man invests Rs. 163000, part in Government 4 per cent. stock at 108 and the remainder in municipal 5 per cent. debenture stock at 109½. Find how much he must invest in each in order that he may have an equal income from the two sources.

16. A man's age is three times that of his son; 10 years hence the father's age will be double that of his son. Find the present age of each.

17. How much tea at 3s. 6d. per lb. should be mixed with 10 lbs. of tea at 2s. 9d. per lb., that the mixture may be worth 3s. 3d. per lb.?

18. *A* and *B* gamble, stipulating that the loser shall pay the winner half the money the former has and 1s. more; they begin with equal sums of money. *A* loses first, but wins afterwards, and now finds that he is richer by 5s. 6d. than when he began gambling. Find the sum with each at first.

19. At a cricket match a contractor provided luncheon for 24 and fixed the price to gain one-eighth of his outlay. Three persons were absent. The remaining 21 paid the fixed price and the contractor lost 2 rupees. What was the charge?

20. A man wants to buy a certain number of mangoes for a certain sum; if he pays 2 as for each, he will have 1 as. left; but if he pays 3 as. for each, he will want 7 as. What sum has he?

21. Two kinds of eggs are selling, one at 5 per anna and the other at 6 per anna. A person wanting to buy a certain number of eggs finds that if he buys up the required number out of the first kind, the money in his pocket will fall short by 2 as., but that if he selects the other kind, he will have 1 a. left. Find the number of eggs he has to buy, and the money in his pocket.

22. In course of 24 hours a clock loses for some hours 6 s. an hour, and for the rest gains 39 s. an hour, but on the whole it neither gains nor loses; how long in the 24 hours does it go fast?

23. *A* and *B* rent a field for Rs. 355 a year. *A* puts in 6 horses for 12 months; *B* puts in 5 horses for 11 months, and three more for 5 months. How much should each contribute towards the rent?

[Let x = charge in Rs. per horse per month. Then charge on 6 horses for 12 months = that on 6×12 horses per month = $6 \times 12x$ in Rs.]

24. A bag contains a certain number of rupees, half as many again two-anna pieces, and four times as many pysas, and the value of the whole is Rs. 300; find how many rupees, how many two-anna pieces, and how many pysas are there?

25. A bag contains 160 coins consisting of half-crowns, shillings, sixpences, and fourpences, and the values of the sums of money represented by each kind of coin are the same ; how many of each are there ?

26. The denominator of a certain fraction exceeds the numerator by 5, and if the numerator be increased by 7, the fraction is increased by unity ; find the fraction.

27. The denominator of a certain fraction exceeds the numerator by 5 ; if 16 be added to each, the fraction is increased by two-thirds the excess of unity over itself Find the fraction.

28. The denominator of a fraction is less than the numerator by a , and if b be added to each, the fraction diminishes by one-fourth of its excess over unity ; find the fraction.

29. A poultry-keeper obtained 150 more eggs in July than in June, and the daily average in July was 4 more than in June ; how many did he get in the two months ?

30. A person bought 34 lbs. of tea of two different sorts, and paid for the whole £3. 6s. ; the better sort cost 2s. 3d. per lb., and the worse 1s. 8d. per lb. ; how many lbs. were there of each sort ?

31. How much are plums a gross when one-fifth the number more for a sovereign lowers the price $2\frac{1}{2}$ d. a score ?

32. The number of months in the age of a person on his birth-day in 1938 will be just two thirds of the number denoting the year in which he was born ; in what year was he born ?

33. In the rains the depth of a river at mid-stream is twice as great as near a bank, but in summer the depth of the river having fallen by 10 ft., the depth at mid-stream is thrice that near the bank. What were the depths in the rainy season ?

34. A having three times as much money as B gave Rs. 10 to B , and then he had twice as much as B had. How much had each at first ?

35. Two brothers own together 200 shares in a railway company. They agree to divide, and one of them takes 90 shares, while the other takes 110 shares and pays £1000 to the first. Find the value of a share.

36. Two farmers own equally 130 cows and 70 sheep ; wishing to separate their business, one farmer takes 70 cows and 40 sheep, and leaves the rest to the other, paying him Rs. 30 ; if a cow be 19 times as valuable as a sheep, find the total value of the flock.

37. A gentleman bequeaths part of £14100 to a charity, and twelve times as much to his eldest son, whose share is half as much again as that of each of his two brothers, and double that of each of his three sisters ; find the sum left to each sister.

38. A farmer buys a flock of sheep at the rate of £7 for every 5 sheep; he afterward loses 9, and sells the remainder at the rate of £16 for every 11, and the sum for which he sells the flock is £24 more than what he gave for it: how many sheep did he buy?

39. A person takes a fancy to dispose of his geese at as many shillings each as the number he has, and paying 1s. to his coolie finds that if he had had one more to sell and had paid 2s. to the coolie, he would have received £1 more by the transaction; what number did he dispose of?

40. A sold a certain number of watches at a guinea each, and gave one-third of the proceeds to B, one-fourth of the remainder to C, and one-fifth of the last remainder to D, after which he had £2.0 remaining: how many did he sell?

41. A reservoir is to be emptied, the rate of discharge of the contents being diminished by 100 gallons every hour. The first half will be emptied in 3 hours, the second in 4 hours. How many gallons does the reservoir contain?

42. In a cricket match the extras in the first innings are one-thirteenth of the score, and in the second innings the extras are one-fifteenth of the score; find the score in each innings, if the grand total be 338, in which there are 22 extras.

43. When the income-tax was 7d. in the £., a person had to pay £63 less than when it was doubled, his income having in the interim diminished by £2250. What was his first income?

44. A railway passenger is allowed to take 30 seers free of charge, and is charged for the excess of luggage at the rate of Rs. 2 per maund; had his luggage been twice as heavy, he would have been charged Rs. 7 more than he was. How much luggage had he?

45. A gentleman travels first class in a railway, while his servant travels third class. A first class passenger is allowed 1 cwt. of luggage and a third class passenger 1 qr. of luggage free of charge, and the excess is charged at a uniform rate of 1s. per cwt. If the servant's amount for luggage exceeds the gentleman's by twice as much as when the luggages are interchanged, and if they have 13 cwts. between them, what are their separate charges on account of luggage?

46. A certain number of persons paid a bill; had there been 12 more, each would have paid as much less than he did as he would have paid more had there been 10 fewer. How many persons were there?

47. A sum of money is to be divided among a number of persons: if 8as. be given to each, there are 12as. short, and if 7as.

4*0*. be given to each, there are Re. 1. 4*as.* over : find the amount to be divided.

48. The price of pictures is raised to 2*s.* 6*d* per dozen, and customers consequently receive 24 less than before for £1 ; what was the original price per dozen ?

49. A man buys a certain number of sheep for £60 : if he had bought one-third more for the same money, each would have cost 10*s.* less. How many sheep did he buy ?

50. A woman finding her apples partly damaged sells one more apple for a rupee than she bought for the same sum, and finds that her loss per apple is one-third its selling price ; how much per dozen does she now get ?

51. A person after paying a poor rate and also an income-tax of 5*d.* in the pound, has £4650 in hand ; the poor-rate amounts to £50 less than the income-tax. Find the original income and the number of pence in the pound in the poor-rate.

52. A merchant increases his property every year by a fourth part, but allows £400 for his annual expenditure, and at the end of two years is £2250 richer than at first ; what property does he begin with ?

53. A farmer transplants a number of cane-cuttings, arranging them in rows so as to put in a row as many cuttings and one more than there are rows ; he could as well have five fewer rows, but seven more cuttings in a row. How many cuttings are there ?

54. A shop keeper sells away his stock of marbles in the following manner : to one person he sells half the number and one more, to a second the number which is just the greater half of the remainder, to a third half the remainder and one more, and to a fourth only three. How many marbles had he ?

55. A purse is divided amongst 4 boys, the first receiving half of it and 1*s.* more, the second half of the remainder and 1*s.* more, the third a similar share, and the fourth 3*s.* 3*d.* ; find the whole value of the purse.

56. A farmer wants to enclose a piece of land with a certain number of hurdles ; if he place them one foot distant from each other, he has not enough hurdles by 28 ; but if he place them a yard apart he has 12 hurdles to spare. How many hurdles has he ?

57. The first edition of a book had 600 pages and was divided into two parts. In the second edition one quarter of the second part was omitted, and 30 pages were added to the first part ; this change made the two parts of the same length. Find the number of pages in each part in the first edition.

58. I bought a certain number of pictures for Re. 1, and a few more for Rs. 3 at 4 *as.* more apiece ; if each picture had been

only 2 as. dearer than in the first case, the same total number of pictures might have been bought for the total sum spent. Find the lowest rate paid per picture, and the total number bought.

59. A certain number of sheep was bought for a certain sum, and thrice as many oxen and two more and seven times as many horses and two more for thrice and fourteen times that sum respectively; if one sheep and one ox could be exchanged for a single horse, find the total number of animals bought.

60. A boy was sent out four times, each time with the same number of pice to buy grapes; he always brought in c fewer grapes than the number of pice he paid for them, and returned first time with a pice, second time with b pice, and third and fourth times with c pice more than in the first and second respectively; it was now found that the sum of the rates per grape in the first and last purchases was the same as that in the others. How many pice had the boy with him each time he went out?

179. **Provisioning.** If Q be the quantity of food consumed by a man in a month, then the quantity that will last a men for b months $= abQ$.

For, Q = quantity eaten in 1 month by 1 man;

$\therefore aQ =$ " " " " " " a men;

$\therefore abQ =$ " " " " b months, a men;

In other words, *the total quantity of food consumed*

$= \text{number of men} \times \text{time} \times \text{rate of allowance per man.}$

Ex. 1. A garrison had sufficient provisions for 30 months, but at the end of 4 months the number of troops was doubled, and 3 months after, it was re-inforced with 400 men more, on which accounts the provisions lasted only 15 months altogether. Required the number of men in the garrison before the augmentation took place. B U. 1871—72.

Let x = the number required.

Let Q = the quantity of food allowed to a man per month.

Then the total quantity of provisions

$=$ that reqd. by x men for 30 months

$=$ " " by $30x$ men per month

$= 30xQ$. [$=$ no. of months \times no. of men \times monthly allowance per man]

This total quantity was eaten by x men for 4 months, by $2x$ men for 3 months, and by $(2x + 400)$ men for 8 months.

\therefore the quantity consumed in the first 4 months $= 4xQ$,

„ „ in the next 3 „ $= 3.2x.Q$

„ „ in the last 8 „ $= 8.(2x + 400).Q$

\therefore the total quantity $= 4xQ + 6xQ + 8(2x + 400)Q = 26xQ + 3200Q$.

But this total quantity has been found to be $30xQ$.

$\therefore 30xQ = 26xQ + 3200Q$; on division by Q , and then by transposition, $30x - 26x = 3200$.

$\therefore 4x = 3200$, whence $x = 800$. *Ans.*

EXAMPLES 96.

1. A ship with 1200 men on board had sufficient provisions to last 17 weeks. The survivors of a wreck having been taken aboard, the provisions were consumed in 15 weeks. How many men were taken aboard?

2. A party of merchants had provisions sufficient to last them a week, in which time they expected to cross a desert; but being detained longer on the way by accidents, they had to reduce their usual rations by a quarter from after the fourth day. How long did the journey take?

3. A shooting party of 12 men has provisions to last 8 days; 2 men leaving immediately, by what part should their daily allowance be diminished in order that the rest may continue 2 days longer?

4. A garrison has provisions for 12 months, but a re-inforcement of 100 men coming in after 3 months, the provisions last altogether for 9 months; find the number of men in the garrison before the re-inforcement.

5. A vessel leaves port on a voyage of 30 days with provisions just sufficient for it. After 12 days have elapsed, it picks up 10 men who have been ship-wrecked, and accordingly the daily allowance of food per man is reduced by one-ninth for the remainder of the voyage. Find the number of men on board at starting.

6. The inmates of a poor-house, consisting of men and women, number 150, and they can be fed for 20 days with the funds in hand, 4 women eating as much as 3 men; but if 55 men be replaced by as many women, and if 5 women eat as much as 3 men, they can be fed for 5 days more. Find the number of each sex.

7. A besieged garrison consists of 300 men, 120 women and 40 children, and has provisions enough for 200 men for 30 days; if a woman eats $\frac{2}{3}$ as much as a man, and a child $\frac{1}{2}$ as much, and if after 6 days 100 men with all the women and children escape, for how long will the remaining provisions last the garrison?

8. A ship left Bombay on a voyage of 3 weeks, with provisions for that time at the rate of 1 seer a day for each man. At the end of a week a storm arose which washed 4 men overboard and so damaged the vessel that the speed was reduced by half, and each man could be allowed only $\frac{3}{4}$ of a seer per diem. What was the original number of the crew?

180. **Work and Cistern.** If w be the amount of work done by a person in a days, then the average daily work of the man $= \frac{w}{a}$. Similarly if Q denote the quantity of water that a given cistern holds, and if it be emptied uniformly by a tap in a hours, the quantity let out per hour $= \frac{Q}{a}$.

Conversely, if $\frac{w}{a}$ or $\frac{Q}{a}$ be the rates as above, the time required for completion $= a$ days or hours.

If w denote the amount of work done by a person in a days working b hours a day, then the *hourly rate* of work $= \frac{w}{ab}$.

Ex 1 A can do a piece of work in 4 hours, and B in 6 hours; how long will it take A and B to do the work together?

Let x = the number of hours required,

and w = the whole work.

By the question,

$$\left. \begin{aligned} 4 \text{ hours' work of } A &= w, & \therefore A's \text{ work per hour} &= \frac{w}{4}; \\ \text{and } 6 \text{ hours' work of } B &= w, & \therefore B's \text{ " " " } &= \frac{w}{6}; \end{aligned} \right\}$$

$$\therefore \text{ the joint work of } A \text{ and } B \text{ per hour} = \frac{w}{4} + \frac{w}{6} = \frac{5w}{12}.$$

Now, by supposition, x hours' joint work of A and $B = w$;

$$\therefore \frac{5w}{12} x = w;$$

$$\therefore x = \frac{12}{5} = 2\frac{4}{5}. \text{ Ans.}$$

Ex. 2. A and B can together do a piece of work in 30 days. B and C in 40 days, and C and A in 50 days. How long will A , B and C work together to do it?

Let x = the number of days required,

and w = the whole work.

$$\left. \begin{aligned}
 \text{By the question, the joint work of } A \text{ and } B \text{ per day} &= \frac{w}{30}, (1) \\
 \text{„ „ „ „ } B \text{ and } C \text{ „ „} &= \frac{w}{40}, (2) \\
 \text{and „ „ „ „ } C \text{ and } A \text{ „ „} &= \frac{w}{50}. (3)
 \end{aligned} \right\}$$

$$\text{By supposition, the joint work of } A, B \text{ and } C \text{ per day} = \frac{w}{x}. (4)$$

Adding up (1), (2) and (3), we have

$$\begin{aligned}
 \frac{w}{30} + \frac{w}{40} + \frac{w}{50} &= \text{twice the joint work of } A, B, C \text{ per day} \\
 &= \frac{2w}{x}, \text{ from (4)};
 \end{aligned}$$

$$\therefore \text{ dividing out } w, \frac{1}{30} + \frac{1}{40} + \frac{1}{50} = \frac{2}{x}$$

$$\text{Multiply both sides by } 600x; \text{ then } 20x + 15x + 12x = 1200;$$

$$\text{whence } x = \frac{1200}{47} = 25\frac{25}{47}.$$

$$\therefore \text{ the time required} = 25\frac{25}{47} \text{ hrs. } \text{Ans.}$$

Ex. 3. A alone can do a piece of work in a hours, A and C together can do it in b hours, and C 's work $= \frac{1}{n}$ of B 's. The work has to be completed in c hours. Find how long after A has commenced, B and C should *relieve* him, so as to finish the work in time? M. U. 1867.

Let x = the number of hours required,
and w = the whole work

$$A's \text{ work per hour} = \frac{w}{a},$$

$$\text{and the joint work of } A \text{ and } C \text{ per hour} = \frac{w}{b}.$$

$$\therefore C's \text{ work per hour} = \frac{w}{b} - \frac{w}{a}.$$

$$\text{By the question, } C's \text{ work} = \frac{1}{n} \text{th of } B's;$$

$$\begin{aligned}
 \therefore B's \text{ work per hour} &= n \text{ times } C's \text{ work per hr.,} \\
 &= n \left(\frac{w}{b} - \frac{w}{a} \right);
 \end{aligned}$$

$$\therefore \text{the joint work of } B \text{ and } C \text{ per hour} = n\left(\frac{w}{b} - \frac{w}{a}\right) + \frac{w}{b} - \frac{w}{a}$$

$$= (n+1)\left(\frac{w}{b} - \frac{w}{a}\right),$$

\therefore A works for x hours, B and C work for $(c-x)$ hours, and x hours' work of A + $(c-x)$ hours' work of B and C = the whole work = w .

$$\therefore x \cdot \frac{w}{a} + (c-x)(n+1)\left(\frac{w}{b} - \frac{w}{a}\right) = w;$$

multiply both sides by $\frac{ab}{w}$; then $bx + (n+1)(c-x)(a-b) = ab$;

$$\therefore bx + (n+1)c(a-b) - x(n+1)(a-b) = ab,$$

$$\therefore x\{b - (n+1)(a-b)\} = ab - (n+1)(a-b)c$$

$$\therefore x = \frac{ab - (n+1)(a-b)c}{(n+2)b - (n+1)a}. \quad \text{Ans.}$$

Ex. 4. A cistern can be filled by two pipes in 3 and 4 hours respectively, and can be emptied by a third in 6 hours; how long will it be in filling, when all three are open?

Let x = the number of hours required.

Let Q = the quantity of water the cistern can hold.

The quantity poured in per hour by one pipe = $\frac{Q}{3}$,

" " " " " by another = $\frac{Q}{4}$,

" " " out " " by the third = $\frac{Q}{6}$;

$$\therefore \text{the total quantity retained per hour} = \frac{Q}{3} + \frac{Q}{4} - \frac{Q}{6} = \frac{5Q}{12}.$$

By supposition, the cistern is filled in x hours.

$$\therefore \frac{5Q}{12}x = Q;$$

$$\therefore x = \frac{12}{5} = 2\frac{2}{5}.$$

\therefore the time required = 2 hrs. 24 min. Ans.

EXAMPLES 97.

1. A can do in 20 days a piece of work which B can do in 30 days; how long would they take to do it working together?

2. *A* and *B* together can reap a field in 4 days, while *B* alone can reap it in 6 days ; in what time can *A* alone reap it ?

3. *A* and *B* together can do a piece of work in 8 days, and *B* alone can do it in 12 days ; supposing *B* works at it for 3 days, how long would *A* now take to finish it alone ?

4. *A* does half as much work again as *B* in the same time, and *B* does one-third as much again as *C* ; working together they can do a certain work in 15 days ; if after working at it alone for 6 days *A* leaves off, how long will *B* and *C* take to finish it ?

5. After a certain number of men had been employed on a certain work for 16 days and had half finished it, 24 more were employed, and the remaining work was completed in 10 days ; how many men were employed at first ?

6. Three men can do as much work as five boys : the wages of 3 boys are equal to those of 2 men. A work on which 10 boys, and 15 men are employed takes 7 weeks and costs £350, how long would it take if 20 boys and 20 men were employed, and how much would it cost ?

7. One man and two boys together do as much work in a time as 3 women ; it takes a woman one day more and a boy three days more than a man to reap a field separately ; how long would it take a man and a boy together to reap it ?

8. A boy and a woman together finish a piece of work in the same time as a man alone ; if a man works $\frac{2}{3}$ ths of a day longer than the time the woman alone takes to do it, he will do a piece of work twice as great, which latter work will take a boy a day more to do than a woman. How long will it take a boy alone to finish it ?

9. A cistern is filled by two pipes in 20 and 30 minutes, respectively ; how long will it be in filling, when both are open ?

10. In course of digging a well a spring is struck and begins feeding the well at a rate which will fill it in 5 hours ; after some time a pump is got to work at a rate sufficient to dry up the full well without the spring in 3 hours. If 7 hours elapse from the time the spring is opened before the well is pumped dry, when did the pump begin work ?

11. Two taps can separately fill a cistern in 5 and 6 hours, and when the waste-pipe is open, the three together take $7\frac{1}{2}$ hours to fill it ; in what time can the waste-pipe empty it ?

12. A bucket is twenty times filled with water and each time emptied into a cistern ; three pipes are then turned on, one of which can fill the empty cistern in 15 minutes, and another in 10 minutes, while the third can empty the cistern, when full, in 12 minutes. If the cistern be now filled in 2 minutes, how many bucketfuls can it hold ?

13. Two pipes, *A* and *B*, fill a cistern in 25 and 30 minutes respectively. Both pipes being opened, find when the first must be closed that the cistern may be just filled in 15 minutes.

14. A bath is supplied with water from two pipes one of which can fill it in 12½ minutes, and the other in 15 min; there is also a discharging pipe, which would empty it, when filled, in 10 minutes. The first pipe is open alone for 4 minutes, and then the first and second are open together for 1 minute; if now the third pipe be opened as well, how long will it take to fill the bath?

181. **Percentage.** Let a quantity, *a*, increase *b* per cent.

Then every 100 increases to $100 + b$;

∴ unity " " $1 + \frac{b}{100}$;

∴ *a* units increase to $a\left(1 + \frac{b}{100}\right)$.

In the case of decrease, the result is evidently $a\left(1 - \frac{b}{100}\right)$.

Thus, if the selling price of an article be a certain per cent. *higher* than its cost, then

$$\text{the selling price} = \left(1 + \frac{\text{rate per cent.}}{100}\right) \times \text{cost.}$$

When the selling price is *lower* than the cost, we have

$$\text{the selling price} = \left(1 - \frac{\text{rate per cent.}}{100}\right) \times \text{cost.}$$

Ex. 1. Of the candidates in a certain examination 45 per cent. passed. If there had been 30 more candidates of whom 19 failed, the number of successful candidates would have been 44·8 per cent. How many candidates were there? C. U. 1890.

Let *x* = the required number of candidates;

then the number that passed = $\frac{45}{100}x = \frac{9}{20}x$.

19 of the additional 30 failing, the increased number of successful candidates would have been $\frac{44\cdot8}{100}x + (30 - 19)$, and this, by the question, would be 44·8 per cent. of the new total, *viz.*, *x* + 30.

$$\therefore \frac{9x}{20} + (30 - 19) = \frac{44\cdot8}{100}(x + 30).$$

Multiply each side by 100; $45x + 1100 = 44\cdot8(x + 30) = 44\cdot8x + 1344$.

$$\therefore 2x = 244. \quad \therefore x = 122. \quad \text{Ans.}$$

Ex 2. A man sells an article at 12 per cent. profit ; if he had bought it at 4 per cent. less, and sold it for Rs. 4 more, he would have gained $22\frac{2}{5}$ per cent. ; what was the cost price ?

Let x = the cost price in rupees.

Then the actual selling price at a profit of 12 per cent. in rupees

$$= x(1 + \frac{12}{100}) = (1 + \frac{3}{25})x = \frac{28x}{25}.$$

If the article had been bought at 4 per cent. less, the cost price in rupees would have been $(1 - \frac{4}{100})$ or $x(1 - \frac{1}{25})$.

The selling price in Rs. at $22\frac{2}{5}$ per cent. profit on this supposed cost

$$= x(1 - \frac{4}{100})(1 + \frac{22\frac{2}{5}}{100}) = x \cdot \frac{24}{25} \cdot \frac{11}{10} = x \frac{88}{125}.$$

But, by the question, this selling price = the actual selling price + Rs. 4.

$$\therefore \frac{88x}{125} = \frac{28x}{25} + 4 ;$$

multiply each side by 125 ; $88x = 84x + 300 ;$

$$\therefore 4x = 300. \text{ whence } x = 75 ;$$

$$\therefore \text{ the required cost } = \text{Rs. } 75. \text{ Ans.}$$

Ex. 3. A tradesman sells two articles together for Rs. 46, making 10 per cent profit on one and 20 per cent. on the other. If he had sold each article at 15 per cent profit, the result would have been the same. At what price does he sell each article ?

Let x = the selling price in Rs. of the article sold at 10 per cent. profit.

Then $46 - x$ = the selling price in Rs. of the article sold at 20 per cent. profit

$$\text{The cost in Rs. of the first article} = \frac{x}{1 + \frac{10}{100}} = \frac{10x}{11}$$

$$\text{“ “ “ “ “ “ second “} = \frac{46 - x}{1 + \frac{20}{100}} = \frac{46 - x}{1 + \frac{1}{5}} = \frac{5}{6}(46 - x)$$

$$\therefore \text{ the total cost of the two in Rs.} = \frac{10x}{11} + \frac{5}{6}(46 - x)$$

$$= \frac{10x}{11} - \frac{5x}{6} + \frac{5 \times 46}{6}$$

$$= \frac{5x}{66} + \frac{115}{3}.$$

If both were sold at 15 per cent. profit, then would the selling price in rupees = the total cost $\times (1 + \frac{15}{100})$

$$= \left(\frac{5x}{66} + \frac{115}{3} \right) \times \left(1 + \frac{15}{100} \right) = \left(\frac{5x}{66} + \frac{115}{3} \right) \frac{23}{20}.$$

By the question, this supposed selling price would have been still the same, *vis.*, Rs. 46.

$$\therefore \left(\frac{5x}{66} + \frac{115}{3} \right) \times \frac{23}{20} = 46;$$

$$\text{multiply each side by } \frac{20}{23}; \quad \frac{5x}{66} + \frac{115}{3} = \frac{46 \times 20}{23} = 40;$$

$$\therefore \frac{5x}{66} = 40 - \frac{115}{3} = \frac{5}{3};$$

$$\therefore x = 22, \text{ and } 46 - x = 24;$$

\therefore the man sells the articles for Rs. 22 and Rs. 24 respectively.

Ans.

EXAMPLES 98.

1. Divide 620 into two parts such that 15 per cent. of one part will exceed 12 per cent. of the other by 12.

2. A number is divided into two parts, so that their difference is 25; 36 per cent. of the higher part added to 35 per cent. of the lower yields 40 per cent. of the remainder after subtracting 25 from the whole number. Find the number.

3. A man sold a ship at a loss of 8 per cent; if he had received Rs. 16800 more for it, he would have gained 6 per cent.; what did the ship cost him?

4. I buy two pictures for Rs. 175, and sell one so as to lose 4 per cent. and the other so as to gain 3 per cent.; and on the whole I neither gain nor lose; what did each picture cost me, and what was each picture sold for?

5. I bought a horse and a carriage for £150; I sold the horse at a gain of 12 per cent. and the carriage at a loss of 4 per cent., and gained on the whole 6 per cent. What was the selling price of each? [First find the prime cost of each.]

6. How many bundles of hay at Rs. 5 per thousand must a ghaswalla mix with 5600 bundles at rupees 6 per thousand in order that he may gain 20 per cent. by selling the whole at 11 annas per hundred?

7. How much water must be added to 6 maunds of milk costing 2 annas a seer, so that by selling the mixture at $1\frac{3}{4}$ annas a seer the milkman may secure a profit $16\frac{2}{3}$ per cent. on his outlay?

8. A merchant buys 1260 cwts. of corn, $\frac{1}{3}$ of which he sells at a gain of 5 per cent., $\frac{1}{3}$ at a gain of 8 per cent., and the remainder at a gain of 12 per cent. If he had sold the whole at a gain of 10 per cent., he would have obtained £23. 2s. more. What was the cost price per cwt. ?

9. Two sorts of mangoes are selling, one of which is 20 per cent. dearer than the other, and I find that for the money in my pocket I can have two more of the cheaper sort than of the other ; how many of each sort can I buy ?

10. By how much per cent. must the price of horses rise in order that only 25 horses can be purchased for a certain sum for which 4 more could be had at the original price ?

11. A boy buys a certain number of oranges at 3 for 2d., and one-third of that number at 2 for 1d. ; at what price must he sell them to get 20 per cent. profit ? If his profit be 5s. 4d., find the number bought.

12. A person buys 5 shares in a company, and sells 3 of them at a gain of 10 per cent. and the remaining 2 at a gain of $16\frac{2}{3}$ per cent. The gain on the latter sale is £2. 19s. $7\frac{1}{2}$ d. more than on the former. Find the price of a single share.

13. How much per cent. must be added to the cost price of goods that a profit of 20 per cent. may be made after throwing off a discount of 10 per cent. from the labelled price ?

14. An article of commerce passes successively through the hands of three dealers, each of whom in selling adds as his profit 10 per cent. of the price at which he bought it. If under these circumstances goods are sold by the third dealer for Rs. 665. 8as., what did the first dealer pay for them ?

15. A whole-sale dealer sells an article at a gain of 20 per cent. to a retail dealer, who selling it for 12 rupees gains 20 per cent. ; what did the whole-sale dealer pay for it ?

16. One-third of a population can read : of the remainder 45 per cent. can read and write ; of what still remains, 9 per cent. can read, write and count : the balance is 500500, who can neither read, write, nor count. Find the total population.

17. A man is 20 per cent. a better hand than a woman, but his wages are 25 per cent. higher ; how much per cent. more money do I spend by having a work done by a man than by a woman ? By how much am I a gainer in respect of time ?

18. A party of coolies consists of 5 per cent. more women than men, and 20 per cent. more boys than women ; a woman does 4 per cent. more work than a boy, but $16\frac{2}{3}$ per cent. less work than a man in the same time ; if a certain work be finished by the party in 13 days, how long would it take the men alone to finish it ?

182. Motion. If a man walks a miles in b hours, his hourly rate of motion $= a/b$ miles. In brief,

$$\text{speed} = \frac{\text{distance gone over}}{\text{time taken}},$$

$$\text{distance} = \text{speed} \times \text{time},$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}.$$

Ex. 1. A who travels $3\frac{1}{2}$ miles an hour starts $2\frac{1}{2}$ hours before B who goes the same road at $4\frac{1}{2}$ miles an hour; where will he overtake A ? A. U. 1889.

Let O be the starting point, O ————— P

and P , the place where they meet;

let $OP = x$ in miles.

The time taken by A to go to $P = \frac{x}{3\frac{1}{2}} = \frac{2x}{7}$ hours,

and " " " " B " " " " $= \frac{x}{4\frac{1}{2}} = \frac{2x}{9}$ " "

By the question, A takes $2\frac{1}{2}$ hours more to travel the distance OP ;

$$\therefore \frac{2x}{7} - \frac{2x}{9} = 2\frac{1}{2} = \frac{5}{2};$$

$$\text{whence } x = 39\frac{3}{8}.$$

\therefore the place sought is $39\frac{3}{8}$ miles distant from the starting point. *Ans.*

Ex. 2. From two towns 561 miles apart two men start, one from each, at the same time: one goes 24 and the other 27 miles a day; in how many days will they meet? C. U. 1879.

A ————— C ————— B

Let x = the number of days required.

A and B are the two towns, and C the place where the men meet, so that $AC = 24x$, and $BC = 27x$;

$$\therefore \text{by addition, } (24x + 27x) \text{ or } 51x = 561;$$

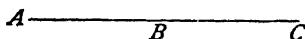
$$\therefore x = 11;$$

\therefore the time required = 11 days. *Ans.*

$$[AC = 24 \times 11 = 264, \text{ and } BC = 27 \times 11 = 297].$$

Ex. 3. A hare is 100 of her leaps before a greyhound, and takes 5 leaps to the greyhound's 4, but 2 of the greyhound's leaps cover as much ground as 5 of the hare's; how many leaps must the greyhound take to catch the hare?

Let x = the number of leaps the greyhound should take.



Draw the line of chase ABC , and suppose the greyhound is at A when the hare is at B , so that $AB = 100$ of the hare's leaps; and suppose that the hound catches the hare at C , so that $AC = x$ of the hound's leaps.

The time in which the hound runs the distance AC is the same in which the hare runs the distance BC .

Now, the hare takes 5 leaps in the time in which the hound takes 4,

$$\therefore \quad \text{ " " " } \frac{5x}{4} \quad \text{ " " " " " " " " " " } x.$$

$$\therefore \quad BC = \frac{5x}{4} \text{ of the hare's leaps.}$$

$$\text{Since } AB + BC = AC,$$

$$\left(100 + \frac{5x}{4}\right) \text{ of the hare's leaps} = x \text{ leaps of the hound,}$$

$$= \frac{5x}{2} \text{ of the hare's leaps, } \therefore 2 \text{ of the}$$

$$\text{hound's leaps} = 5 \text{ of the hare's.}$$

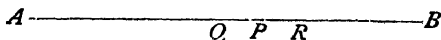
$$\therefore \quad 100 + \frac{5x}{4} = \frac{5x}{2}.$$

$$\therefore \quad 100 = \frac{5x}{4}.$$

$$\therefore \quad x = 80.$$

[The student should cut short the above work after he has thoroughly understood it.]

Ex. 4. AB is a railway 220 miles long, and three trains (P, Q, R) travel upon it at the rate of 25, 20 and 30 miles per hour respectively. P and Q leave A at 7 A. M. and 8-15 A. M. respectively, and R leaves B at 10-30 A. M. When and where will P be equidistant from Q and R . C U. 1870.



Let x = the number of hours after 10-30 A. M., after which P is equidistant from Q and R .

In the annexed diagram, $PQ = PR$.

$$\therefore AP - AQ = AR - AP.$$

$$\therefore \quad 2AP = AQ + AR \dots \dots \dots (1)$$

Now, P has been running for $(3\frac{1}{2} + x)$ hours, $\therefore AP = 25(3\frac{1}{2} + x)$ miles ;

Q " " " " $(2\frac{1}{4} + x)$ " , $\therefore AQ = 20(2\frac{1}{4} + x)$ " ;

R " " " " x " , $\therefore BR = 30x$ " ;

$$AR = AB - BR = (220 - 30x) \text{ miles.}$$

\therefore by substitution in (1) $50(3\frac{1}{2} + x) = 20(2\frac{1}{4} + x) + 220 - 30x$.

$$\therefore 175 + 50x = 45 + 20x + 220 - 30x ;$$

$$\therefore 60x = 265 - 175 = 90 ;$$

$$\therefore x = 1\frac{1}{2}.$$

$$\therefore AP = 25(3\frac{1}{2} + 1\frac{1}{2}) \text{ miles} = 125 \text{ miles ;}$$

$\therefore P$ is equidistant from Q and R at 12 o'clock (noon) at a distance of 125 miles from A .

Ex. 5. A person sets out to walk to a certain town. But when he has accomplished a quarter of his journey, he finds that if he continues at the same pace, he will have gone only $\frac{3}{8}$ ths of the whole distance when he ought to be at his destination. He therefore increases his speed by a mile per hour, and arrives just in time. Find his rates of walking. M. U. 1875.

Let his speed at first = x miles per hour,

and let the length of the journey = d miles.

By the question, the time that he will have taken to go $\frac{5}{8}$ ths of the whole journey at his original speed = the time taken to travel $\frac{1}{4}$ th of the journey at his original speed + the time taken by him to travel $\frac{3}{8}$ ths of the journey at the increased rate.

$$\therefore \frac{\frac{5}{8}d}{x} = \frac{\frac{1}{4}d}{x} + \frac{\frac{3}{8}d}{x+1} \quad \left[\text{time} = \frac{\text{distance}}{\text{speed}} \right].$$

Multiply both sides by $\frac{12}{d}$; then $\frac{10}{x} = \frac{3}{x} + \frac{9}{x+1}$;

$$\therefore \frac{7}{x} = \frac{9}{x+1} ;$$

$$\therefore 9x = 7(x+1) ;$$

$$\therefore x = 3\frac{1}{2}.$$

\therefore the man's original speed = $3\frac{1}{2}$ miles per hour, } *Ans.*
and his increased speed = $4\frac{1}{2}$ miles per hour. }

EXAMPLES 99.

1. A train going 20 miles an hour leaves Bristol for London, and another going 30 miles per hour leaves London for Bristol at the same time ; when and where will they meet ? [London to Bristol is 150 miles.]

2. A train going 25 miles per hour leaves Howrah for Madhupur at 1-12 P. M., and another going 30 miles per hour leaves Madhupur for Howrah at 2-6 P. M.; when and where will they meet, the distance between Howrah and Madhupur being 182 miles?

3. A mail train leaves Howrah for Jubbulpore at 10-30 A. M., and travels 30 miles per hour; an express leaves Howrah for Jubbulpore at noon and travels 40 miles per hour; when and where will it overtake the mail?

4. Two persons started at the same time from *A*. One rode on horse back at the rate of $7\frac{1}{2}$ miles an hour and arrived at *B* 30 minutes later than the other, who travelled the same distance by train at the rate of 30 miles an hour. Find the distance between *A* and *B*.

5. Two cyclists start to run to a post and back. The quicker meets the other 25 yards from the post on his way back, and arrives at the starting point 2 minutes sooner. How long does each take to run to the post, the distance being 250 yards?

6. A train, 176 ft. long, runs at the rate of 20 miles per hour; another, 264 ft. long, runs on a parallel rail in the opposite direction at the rate of 30 miles per hour; how long will they take to pass each other?

$A \xrightarrow{\quad} B \quad \quad \quad C \xleftarrow{\quad} D$ The position when the trains just meet.

$C \xleftarrow{\quad} D \quad \quad \quad A \xrightarrow{\quad} B$ The position when the trains just separate.

The first distance from *A* to *D* = 176 ft + 264 ft. = 440 ft = $\frac{1}{2}$ mile.

The time reqd. = that in which *A* and *D* meet, [see Ex. 2, Art. 182.]

7. If in Ex 6 the trains run in the same direction, how long will they take to pass each other?

$\left[\begin{array}{c} A \xrightarrow{\quad} B \\ C \xrightarrow{\quad} D \end{array} \right.$ 1st position; $\begin{array}{c} A \xrightarrow{\quad} B \\ C \xrightarrow{\quad} D \end{array}$ 2nd position.

[The time required = that in which *C* and *B* meet from a distance of $\frac{1}{2}$ mile.]

8. At a paper chase the hares had 8 minutes' start, and ran the whole distance at the rate of 3 miles per hour. A hound that ran the same course at the rate of 5 miles per hour, arrived 2 minutes after the hares. How long was the run?

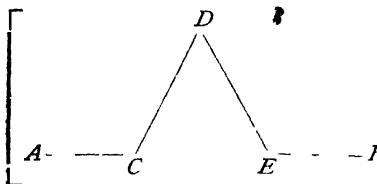
9. A thief spies a constable in pursuit from a distance of 40 yds., and tries to make off, taking 4 steps to the constable's 3;

each step of the constable covers a yard, while the thief's covers 2 feet; how many steps does the constable take before he catches the thief?

10. A person ran out a certain distance at the rate of 5 miles per hour, and then rode part of the way back at the rate of 12 miles per hour, running the remaining distance at the same rate as before in 3 minutes; he was out 8 min. 50 sec. How far did he go?

11. A person sets out to ride from A to B , but after proceeding one-third of the distance, slackens his speed by one-fourth; on his way back he rides at a uniform rate less by a mile per hour than his rate at starting, and finds that the return journey takes as much time as the first. Find his speed at starting.

12. A person walks from A to B , a distance of $7\frac{1}{2}$ miles, in 2 hours 17 $\frac{1}{2}$ minutes, and returns in 2 hours 20 minutes, his rates of walking up-hill, down-hill, and on a level road being 3, 3 $\frac{1}{2}$ and 3 $\frac{1}{2}$ miles per hour respectively. Find the length of the level road between A and B . (B. U. 1884).



In going from A to B , CD is up-hill, and in coming from B to A , ED is up-hill; \therefore the distance walked at the up-hill rate in the whole journey $= 7\frac{1}{2} - x$. Find the whole time taken in going and coming back.]

13. The distance from P to Q is $3\frac{1}{2}$ miles; two persons A and B start together from P to go to Q , the former by carriage which travels at the rate of 6 miles per hour, the latter walking at the rate of 3 miles an hour. If A remains at Q for 15 minutes and then returns by the carriage to P , find where he will meet B .

14. A fugitive having a start of 16 miles was pursued so as to be gained upon 2 miles an hour. After the pursuers had travelled 2 hours, they met a cyclist coming at the same rate as themselves, who had met the fugitive $1\frac{1}{2}$ hours before. Find his rate of flight.

15. A cloud is estimated to be at a height of a ft. above the ground; whenever a flash of lightning is seen, the accompanying peals of thunder are heard b seconds later, although the flash and the peals are known to originate simultaneously; supposing sound to travel c ft. per second, find the rate at which the light of a flash travels.

16. Two persons start simultaneously from A and B respectively, and continue going to and fro between those places at the respective rates of a miles and b miles per hour; if the distance AB be c miles, when and where will they meet for the second

time, supposing that they do so while going in opposite directions. What will be the answers, if they be moving in the same direction at the time?

183. Motion up or down a river.

When a boat is rowed on still water, the rate at which it moves gives the power of its crew

When it is rowed *down a river*, the current helps it forwards, so that the resultant *rate of motion down stream*

$$= \text{rate on still water} + \text{rate of current.}$$

When a boat is rowed *up a river*, the current retards the motion, so that the resultant *rate of motion up stream*

$$= \text{rate on still water} - \text{rate of current.}$$

Ex. 1. A boat goes up a river a certain distance in the time in which it goes down the river $\frac{4}{3}$ of the same distance; the current flowing at the rate of 3 miles per hour, find the rate at which the crew can pull on still water.

Let x miles per hour = the rate at which the crew can pull on still water, i.e., the rate of rowing on still water.

Then, the rate of motion up the river = $(x - 3)$ miles per hour, and the time taken to go up a distance, d , = $\frac{d}{x-3}$ hours; the rate of motion down the river = $(x + 3)$ miles per hour, and the time taken to go down a distance, $\frac{4}{3}d$, = $\frac{\frac{4}{3}d}{x+3}$ hours. Since, by the question, the two times are equal,

$$\frac{\frac{4}{3}d}{x+3} = \frac{d}{x-3};$$

$$\therefore \frac{4}{3}(x-3) = x+3;$$

$$\text{multiply each side by } 3; \quad 4x - 12 = x + 9$$

$$\therefore x = 21$$

\therefore the crew can row on *still water at the rate of 21 miles per hour.* *Ans.*

EXAMPLES 100.

1. A boat goes up stream 20 miles in the same time in which it goes down stream 30 miles; the current flowing at the rate of 2 miles per hour, find the rate at which the boat could be rowed on still water; find also the rate of motion up stream.

2. A boat is rowed 3 miles up a river, after which it enters a pool of still water, and is rowed 4 miles further on in 20 minutes; if the whole trip occupies 50 minutes, find the rate at which the current flows.

3. A piece of cork, when thrown on a river, is seen to float down 11 yards in 5 seconds. A steamer, which runs against the current at the rate of 20 miles per hour, is found to go from *A* to *B* and back again to *A* in 4 hours 5 minutes. Find the distance from *A* to *B*.

4. A boat has 12 rowers and starts from *A* towards *B* downstream; another has 9 equally good rowers, and starts at the same time (10 A.M.) from *B* towards *A*. The boats pass each other on the way at 11:30 A.M., the men rowing as hard as they can. If the distance from *A* to *B* is 42 miles, and if the current flows at the rate of 4 miles per hour, how much sooner will one of the boats reach its destination than the other?

5. It is found that between two given stations 12 rowers pull a boat up-stream in the same time as 8 rowers pull it down-stream; what fraction of the time may be saved in the down-stream trip, if all the twelve be set to work, supposing the rate of the current is 1 mile per hour?

6. A boat sails up 3 miles in 12 minutes more time than it takes to sail down the same distance; supposing the wind to be blowing uniformly in the same direction, and the rate of the stream to be 2 miles per hour, how far can the wind move the boat on still water in an hour?

7. A boat has to be rowed 4 miles up-stream; after proceeding a certain distance, the rate of the current which has hitherto been 2 miles per hour, is seen to fall suddenly by half the amount and to continue the same thenceforward; had the weaker current been met a mile further back, and while on it had the crew been putting forth only two-thirds their usual strength, the time required for the trip would have been the same. Find this time, supposing the crew to pull at the rate of $2\frac{1}{2}$ miles per hour on still water.

8. A boat has to be rowed with the stream just as many miles as it can be rowed on still water in an hour; the actual time required is the same in which the boat can be rowed 3 miles further down with the strength of the current trebled; if the rate of the current be 2 miles per hour, find the time taken.

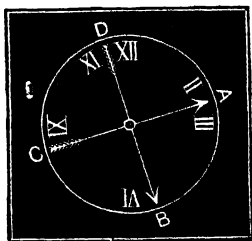
184. Clocks and Watches. The minute-hand of a clock moves over 60 minute-spaces while the hour-hand moves over 5.

Therefore the minute-hand moves 12 times as fast as the hour-hand. Hence, *in x minutes the hour-hand moves over $\frac{x}{12}$ minute-spaces.*

When the hands are exactly opposite each other, they are in a straight line having the position of a diameter of the dial. In this case the circumference of the dial containing 60 minute-spaces is divided into two equal parts by the line of the hands. Hence, *when the hands are in exactly opposite directions, they are separated by 30 minute-spaces.*

Of course when the hands are in *the same direction, i. e., coincident in position, there are no spaces between them.*

In the annexed diagram, AC and BD are two diameters of the dial of the clock, at right angles to each other. They evidently divide the whole circumference into four equal parts, each of which contains 15 minute-spaces. Now, when the hour-hand points against A , the minute-hand, in order to make a right angle with the hour-hand, must point against B or D ; that is, the hands must be separated by the arcs AB or ADB , which respectively contain 15 and 45 minute-spaces. Hence, *whenever the hands are at right angles, they are separated by 15 or 45 minute-spaces.*



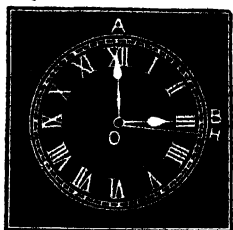
It should be carefully borne in mind that *the hands of a clock are at right angles to each other twice, but are in one line either in the same direction or in opposite directions only once, in course of an hour.*

The following statements deserve attention, and will be apparent to the student on a little reflection. When the hands are at right angles to each other

- (1) between 12 and 3 o'clock, the minute-hand is *in advance* of the hour-hand by 15 or 45 minute-spaces ;
- (2) between 3 and 9 o'clock, the minute-hand is *either behind or in advance* of the hour-hand by 15 minute-spaces ;
- (3) between 9 and 12 o'clock, the minute-hand is *behind* the hour-hand by 45 or 15 minute-spaces.

Ex. 1. Find (a) the instant of time between 3 and 4 o'clock at which the hour hand and minute-hand are exactly in the same direction, (b) that at which they are exactly opposite each other.

(a) Let x = the required number of minutes past 3, in the 1st case. In the annexed figure, let A denote the 12 o'clock mark, and B the 3 o'clock mark, and let OH represent the position of the two hands when they are coincident.



In x minutes the hour-hand passes over $\frac{x}{12}$ minute-spaces;

\therefore the arc BH contains $\frac{x}{12}$ minute-spaces,
while „ „ AH „ „ „ „ „ „ „
and „ „ AB „ 15 „ „ „

$$\text{Arc } AH = \text{arc } AB + \text{arc } BH,$$

$$\therefore x = 15 + \frac{x}{12};$$

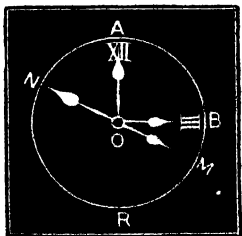
$$\therefore x - \frac{x}{12} \text{ or } \frac{11}{12}x = 15;$$

$$\therefore x = \frac{180}{11} = 16\frac{4}{11}$$

\therefore the hands are in the same direction at 16 $\frac{4}{11}$ minutes past 3. *Ans.*

(b) Let x = the required number of minutes past 3, in the 2nd case. In the annexed figure, let OM represent the position of the hour-hand, and ON that of the minute-hand, when the hands are opposite each other, so that MON is a straight line, and the arc MNR contains 30 minute-spaces. In x minutes the hour-hand passes over $x/12$ minute-spaces.

\therefore the arc BM contains $\frac{x}{12}$ minute-spaces,
while „ „ AMN „ „ „ „ „ „ „
and „ „ AB „ 15 „ „ „ „
Arc $AMN = \text{arc } AB + \text{arc } BM + \text{arc } MNR,$



$$\therefore x = 15 + \frac{x}{12} + 30;$$

$$\therefore \frac{11x}{12} = 45;$$

$$\therefore x = \frac{540}{11} = 49\frac{1}{11}.$$

\therefore the hands are opposite each other at 49 $\frac{1}{11}$ minutes past 3. *Ans.*

Ex. 2. Find the time when the hour and minute hands of a clock are at right angles to one another (a) between 1 and 2 o'clock, (b) between 4 and 5 o'clock, (c) between 10 and 11 o'clock.

(a) Let x = the number of minutes past 1, when the hands are at right angles to one another.

At 1 o'clock the hour-hand is separated from the 12 o'clock mark by 5 minute-spaces

\therefore in x minutes more the hour-hand is separated from the 12 o'clock mark by $\left(5 + \frac{x}{12}\right)$ minute-spaces, while the minute-hand is separated from it by x minute-spaces. Since in this problem the minute-hand is in advance of the hour-hand, when they are at right angles,

$$x - \left(5 + \frac{x}{12}\right) = 15 \text{ or } 45. \quad \therefore \frac{11x}{12} = 20 \text{ or } 50$$

$$\therefore x = \frac{240}{11} \text{ or } \frac{600}{11} = 21\frac{9}{11} \text{ or } 54\frac{6}{11}.$$

\therefore the hands are at right angles once at $21\frac{9}{11}$ minutes past 1, and again at $54\frac{6}{11}$ minutes past 1 (i.e., $5\frac{6}{11}$ minutes to 2) *Ans.*

(b) Let x = the number of minutes past 4, when the hands are at right angles to one another.

At 4 the hour-hand is separated from the 12 o'clock mark by 20 minute-spaces

\therefore at x minutes past 4 the hour hand is separated from the same mark by $\left(20 + \frac{x}{12}\right)$ minute-spaces.

\therefore when the hands are at right angles,

$$x - \left(20 + \frac{x}{12}\right) = 15 \text{ or } -15, \text{ according as the minute-hand is the more or less advanced of the two.}$$

$$\therefore \frac{11x}{12} - 20 = 15 \text{ or } -15;$$

$$\therefore \frac{11x}{12} = 35 \text{ or } 5;$$

$$\therefore x = \frac{420}{11} \text{ or } \frac{60}{11}; \text{ i.e., } 38\frac{2}{11} \text{ or } 5\frac{5}{11}$$

\therefore the hands are at right angles at $5\frac{5}{11}$ minutes past 4, and again at $38\frac{2}{11}$ minutes past 4. *Ans.*

(c) Let x = the number of minutes past 10, when the hands are at right angles.

At x minutes past 10 the hour-hand is separated from 12 o'clock mark by $\left(50 + \frac{x}{12}\right)$ minute-spaces.

Since in this problem *the hour hand is in advance of the minute-hand*, when the hands are at *rt. ∠s*,

$$\left(50 + \frac{x}{12}\right) - x = 45 \text{ or } 15;$$

$$\therefore \frac{11x}{12} = 5 \text{ or } 35;$$

$$\therefore x = 5\frac{5}{11} \text{ or } 38\frac{2}{11}.$$

\therefore the hands are at *rt. ∠s* at $5\frac{5}{11}$ minutes, and again at $38\frac{2}{11}$ minutes past 10. *Ans.*

EXAMPLES 101.

- ✓ 1. Find the time when the hour and minute-hands of a watch are exactly in the same direction, (a) between 4 and 5 o'clock, (b) between 9 and 10 o'clock.
- ✓ 2. Find the time when the hour and minute-hands of a watch are exactly opposite each other, (a) between 2 and 3 o'clock, (b) between 7 and 8 o'clock.
- ✓ 3. When are the hour and minute-hands of a clock in the same line between 1 and 2 o'clock?
- ✓ 4. When are the hour and minute-hands of a watch at right angles to one another, (a) between 1 and 2 o'clock, (b) between 5 and 6 o'clock, (c) between 8 and 9 o'clock, (d) between 11 and 12 o'clock?
- ✓ 5. When does the hour-hand make with the minute-hand, between 4 and 5 o'clock, an angle equal to that of an equilateral triangle? What is the time in the same case between 11 and 12 o'clock?
6. Find the interval that elapses between the two times at which the hands of a clock are
 - ✓ (1) at right angles to each other between 3 and 4 o'clock;
 - ✓ (2) 20 minutes apart between 2 and 3 o'clock.
- ✓ 7. It is between 4 and 5 o'clock that the hands of a watch are 25 min. apart; how long after will they next be so far apart again?
- ✓ 8. How many minutes does it want to 5 o'clock if half an hour ago it was thrice as many minutes past 3 o'clock?
- ✓ 9. It is between 10 and 11 o'clock when the hands of a watch are separated by two thirds the number of minute-spaces by which they were separated 10 minutes ago; find the time. What is the answer, when the time is between 11 and 12 o'clock?

10. Find the time between 5 and 6 o'clock, so that in 20 minutes more the minute-hand will be as much in advance of, as it is now behind, the hour-hand.

11. It is between 4 and 5 o'clock when, looking at a chronometer, I find the long hand behind the short hand by 10 minute-spaces ; looking at it again in course of the same hour, I find their relative positions exactly reversed : find the interval between the two observations.

12. It is between 1 and 5 o'clock when a man is out for a long walk ; when he returns and asks the time, he is told that it is between 7 and 8 o'clock and the hands of the clock have exactly changed places since the time he went out. When did he go out, and how long did he spend out-doors ?

185. **Arrangement into squares.** When a number of men is arranged into a solid square, a men in each side, there are a rows with a men in each row ; therefore the total number of men $= a^2$.

In the annexed diagram 4 men are on each side, so that there are 4 rows and 4 men in each row ; therefore the total number of men in the solid square $= 4 \times 4 = 4^2$.

```

x x x x
x x x x
x x x x
x x x x

```

Hence, *no. of men in a solid square* $= (\text{no. in front})^2$.

Suppose there are 9 men in front of a hollow square, 3 deep. In the annexed figure the positions occupied by the men are marked with cross lines \times , while the vacant places are indicated by dots \circ .

```

x x x x x x x x x
x x x x x x x x x
x x x x x x x x x
M x x x o A o o B x x x N
x x x o o o x x x
x x x o D o C o x x x
x x x x x x x x x
x x x x x x x x x

```

If there were no vacant places on the line MN , there would have been 9 men in it ; but there are 3 occupied positions on one side

and three on the other of AB . Therefore there could have been $(9-2 \times 3)$ men in the part AB of the line MN .

Therefore the square $ABCD$ could have contained $(9-2 \times 3)^2$ men. But the square, when full, would contain 9^2 men altogether. Therefore the square, when hollow and only three deep, contains $9^2 - (9-2 \times 3)^2$ men. Since the above reasoning is perfectly general, we may infer that when there are x men in the front line of a hollow square, a deep, the number of men in the square $= x^2 - (x-2a)^2$.

Hence, *no. of men in a hollow square*
 $= (\text{no. in front})^2 - (\text{no. in front} - \text{twice depth})^2$.

Ex. 1. A person has a number of rupees which he tries to arrange in the form of a square. On the first attempt he has 116 over. When he increases the side of the square by 3 rupees, he wants 25 rupees to complete the square. How many rupees has he? B. U. 1875.

Let x = the number of rupees in the front line of the 1st square.

Then the total number of rupees $= x^2 + 116$, (x^2 being the number in the square).

Again, by the question, the total number of rupees $+ 25 = (x+3)^2$, $x+3$ being the number in a side of the new square;

$$\begin{aligned} \therefore (x^2 + 116) + 25 &= (x+3)^2; \\ \therefore x^2 + 141 &= x^2 + 6x + 9; \\ \therefore 6x &= 132; \\ \therefore x &= 22. \end{aligned}$$

\therefore the number of rupees required $= (22)^2 + 116 = 600$. *Ans.*

Ex. 2. An officer can form his men into a hollow square 5 deep, and also into a hollow square 6 deep, but the front in the latter formation contains 4 men fewer than in the former; find the number of men. C. U. 1887.

Let x = the number of men in the front of the hollow square, 5 deep

Then $x-4$ = the number of men in front in the other formation.

The total number of men in the first case $= x^2 - (x-2 \times 5)^2 = 20x - 100$,

$$\begin{aligned} \text{and that in the second case} &= (x-4)^2 - \{(x-4) - 2 \times 6\}^2 \\ &= (x-4)^2 - (x-16)^2 \\ &= 24x - 240 \end{aligned}$$

Since the number of men in each case is the same,

$$24x - 240 = 20x - 100.$$

$$\therefore 4x = 140.$$

$$\therefore x = 35.$$

\therefore the number of men required $= 20 \times 35 - 100 = 600$. *Ans.*

EXAMPLES. 102.

1. A number of men is arranged into a solid square. Subsequently, 32 men being sent away, the rest are re-arranged into a solid square, having 2 men fewer in each side than before. How many men were there at first?

2. A number of men is arranged into a hollow square, 3 deep; if the depth be increased by 2 men, there will be 2 men fewer in each side. Find the number of men.

3. The men of a regiment are arranged into a hollow square, 16 deep, upon an order to deploy, it spreads out into a line 48 times as long as the front in the square formation. If the men are as close together as possible in each formation, how many are there?

4. A number of men is arranged into a hollow square, a deep, with $4bc$ men over; had there been $4ac$ men fewer, they might have been re-arranged into a hollow square, b deep, having the same front. How many men are there?

186. Special cases of Mixture

Ex. 1. A vessel contains a mixture of wine and water, so that for every 3 gallons of wine there are 2 gallons of water; another vessel contains a mixture of 7 gallons of wine and 3 gallons of water. What quantity should be taken of each mixture so as to produce a new mixture of 12 gallons containing twice as much wine as water?

Let x = the number of gallons taken from the first vessel.

Then $12 - x$ = the number of gallons from the 2nd vessel.

In every 5 gal. from the 1st vessel, we take 3 gal. of wine, and 2 gal. of water

\therefore in x gal. from the 1st vessel, we take $\frac{3}{5}x$ gal. of wine, and $\frac{2}{5}x$ of water.

In 10 gal. from the 2nd vessel, there are 7 gal. of wine and 3 gal. of water,

in $12 - x$ gal. from the 2nd vessel, there are $\frac{7}{10}(12 - x)$ gal. of wine and $\frac{3}{10}(12 - x)$ gal. of water,

\therefore the quantity of wine in the new mixture = $\{\frac{3}{5}x + \frac{7}{10}(12 - x)\}$ gal., and that of water " " " " = $\{\frac{2}{5}x + \frac{3}{10}(12 - x)\}$ gal.

Since the new mixture contains twice as much wine as water,

$$2\{\frac{3}{5}x + \frac{7}{10}(12 - x)\} = \frac{2}{5}x + \frac{3}{10}(12 - x);$$

multiply each side by 10; $8x + 72 - 6x = 6x + 84 - 7x$;

$$\therefore 3x = 12;$$

$$\therefore x = 4, \text{ and } 12 - x = 8.$$

\therefore 4 gallons of the first mixture, and 8 gallons of the other are to be taken. *Ans.*

Ex. 2. A goldsmith was given 10 tolas of pure gold to prepare a bangle, but he fraudulently replaced part of it by silver. The bangle was found to weigh $8\frac{1}{2}$ tolas under water; knowing that a tola of pure gold weighs only $\frac{7}{8}$ tola under water, and a tola of pure silver only $\frac{3}{4}$ tola in the same case, find out the quantity of gold abstracted.

Let x denote the quantity in tolas of gold abstracted;

then the quantity of gold in the bangle $= 10 - x$,

and „ „ „ silver „ „ „ $= x$.

Since a tola of gold weighs $\frac{7}{8}$ of a tola under water, and a tola of silver $\frac{3}{4}$ of a tola in the same case, the weight of the whole bangle in tolas under water

$$= \frac{7}{8}(10 - x) + \frac{3}{4}x = \frac{35}{4} - \frac{1}{8}x, \text{ simplifying;}$$

by the question, the same $= 8\frac{1}{2} = \frac{17}{2}$;

$$\therefore \frac{17}{2} = \frac{35}{4} - \frac{1}{8}x;$$

$$\therefore \frac{x}{8} = \frac{35}{4} - \frac{17}{2} = \frac{1}{4};$$

$$\therefore x = 2.$$

\therefore the quantity of gold replaced $= 2$ tolas. *Ans.*

Ex. 3. Lard is 3 times and castor oil is $1\frac{1}{2}$ times as heavy as water, bulk for bulk; find the proportion in which they should be mixed in order that the mixture may be just twice as heavy as water.

Let there be x parts by volume of castor-oil for 1 part of lard, and let w denote the weight of 1 part of water.

Then the weight of 1 part of lard $= 3$ times that of water $= 3w$;

„ „ „ x parts of oil $= 1\frac{1}{2} \times$ that of x parts of water
 $= \frac{3}{2}xw$;

$$\therefore \text{the total weight} = 3w + \frac{3}{2}xw \quad (A)$$

Since the mixture consists of $x + 1$ parts, and since by the question it is twice as heavy as water,

$$\text{the total weight} = 2(x + 1)w. \quad (B)$$

$$\therefore \text{from (A) and (B), } 2(x + 1)w = 3w + \frac{3}{2}xw;$$

$$\therefore 4(x + 1) = 6 + 3x;$$

$$\therefore x = 2.$$

\therefore the vols. of oil and lard are in the proportion of 2 : 1. *Ans.*

EXAMPLES 103.

1. A vessel contains 6 gallons of wine and 6 gallons of water ; what quantity of the mixture must be drawn off and replaced by an equal quantity of wine in order that there may be twice as much wine as water.

2. A vessel contains a mixture of 24 seers of milk and 12 seers of water, and in a second there are 9 seers of milk and 7 seers of water ; what quantity should be drawn off from each so as to make a mixture of $20\frac{1}{2}$ seers of milk and $11\frac{1}{2}$ seers of water ?

3. Two vessels contain mixtures of wine and water ; in one there is twice as much wine as water, and in the other three times as much water as wine. Find how much must be drawn off from each to fill a third vessel which holds 45 gallons, in order that its contents may be half wine and half water.

4. Each of two vessels holds 15 seers of adulterated milk, so that there are as much milk and water in one vessel as there are water and milk respectively in the other. 7 seers from the vessel richer in milk and 12 seers from the other are mixed up ; if the new mixture be estimated to contain $9\frac{1}{2}$ seers of milk, find the quantities of milk and water in each vessel.

5. A mixture of milk and water contains a gallons of milk and b gallons of water ; on replacing c gallons of the mixture by c gallons of milk, the quantities of milk and water in it are reversed.

Shew that $bc = b^2 - a^2$.

6. The total quantity of a mixture of wine and water is a gallons ; b gallons are now replaced by an equal quantity of wine, and this operation is repeated a second time ; it is found at last that the proportion of wine and water has been reversed. Find the quantities of each at first.

7. In a twelve-gallon mixture of wine and water, 3 gallons are drawn off thrice and are each time replaced by an equal quantity of wine ; if the mixture contains at last $7\frac{3}{5}$ gallons of wine, what quantity of wine was in it at first ?

8. A certain metal is thrice as heavy, and cork is half as heavy as water, bulk for bulk ; find the quantity of each in a combination of cork and the metal, whose volume is 10 cubic inches, and which is as heavy as an equal bulk of water.

9. If 19 lbs. of gold weigh 18 lbs. in water, and 10 lbs. of silver weigh 9 lbs. in water, find the quantities of gold and silver in a mass of gold and silver weighing 106 lbs. in air and 99 lbs. in water.

10. An alloy consists of 4 parts of gold and 3 parts of silver, and another consists of 4 parts of gold and 5 parts of silver ; how

should the two be mixed so as to produce a new alloy containing $2\frac{1}{2}$ parts of gold and $2\frac{1}{2}$ parts of silver?

187. Problems of a Geometrical nature.

Ex. 1. The height of a triangle is greater than its base by 1 ft.; if the base be diminished by 3 ft., and the height increased by 5 ft., the area remains unaltered. Find this area.

Let x denote the length of the base in feet; then $x+1$ = the length of the height in feet

Since the area of a triangle = $\frac{1}{2}$ height \times base,

the area of the given triangle = $\frac{1}{2}x(x+1)$ sq. ft.

When the base is diminished by 3 ft., and the height increased by 5 ft., the area = $\frac{1}{2}(x-3)(x+1+5) = \frac{1}{2}(x-3)(x+6)$.

Since, by the question, the area remains the same,

$$\frac{1}{2}(x-3)(x+6) = \frac{1}{2}x(x+1);$$

$$\therefore x^2 + 3x - 18 = x^2 + x;$$

$$\therefore 2x = 18;$$

$$\therefore x = 9,$$

and

$$x+1 = 10$$

Since the area = $\frac{1}{2}x(x+1)$ sq. ft.,

\therefore the area sought = $\frac{1}{2} \times 9 \times 10$ sq. ft.

$$= 45 \text{ sq. ft.}$$

$$= 5 \text{ sq. yds. } \textit{Ans.}$$

Ex. 2. The interior angles of a closed polygon are together thrice the interior angles of a second polygon the number of whose sides is just half that of the other. Find the numbers of their sides.

Let x denote the number of sides of the second polygon;

then $2x$ = the number of sides of the other.

Since by Euclid I. 32, Cor., the interior angles of a polygon = (twice the number of sides - 4) rt. \angle s,

the angles of the first polygon = $(2 \times 2x - 4)$ right angles,

and „ „ „ „ second „ = $(2x - 4)$ „ „ ;

\therefore by the question, $2 \times 2x - 4 = 3(2x - 4)$;

$$\therefore 4x - 4 = 6x - 12;$$

$$\therefore -2x = -8;$$

$$\therefore 2x = 8, \text{ and } x = 4;$$

\therefore the sides of the polygons are 8 and 4 in number. *Ans.*

Ex. 3. The top of a lotus-bud was seen half a cubit above the water of a lake ; a strong breeze, commencing to blow, it slowly advanced and got under water at a distance of two cubits Find the depth of the water. (Lilavati).

Let OA , the depth of the water $= x$ cubits, AB , the part of the stem above water $= \frac{1}{2}$ cubit ; C is the point where the bud was submerged.

$$AC = 2 \text{ cubits.}$$

$$OC = OB = (x + \frac{1}{2}) \text{ cubits}$$

$\therefore OA$ is vertical, and AC horizontal,

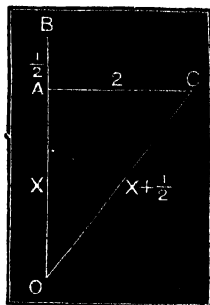
$$\angle OAC = \text{a rt. } \angle ;$$

\therefore by Euclid I. 47, $OA^2 + AC^2 = OC^2$;

$$\text{i.e., } x^2 + 4 = (x + \frac{1}{2})^2 = x^2 + x + \frac{1}{4} ;$$

$$\therefore x = 4 - \frac{1}{4} = 3\frac{3}{4}.$$

\therefore the required depth $= 3\frac{3}{4}$ cubits. *Ans.*



EXAMPLES 104.

1. The height and the base of a triangle are together equal to 6 ft. ; if the height be increased by 3 ft., and the base diminished by the same amount, the area of the triangle remains the same. Find the height and the base.

2. A rectangular park consists of a tank surrounded by a walk of the uniform breadth of 1 foot ; the unequal sides of the park differ by 1 foot, and the cost of paving the walk at 1s. per sq. foot is £11. 18s. ; find the dimensions of the park.

3. The angles of a polygon are together equal to as many right angles as the figure has sides increased by six right angles. Find the number of its sides.

4. The sides of a polygon are three fewer than those of another, and the angles of the latter are together double of those of the other ; how many sides are there in each polygon ?

5. Each angle of a regular polygon is three-fifths of two right angles ; find the number of its sides.

6. Three times the angle of a regular polygon equal eight times the angle of an equilateral triangle ; find the sum of the angles of the polygon.

7. In a right-angled triangle, the base is 8, and the sum of the hypotenuse and the perpendicular is 12 ; find them.

8. A ladder is placed against a vertical wall, so that the length of the ladder plus the distance of its foot from the foot of

the wall = 36 ft.; if the top of the ladder be at a height of 12 ft. from the ground, find the length of the ladder.

9. A rope is made straight by fastening one end at the top of a vertical lamp-post, and fixing the other end on the ground at a distance from the foot of the post exceeding the length of the post by 3 ft. Had the post been 2 ft. shorter, the end of the rope touching the ground would have been 1 ft. more distant from the foot of the post. How high is it?

10. The perimeter of an isosceles triangle is 36 ft., and its height is 12 ft. Find the base and the equal sides.

11. One side of a rectangle exceeds the other by 3 ft., and the diagonal remains unaltered by increasing the longer side by 2 ft. and diminishing the smaller side by 4 ft. Find the sides.

12. If a square be transformed into a rectangle by increasing two opposite sides and diminishing the other two, each by 1 ft., the diagonal is increased by $\frac{1}{2}\sqrt{2}$ of a foot. Find a side of the square.

CHAPTER XXIX.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

188. Nature of Simultaneous Equations.

Suppose we are given the equation $2x + 3y = 5$.

By transposition, $2x = 5 - 3y$,

$$\therefore x = \frac{5 - 3y}{2}.$$

If we put $y = 1$, then will $x = \frac{5 - 3 \times 1}{2} = 1$;

" " " $y = 2$, " " $x = \frac{5 - 3 \times 2}{2} = -\frac{1}{2}$;

" " " $y = 7$, " " $x = \frac{5 - 3 \times 7}{2} = -8$.

Thus giving y as many values as we please, we can make the values of x equally numerous. In fact there will be an infinite number of values of x corresponding to an infinite number of values given to y . Hence we cannot say what particular value x has in the given equation. But suppose we are given another equation along with the first one, say

$$x + y = 2.$$

$$\therefore x = 2 - y.$$

Thus, being given the two equations,

$$\left. \begin{array}{l} \text{we must have } x = \frac{5-3y}{2} \text{ (1),} \\ \text{and } x = 2-y \text{ (2),} \end{array} \right\} \text{ at one and the same time.}$$

Now it is clear that y cannot have *any* value, if the equations (1) and (2) be simultaneously true. y must have such a value, as, when substituted for it in (1) and (2), will make the right side of each of the equations the same, *viz.*, the value of x .

Hence y must be found from the equation

$$\begin{aligned} \frac{5-3y}{2} &= 2-y; \\ \therefore 5-3y &= 4-2y; \\ \therefore -y &= -1; \\ \therefore y &= 1. \end{aligned}$$

Having found y , we can find x from any one of the

$$\text{equations } x = \frac{5-3y}{2}, \text{ (3)}$$

$$\text{and } x = 2-y.$$

$$\text{Thus, by substitution in (3), } x = \frac{5-3 \times 1}{2} = 1.$$

Therefore $x=1$ and $y=1$ will satisfy the given equations, $2x+3y=5$ and $x+y=2$, simultaneously. For, if in $2x+3y=5$, we put $x=1$, and $y=1$, we get $2+3=5$, which is true; again, if in $x+y=2$, we put the same values for x and y , we get $1+1=2$, which is also true. No other values of x and y will satisfy both the given equations. For, suppose $x=4$, and $y=-1$.

Then $2x+3y=5$ becomes $2 \times 4 + 3(-1) = 5$, which is true; but $x+y=2$, $4+(-1)=2$, which is not true.

Thus $2x+3y=5$ and $x+y=2$ are not true *at one and the same time* for values of x and y other than those found above. We have seen then that if we are given only one equation, $2x+3y=5$, we cannot say what particular values x and y may not have, the number of their values being infinite. But if we are given another equation, say $x+y=2$, along with the first, we can definitely say what values x and y will have, as in the present case the values are 1 and 1.

Hence to find two unknown quantities we must have two *different* equations.

Similarly we can show that we must be given *three equations* to determine *three unknown* quantities, and so on.

189. Definition. The Equations that involve two or more unknown quantities, and are simultaneously satisfied by the same values of the unknowns are called **Simultaneous Equations**.

190. Two Unknowns. The solution of Simultaneous Equations involving two variables depends upon the **elimination** of one of the variables so as to form an equation in the other only. The elimination may be effected in three ways, each equation being supposed to be in the typical form $ax + by = c$. These are as follow :

(I). Method of Comparison :

Obtain separately from the two given equations two expressions each giving the value of one and the same unknown in terms of the other. Next equate these expressions.

$$\begin{array}{ll} \text{Let us solve} & 11x + 15y = 27, \quad (1) \\ \text{and} & 3x + 5y = 11. \quad (2) \end{array}$$

$$\text{From (1),} \quad 11x = 27 - 15y ;$$

$$\therefore \quad x = \frac{27 - 15y}{11} \dots\dots\dots(3)$$

$$\text{From (2),} \quad 3x = 11 - 5y ;$$

$$\therefore \quad x = \frac{11 - 5y}{3} \dots\dots\dots(4)$$

$$\therefore \text{ from (3) and (4),} \quad \frac{27 - 15y}{11} = \frac{11 - 5y}{3} ;$$

$$\text{clearing of fractions,} \quad 81 - 45y = 121 - 55y ;$$

$$\therefore \quad 10y = 40 ;$$

$$\therefore \quad y = 4 ;$$

$$\therefore \text{ from (4), by substitution, } x = \frac{11 - 5 \times 4}{3} = -3 ;$$

$$\therefore \text{ we have } \underline{x = -3, \quad y = 4. \quad \text{Ans.}}$$

N. B. This method is less generally used than the two following.

(II). Method of Substitution. *Find the value of one of the variables from one of the equations in terms of the other variable ; substitute this value in the other equation.*

$$\begin{array}{ll} \text{Let us solve} & 11x + 15y = 27, \quad \dots \quad (1) \\ & \text{and } 3x + 5y = 11. \quad \dots \quad (2) \end{array}$$

$$\text{From (2),} \quad 5y = 11 - 3x ;$$

$$\therefore \quad y = \frac{11 - 3x}{5} ; \dots \quad (3)$$

∴ by substitution for y in (1),

$$11x + 15 \times \frac{11-3x}{5} = 27;$$

$$\text{i.e., } 11x + 33 - 9x = 27;$$

$$\therefore 2x = 27 - 33 = -6;$$

$$\therefore x = -3;$$

$$\therefore \text{ from (3), } y = \frac{11-3(-3)}{5} = 4.$$

$$\text{Hence } \left. \begin{array}{l} x = -3, \\ y = 4, \end{array} \right\} \text{ as found before.}$$

(III). **Method of Multipliers.** *Multiply the two given equations by proper multipliers so as to make the coefficients of one of the unknowns the same, at least numerically, in the two equations. Then add or subtract one of the equations to or from the other so as to obtain an equation in one unknown only.*

$$\text{Let us solve } 4x + 5y = 23, \quad (1)$$

$$\text{and } 3x + 2y = 12. \quad (2)$$

$$\text{Multiply (1) by 2; then } 8x + 10y = 46.$$

$$\text{Multiply (2) by 5; then } 15x + 10y = 60.$$

$$\therefore \text{ by subtraction, } \begin{array}{r} -7x \\ x \end{array} = 46 - 60 = -14;$$

$$\therefore x = 2.$$

To find y ;

$$\text{Multiply (1) by 3; then } 12x + 15y = 69.$$

$$\text{Multiply (2) by 4; then } 12x + 8y = 48.$$

$$\therefore \text{ by subtraction, } 7y = 21;$$

$$\therefore y = 3.$$

$$\therefore \underline{x = 2, y = 3. \text{ Ans.}}$$

N.B. Having found x , we may easily find y by substitution thus:

$$\text{Since } x = 2, \text{ from (1), } 4 \times 2 + 5y = 23;$$

$$\therefore 5y = 15;$$

$$\therefore y = 3.$$

This method is very commonly used.

Ex. 1. Solve $\frac{x}{a+b} + \frac{y}{b+c} = a+b$, and $\frac{x}{a} - \frac{y}{b} = a-c$.

$$\left. \begin{array}{l} \text{We have } \frac{x}{a+b} + \frac{y}{b+c} = a+b, \dots\dots\dots(1) \\ \frac{x}{a} - \frac{y}{b} = a-c, \dots\dots\dots(2) \end{array} \right\}$$

Multiply (1) by $b+c$, and (2) by b ; then

$$x \cdot \frac{b+c}{a+b} + y = (a+b)(b+c), \dots\dots\dots(3)$$

$$\text{and } x \cdot \frac{b}{a} - y = b(a-c) \dots\dots\dots(4)$$

Adding (3) and (4), $x \left(\frac{b+c}{a+b} + \frac{b}{a} \right) = (a+b)(b+c) + b(a-c)$;

$$\therefore x \cdot \frac{a(b+c) + b(a+b)}{a(a+b)} = (ab + b^2 + ac + bc) + (ab - bc) ;$$

$$\text{i.e., } x \cdot \frac{2ab + ac + b^2}{a(a+b)} = 2ab + ac + b^2 ;$$

dividing by $2ab + ac + b^2$, $\frac{x}{a(a+b)} = 1$,

$$\therefore x = a(a+b).$$

Substitute the value of x in (2), [(2) being the simpler of the two given equations]; then

$$a+b - \frac{y}{b} = a-c ;$$

$$\therefore -\frac{y}{b} = -b-c ;$$

$$\text{changing signs, } \frac{y}{b} = b+c ;$$

$$\therefore y = b(b+c).$$

Hence $x = a(a+b)$, and $y = b(b+c)$. *Ans.*

N.B. Having found one of the unknowns, it is always practically best to substitute its value in the *simpler of the two given equations*.

$$\text{Ex. 2. Solve } \left. \begin{array}{l} 16x + 65y + 34 = 0, \\ 24x - 85y = 314. \end{array} \right\}$$

Putting the given equations in the usual form, we have

$$\left. \begin{array}{l} 16x + 65y = -34, \dots\dots(1) \\ 24x - 85y = 314, \dots\dots(2). \end{array} \right\}$$

Multiplying (1) by 3, and (2) by 2, we have

$$\left. \begin{array}{l} 48x + 195y = -102, \\ 48x - 170y = 628. \end{array} \right\} \dots\dots\dots(A).$$

From the last two equations, by subtraction,

$$365y = -730;$$

$$\therefore y = -2.$$

Substituting for y in (1), we get

$$16x - 65 \times 2 = -34;$$

$$\therefore 16x = 130 - 34 = 96;$$

$$\therefore x = 6.$$

$$\therefore \underline{x = 6, \text{ and } y = -2. \text{ Ans.}}$$

N. B. In the above work we have first attempted to transform by multipliers the equations (1) and (2) into two others in which the co-efficient of x is the same. This last co-efficient is then some common multiple of the original co-efficients of x in (1) and (2). Hence it is *practically most convenient to choose the L. C. M. of the original co-efficients*. The co-efficient of x in (1) and (2) are 16 and 24, of which the L. C. M. is 48. By suitably multiplying (1), and also (2), we have made 48 the co-efficient of x in the transformed equations (A). We further press another point upon the student's attention. We have in the above example eliminated x so as to find y first. Suppose we want to eliminate y from (1) and (2) so as to find x first. The L. C. M. of 65 and 85 is $5 \times 13 \times 17$. We should then multiply (1) by 17, and (2) by 13, and solve the resulting equations. But these large multipliers necessarily lead to too much multiplication, which means much trouble besides being a fruitful source of error. We have therefore preferred to find y first instead of x . The student should therefore always *begin by first determining by simple inspection which of the unknowns can be conveniently eliminated from the given equations*.

EXAMPLES 105.

Solve by each of the fore-going methods :

- | | | |
|---|--|---|
| 1. $x + y = 13,$
$x - y = 7.$ | 2. $2x + 3y = 36,$
$3x + y = 35.$ | 3. $5x + 2y = 7,$
$7x + 3y = 9.$ |
| 4. $8x - 7y = 42,$
$4x + 3y = 34.$ | 5. $12x - 7y = 44,$
$8x - 3y = 36.$ | 6. $14x - 5y = 100,$
$13x - 10y = 50.$ |
| 7. $16x + 24y = 104,$
$15x - 2y = 24.$ | 8. $13x + 24y = 610,$
$36y - x = 710.$ | 9. $6x - 12y = 1,$
$8x + 9y = 18.$ |
| 10. $22x - 11y = 0,$
$5x + y = 35.$ | 11. $\frac{1}{8}(x - 3) = \frac{1}{2}(y - 7),$
$11x = 13y.$ | 12. $16x - y = 4x + 2y$
$= 6.$ |

13. $7x + 8y = 51$, $9x - 16y = 13$. 14. $x + 6y = 37$, $19x - 2y = 7$. 15. $12x + 13y = 34$, $17x + 39y = 7$.
16. $16x - 9y = 31$, $13x - 12y = -17$. 17. $22x - 15y + 16 = 0$, $12y - 25x + 2 = 0$. 18. $23x - 5y = -6\frac{1}{2}$, $15y + 7x = \frac{5}{4}$.
19. $20x + 9y = 218$, $30x + 12 = 156y$. 20. $35x = 13y + 48$, $26y = 45x - 71$. 21. $11x - 15y = 7$, $44x - 19y = 41\frac{1}{3}$.
22. $21x - 17y + 9 = 0$, $56x - 96y + 176 = 0$. 23. $21x - 9y = 0$, $31x - 15y + 12 = 0$. 24. $27x - 55y + 23 = 0$, $99y + 25x + 7\frac{2}{3} = 0$.
25. $129x - 243y = 445y - 215x = 50$
26. $ax - by = 2ab$, $2bx + 2ay = 3b^2 - a^2$. 27. $(a+b)x + (a-b)y = 2ac$, $(b+c)x + (b-c)y = 2bc$.
28. $\frac{x}{2} + \frac{y}{3} = 8$, $\frac{x}{2} + \frac{y}{5} = 5$. 29. $\frac{x}{4} + \frac{y}{5} = 2$, $\frac{3x}{2} + \frac{7y}{10} = 9\frac{1}{2}$. 30. $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{3a} + \frac{y}{6b} = \frac{2}{3}$.
31. $\frac{x}{a} + \frac{y}{b} = m$, $\frac{x}{b} + \frac{y}{a} = n$. 32. $\frac{x}{a} + \frac{3y}{a+b} = 4$, $(a+b)x - ay = 0$. 33. $\frac{x}{a+b} + \frac{y}{a-b} = \frac{1}{c}$, $\frac{x}{a+c} + \frac{y}{a-c} = \frac{1}{b}$.
34. $\frac{x}{a} + \frac{y}{c} = a+b$, $\frac{x}{b} + \frac{y}{a} = a+c$. 35. $\frac{bx}{c} - \frac{cy}{b} = \frac{a}{c} - \frac{c}{a}$, $\frac{x}{a} + \frac{y}{b} = \frac{1}{a} + \frac{1}{b}$. 36. $\frac{x}{4} + \frac{y}{10} = -\frac{y}{3} - \frac{x}{9} = 1$.
37. $\frac{x}{a} + \frac{y}{a+c} = a$, $\frac{y}{a-c} - \frac{x}{c} = c$. 38. $\frac{x}{b} - \frac{y}{a} = \frac{v}{b+c} - \frac{x}{a+c} = a-b$. 39. $\frac{2x}{5} - 3y = 1$, $x + \frac{6y}{5} = 2\frac{1}{2}$.
40. $2x + 3y = 2\frac{1}{2}$, $5x - 4y = 5$.

191. **Reduction to the standard form.** Very often equations have to be reduced to the form $ax + by = c$, before applying the usual methods of solution.

Ex. 1. Solve $(x+7)(y-3)+7=(y+3)(x-1)+5$, } C. U. 1888.
 $5x-11y+35=0.$

Perform the multiplication in the first equation ;

then $xy+7y-3x-21+7=xy+3x-y-3+5$,

cancel xy ; then $7y-3x-14=3x-y+2$;

by transposition, $8y-6x=16$;

dividing by 2, $4y-3x=8$(1)

From the second of the given equations, we have

$$11y-5x-35=0 \dots \dots \dots (2)$$

Now solve (1) and (2)

Multiply (1) by 5 and (2) by 3 ;

then $20y-15x=40$,

and $33y-15x=105$,

∴ by subtraction, $-13y = -65$, $y = 5$

Then by substitution for y in (1),

$$4 \times 5 - 3x = 8 ;$$

$$20 - 3x = 8 ;$$

$$-3x = -12$$

$$x = 4$$

$$\therefore x = 4, y = 5 \quad \text{Ans.}$$

Ex 2 Solve $\left. \begin{array}{l} \frac{x-2}{2} + \frac{x+1}{14} = \frac{x-y-1}{8} - \frac{y+12}{4} \\ \frac{x+7}{3} + \frac{y-5}{10} = \frac{x-2}{7} - \frac{5(y+1)}{7} \end{array} \right\}$ C. U 1882

Multiply the first of the given equations by

$$56 (= \text{L. C. M. of } 2, 14, 8 \text{ and } 4)$$

Thus $28(x-2) + 4(x+1) = 7(x-y-1) - 14(y+12)$;

$$28x - 56 + 4x + 4 = 7x - 7y - 7 - 14y - 168 ;$$

$$32x - 4y - 50 = 7x - 21y - 175, \text{ simplifying,}$$

∴ by transposition, $17x + 17y - 56 - 175 = -119$;

$$\text{dividing by } 17, \quad x+y = -7 \dots \dots \dots (1)$$

Multiply the second of the given equations by

$$210 (= \text{L. C. M. of } 3, 10 \text{ and } 7) ; \text{ then}$$

$$70(x+7) + 21(y-5) - 210(x-2) - 30 \times 5(y+1) ;$$

$$70x + 490 + 21y - 105 - 210x - 210x - 150y - 150 ;$$

$$\text{simplifying, } 70x + 21y + 385 = -210x - 150y + 60 ;$$

∴ by transposition, $280x + 171y = -325$(2)

Multiply (1) by 171 : $171x + 171y = -1197$(3)

Subtracting (3) from (2), $109x = 1197 - 325 = 872$;

$$\therefore x = 8 ;$$

∴ by substitution in (1), (which is the simplest of the equations obtained), we have

$$8 + y = -7 ;$$

$$\therefore y = -15.$$

$$\therefore \underline{x = 8, y = -15.} \quad \text{Ans}$$

EXAMPLES 106.

Solve

1. $5x - 2y = 7$ and $2y = x + y + 11$.

2. $3x + 2y - 1 = 2x + 5y - 18 = x + 4y - 11$.

3. $4x - 6y - 3 = 7x + 2y - 4 = -2x + 3y + 24$.

4. $ax + y = r + by = \frac{1}{2}(x + y) + 1$.

5. $17x + 5y - 3 = 40 - 13x - 4y$,

$$11x - 9y + 1\frac{1}{2} = 1 - 14x - y.$$

6. $5x - (2x - 6y) = 14 + 7x$,

$$3x + 2y = 24 - (6x + 3y).$$

7. $\frac{1}{5}x - \frac{2}{3}y = \frac{1}{2}y$,

$$\frac{1}{3}x - \frac{3}{5}y + 10 = 0.$$

8. $\frac{x}{6} + y = 6$,

$$\frac{1}{4}(y - 1) + x = 8.$$

9. $\frac{2x - y}{7} - \frac{5x - 2y}{4} = 3x - 5y + 3$,

$$x + y - 11(x - y).$$

10. $\frac{5x - 2y}{17} - \frac{2x + 3y}{3} = x + y - 2$, $\frac{3x - 4y}{13} = \frac{3x + 4y}{15}$.

11. $\frac{7}{2}x - (\frac{1}{3}x + 2y) = 10 + \frac{1}{6}x$,

$$\frac{1}{8}x + \frac{5}{6}y = 7 - (\frac{1}{8}x + \frac{1}{6}y).$$

12. $\frac{13x}{6} - (\frac{1}{2}x - 5y) = 43 - \frac{x}{3}$,

$$\frac{7}{18}x + \frac{4}{9}y = \frac{8}{9}y - 10 - \frac{1}{18}x.$$

3. $\frac{3x}{12} - \frac{y}{15} - \frac{7x - 3y}{12} = \frac{x}{12} - \frac{y}{30} = \frac{1}{10}$.

4. $\frac{x - 3y}{2} - \frac{y - 3x}{2} + 8 = 0$,

$$x - 2y = \frac{1}{2}(x - \frac{1}{2}y)$$

15. $x - 2y + 4 = \frac{1}{2}\{2x + 3(y - \frac{1}{2})\}$,

$$\frac{1}{2}(y + \frac{1}{2}x) - \frac{1}{2}(x + 2) = 1\frac{1}{10}.$$

16. $(5x + 2)(y + 3) - 27 = 5xy + 38$, $25x - 8y = 19$.

17. $\frac{x+1}{y-2} = \frac{x+2}{y-3}$, $3x = 4y + 5$. 18. $\sqrt{\frac{x}{a+b} - \frac{y}{b+c}} = 1$, $\frac{a-c}{x-2y} = \frac{1}{2}$.

$$\checkmark 19. \frac{a-bx}{a+bx} = \frac{1-v}{1+v},$$

$$\frac{c}{c+dx} = \frac{d}{c+dy}.$$

$$\checkmark 20. \frac{x}{2} + \frac{9(2x-y)}{8x-7} = 5 - \frac{6-x}{2},$$

$$\frac{x}{2} + \frac{y+1}{3} = 2.$$

$$21. \frac{17y-6x}{11x-20} + \frac{1}{3}y + \frac{1}{3}x = \frac{x+5}{2} + \frac{6y-5x}{30},$$

$$\frac{27x-4}{19y-9x} + \frac{2}{87}(14x-15y) = 50 - \frac{1}{87}(x+y) + \frac{1}{3}(x-y)$$

$$22. \frac{x^2-y^2-7x+7y+2}{x+y-4} + 5x+2y = \frac{12x+2y-7}{2}, \quad \frac{x}{3} + \frac{y}{6} = \frac{1}{2}.$$

$$23. \frac{\sqrt{(x+y)} + \sqrt{(x-y)}}{\sqrt{(x+y)} - \sqrt{(x-y)}} = \frac{5}{3}, \quad 3x+5y=63$$

$$24. \sqrt{(x+y+6)} - \sqrt{(x+y-1)} = 1,$$

$$\frac{2y}{\sqrt{(x+y)} - \sqrt{(x-y)}} = \sqrt{10} + \sqrt{6}. \quad (\text{Rationalize}).$$

192. Reciprocals of unknowns The mode of treatment here is generally the same as usual.

$$\text{Ex. 1. Solve } \left. \begin{array}{l} \frac{4}{x} + \frac{10}{y} = 2, \dots (1) \\ \frac{3}{x} + \frac{2}{y} = \frac{19}{20} \quad (2) \end{array} \right\} \quad \text{C U. 1879}$$

$$\text{Multiplying (2) by 5,} \quad \frac{15}{x} + \frac{10}{y} = \frac{19}{4}; \dots (3),$$

$$\text{subtracting (1) from (3),} \quad \frac{15}{x} - \frac{4}{x} = \frac{19}{4} - 2.$$

$$\text{i.e.,} \quad \frac{11}{x} = \frac{11}{4};$$

$$\therefore x = 4$$

Substitute the value of x in (1); then

$$\frac{4}{4} + \frac{10}{y} = 2;$$

$$\therefore \frac{10}{y} = 1\frac{1}{2};$$

$$\therefore y = 10.$$

Hence, $x=4, y=10$. *Ans.*

EXAMPLES 107.

Solve

✓ 1. $\frac{7}{x} + \frac{1}{y} = \frac{1}{x} + \frac{3}{y}$,

$\frac{2}{x} + 3 = \frac{1}{y} + 1$.

4. $\frac{a}{x} + \frac{b}{y} = \frac{b}{x} + \frac{a}{y}$,

$= \frac{1}{a} + \frac{1}{b}$.

7. $\frac{a}{x+1} - \frac{b}{y+1} = 1$,

$(x-d) = dy - c$.

2. $\frac{4}{x} + \frac{5}{y} = \frac{5}{8}$,

$\frac{5}{x} + \frac{6}{y} = 1\frac{1}{6}$.

✓ 5. $\frac{m}{x} - \frac{n}{y} = a$,

$px - qy = 0$.

✗ 3. $\frac{a}{x} + \frac{b}{y} = c$,

$\frac{a}{x} - \frac{b}{y} = d$.

6. $17x + \frac{4}{y} = 59$,

$19x - \frac{4}{3y} = 153$.

✗ 8. $\frac{a}{mx} + \frac{b}{ny} = c$,

$\frac{b}{mx} + \frac{a}{ny} = 1$.

✓ 9. $\frac{2a}{x+y} + \frac{3b}{x-y} = 5$,

$\frac{5a}{x+y} - \frac{2b}{x-y} = 3$.

10. $\frac{4x+6y+3}{2x+3y-1} + \frac{2x+3y-7}{x+y+1} = 2$,

$\frac{2x+3y-4}{2x+3y-1} + \frac{x+y-1}{x+y+1} = 3$.

103. **Exponential Equations.** The Index Law is specially needed here.

Ex. 1. Solve $a^x \cdot a^{y+1} = a^7$, $a^{2y} \cdot a^{3x+5} = a^{20}$. } C. U. 1879.

Since $a^x \cdot a^{y+1} = a^{x+y+1}$,

and $a^{2y} \cdot a^{3x+5} = a^{2y+3x+5}$,

the given equations reduce to

$a^{x+y+1} = a^7$(1)

and $a^{2y+3x+5} = a^{20}$(2)

\therefore from (1), $x+y+1=7$ See Art. 176.

i.e., $x+y=6$(3)

and from (2), $2y+3x+5=20$,

i.e., $3x+2y=15$(4)

Multiply (3) by 2; $2x+2y=12$;.....(5)

subtracting (5) from (4), $x=3$;

by substitution in (3), $3+y=6$; *i.e.*, $y=3$.

Hence $x=3$, $y=3$. *Ans.*

Ex. 2. Solve $2^{x+1} 3^{y+2} = \frac{1}{6}$, $2^{2x+1} 3^{8y+6} = \frac{1}{8}$.

Break up terms thus: $2^{x+1} = 2 \cdot 2^x$; $3^{y+2} = 3^2 \cdot 3^y$, &c.

Thus the first given eqn. is $(2 \cdot 2^x)(3^2 \cdot 3^y) = \frac{1}{6} = \frac{1}{2} \times \frac{1}{3} = 2^{-1} \cdot 3^{-1}$;
dividing by 2×3^2 ; $2^x \cdot 3^y = 2^{-2} \cdot 3^{-3}$. (1)

The second given equation becomes

$$(2 \cdot 2^{2x})(3^6 \cdot 3^{8y}) = \frac{1}{8} = \frac{1}{2^3 \cdot 3^1} = 2^{-3} \cdot 3^{-1};$$

dividing by 2×3^6 , $2^{2x} \cdot 3^{8y} = 2^{-4} \cdot 3^{-7}$; (2)

cubing (1), $2^{3x} \cdot 3^{3y} = 2^{-6} \cdot 3^{-9}$; (3)

dividing (3) by (2); $2^x = 2^{-2}$, i.e., $x = -2$;

\therefore from (1), $2^{-2} \cdot 3^y = 2^{-2} \cdot 3^{-3}$; $\therefore y = -3$.

$\therefore \underline{x = -2, y = -3}$. Ans.

EXAMPLES 108.

Solve

1. $a^{x+1} a^{y+2} = a^8$,

2. $a^{x-3} a^{y+2} = a^2 \cdot a^7$,

$a^{x+1} a^{y+1} = a^{11}$.

$a^x a^y = a^4$.

3. $a^{2x+1} a^{y+1} = a^3 = a^{x+1} a^{y+1}$.

4. $2^{x+y} = 2^{2x-y} = \sqrt{8}$

5. $a^{2x} a^{2y} = \frac{1}{a^6}$,

6. $2^x \cdot 3^y = 18$,

$a^{3x} a^y = \sqrt{a^{-18}}$.

$2^{2x} \cdot 3^y = 36$.

7. $2^{x+1} 3^{y+1} = 3^{x+1} 2^{y+1}$,

8. $2^{2x-1} 3^{y-1} = \frac{1}{3} \cdot 2^{x+1} 3^y$.

$2^{x+1} 3^y = 72$.

$2^{x-3} 3^y = \frac{9}{2}$.

9. $x^{y+ns} = y^{x+nb}$,

10. $x^{2c+1} = 2y^x$,

$x^a = y^b$.

$x^{y-4} = 1$.

104. More than Two Unknowns. First Method:

When there are more than two unknowns, connected of course by an equal number of equations, it is usual to adopt the following method of successive elimination: If there be three given equations involving three unknowns, eliminate one of the unknowns between any two of the equations, and next eliminate the same unknown between any of these equations and the third; we thus obtain two equations in the other two unknowns only, to which equations now apply the method of elimination in the usual way to find these unknowns. Lastly substitute the values

of the two unknowns so obtained in any of the given equations to find the third unknown.

This method can evidently be extended to four or more given equations.

Ex. 1. Solve
$$\left. \begin{aligned} 4x - 5y + 6z &= 3, & (1) \\ 8x + 7y - 3z &= 2, & (2) \\ 7x + 8y + 9z &= 1. & (3) \end{aligned} \right\} \quad \text{B. U. 1862.}$$

We shall first eliminate z from (1) and (2).

Multiplying (2) by 2, $16x + 14y - 6z = 4$;
adding this product to (1), $20x + 9y = 7$(4)

We shall next eliminate z from (2) and (3).

Multiplying (2) by 3, $24x + 21y - 9z = 6$;
adding this product to (3), $31x + 29y = 7$(5)

We shall now find x and y from (4) and (5).

From (4) and (5), $31x + 29y = 20x + 9y$, \therefore each $= 7$
 $\therefore 11x + 20y = 0$;
 $\therefore 20y = -11x$;
 $\therefore y = -\frac{11}{20}x$(6)

Substituting for y in (4), $20x - \frac{11}{20}x = 7$;
 $\therefore \frac{391}{20}x = 7$;
 $\therefore x = 7 \times \frac{20}{391} = \frac{20}{43}$;
 \therefore from (6), $y = -\frac{11}{20} \times \frac{20}{43} = -\frac{11}{43}$.

To find z substitute the values of x and y in any one of the given equations, say (1). (In practice the simplest of the equations should be chosen as it gives the result very readily).

Thus
$$\frac{4 \times 20}{43} - 5\left(-\frac{11}{43}\right) + 6z = 3$$
 ;
i.e., $\frac{80}{43} + \frac{55}{43} + 6z = 3$;
 $\therefore 6z = 3 - \left(\frac{80}{43} + \frac{55}{43}\right)$
 $= 3 - \frac{135}{43} = \frac{-6}{43}$;
 $\therefore z = -\frac{1}{43}$.
 $\therefore \underline{x = \frac{20}{43}, y = -\frac{11}{43}, z = -\frac{1}{43}. \text{ Ans.}}$

EXAMPLES 109.

$$\begin{aligned} 1. \quad & 2x + 3y + z = 11, \\ & 3x + 4y + 3z = 17, \\ & 5x + 7y + 3z = 27. \end{aligned}$$

$$\begin{aligned} 2. \quad & 4x - 5y + 3z = 5, \\ & 3x + 5y - 2z = 13, \\ & x + y + z = 9. \end{aligned}$$

$$\begin{aligned} 3. \quad & 4x - 3y + 2z = 0, \\ & 6x + 4y - z = 31, \\ & 5x - 9y + z = -39. \end{aligned}$$

$$\begin{aligned} 4. \quad & 2x + 5y - 6z = 3, \\ & x - 6y + 3z = -9, \\ & y - 2z + 2 = 0. \end{aligned}$$

$$\begin{aligned} 5. \quad & 6x - 7y + 9z + 47 = 0, \\ & 2x - 3y + 14 = 0, \\ & 4z - 5z + 11 = 0. \end{aligned}$$

$$\begin{aligned} 6. \quad & 8x + z = 2(y - z), \\ & 4x + 5y = \frac{3}{5}(3y + z), \\ & 2x + 6y - 11z = 2(2x - 15). \end{aligned}$$

$$\begin{aligned} 7. \quad & \frac{x}{3} + \frac{y}{4} + \frac{z}{2} = 3\frac{5}{12}, \\ & \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 3, \\ & 2x - 3y + 4z = 11. \end{aligned}$$

$$\begin{aligned} 8. \quad & \frac{x}{14} + \frac{y}{17} + \frac{z}{21} = 3, \\ & x - y + z = 18, \\ & 2x + y - 2z = 3. \end{aligned}$$

$$9. \quad \frac{x + 2y + 1}{2} = \frac{y + 3z + 2}{3} = \frac{z + 4x + 3}{4} = 2.$$

$$10. \quad 2x - y = 2y - \frac{1}{z} = \frac{3}{z} - 7z = 1.$$

$$\begin{aligned} 11. \quad & \frac{1}{2x} + \frac{1}{3y} + \frac{1}{4z} = 1\frac{1}{12}, \\ & \frac{2}{x} + \frac{3}{y} + \frac{4}{z} = 9, \\ & \frac{1}{3x} - \frac{2}{y} + \frac{3}{z} = 1\frac{1}{3}. \end{aligned}$$

$$\begin{aligned} 12. \quad & \frac{3}{2x} - \frac{4}{5y} + \frac{1}{2z} = 7\frac{3}{10}, \\ & \frac{1}{3x} + \frac{1}{y} + \frac{2}{z} = 20\frac{1}{3}, \\ & \frac{1}{5x} - \frac{1}{4y} + \frac{1}{z} = 8\frac{1}{15}. \end{aligned}$$

$$\begin{aligned} 13. \quad & \frac{x}{2} + \frac{3}{y} + \frac{4}{z} = 1\frac{1}{2}, \\ & x + \frac{1}{y} + \frac{1}{z} = 1, \\ & 11x + \frac{14}{z} = 25. \end{aligned}$$

$$\begin{aligned} 14. \quad & \frac{ax}{b+c} + \frac{by}{c+a} = 1, \\ & \frac{2x}{b+c} + \frac{2y}{c+a} = \frac{1}{a} + \frac{1}{b}, \\ & ax + by + cz = a + b + c. \end{aligned}$$

$$15. \frac{y}{b} + \frac{z}{c} - \frac{x}{a} = a,$$

$$\frac{z}{c} + \frac{x}{a} - \frac{y}{b} = b,$$

$$\frac{x}{a} + \frac{y}{b} - \frac{z}{c} = c.$$

$$16. \frac{a}{x} + \frac{b}{y} - \frac{a}{z} = \frac{1}{b},$$

$$\frac{a^2}{x} + \frac{b^2}{y} - \frac{ab}{z} = \frac{a}{b},$$

$$cx = ay.$$

$$17. x + 2y + z = 12,$$

$$x - 3y + 2z = 2,$$

$$2x - y + 3z = 5,$$

$$2y - z = 4.$$

$$18. x + y - z + 2u = 0,$$

$$2x - y + z - u = 4,$$

$$x + y = 7,$$

$$y - u = 7.$$

$$19. 2x - 3y + 4z - u = 10,$$

$$4y - 3z + u = 1,$$

$$3z - 2x + v = 7,$$

$$x - v = 5,$$

$$x + v + u = 2.$$

$$20. x + y - 3v + u = 4,$$

$$2x - y - 2v + u = -1,$$

$$x + v - 2u = 7,$$

$$x + v + 3v = 2,$$

$$x + u = -2.$$

195. **Special forms.** The following examples are intended to illustrate how some equations having special forms may be very readily and neatly solved.

$$\text{Ex. 1. Solve } \left. \begin{aligned} \frac{1}{x} + \frac{1}{y} &= 3, (1) \\ \frac{1}{y} + \frac{1}{z} &= 4, (2) \\ \frac{1}{z} + \frac{1}{x} &= 5, (3) \end{aligned} \right\}$$

M. U. 1863.

Add together (1), (2) and (3); then $\frac{2}{x} + \frac{2}{y} + \frac{2}{z} = 12$;

dividing by 2, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 6 \dots (4).$

Subtract (1) from (4); thus $\frac{1}{z} = 3, \therefore z = \frac{1}{3}.$

„ (2) „ (4) „ $\frac{1}{x} = 2, \therefore x = \frac{1}{2}.$

„ (3) „ (4) „ $\frac{1}{y} = 1, \therefore y = 1.$

Hence $x = \frac{1}{2}, y = 1, z = \frac{1}{3}.$ Ans.

Ex. 2. Solve $u+2x+y+2z = -13, \dots (1)$
 $2u+x+2y+z=3, \dots (2)$
 $u+2x+12y+z=21, \dots (3)$
 $u+x+6y+z=10, \dots (4)$ } B. U 1872

Add (1) and (2); thus $3(u+x+y+z)=0$;

$$\therefore u+x+y+z=0; \dots (5)$$

subtracting (5) from (1), $x+z=-3; \dots (6)$

" " (2), $u+y=3; \dots (7)$

" " (3), $x+11y-21; \dots (8)$

" " (4), $5y-10; \dots (9)$

\therefore from (9), $y=2$.

Substituting for y in (7) and (8), $u+2=3$,

and $x+22=21$;

$$\therefore u=1, x=-1.$$

Substituting for x in (6), $-1+z=-3$;

$$\therefore z=-2$$

Hence $x=-1, y=2, z=-2, u=1$. Ans

EXAMPLES 110

1. $x+y=6,$

$$y+z=10,$$

$$z+x=8$$

2. $2x+y+z=7,$

$$x+2y+z=8,$$

$$x+y+2z=9$$

3. $x+y-z=a,$

$$x+z-y=b,$$

$$y+z-x=c.$$

4. $2x+3y-z=9,$

$$x+y-2z=-3,$$

$$3x+4y-3z=6.$$

5. $\frac{1}{y} + \frac{1}{z} - \frac{1}{x} = \frac{1}{a},$

$$\frac{1}{z} + \frac{1}{x} - \frac{1}{y} = \frac{1}{b},$$

$$\frac{1}{x} + \frac{1}{y} - \frac{1}{z} = \frac{1}{c}.$$

6. $2x + \frac{1}{y} - \frac{1}{z} = 4\frac{1}{2},$

$$\frac{2}{y} + \frac{1}{z} - x = -1\frac{1}{2},$$

$$\frac{2}{z} + x - \frac{1}{y} = 2\frac{1}{2}.$$

7. $ax+by=a+b,$

$$by+cz=b+c,$$

$$cz+ax=c+a.$$

8. $2x+3y=5,$

$$3y+4z=7,$$

$$2z+x=3.$$

$$9 \quad \frac{a}{x} + \frac{b}{y} = \frac{a}{y} + \frac{b}{z} = \frac{a}{z} + \frac{b}{x} = 1.$$

$$10. \quad \frac{b}{x} - \frac{c}{y} = \frac{b}{y} - \frac{c}{z} = \frac{b}{z} - \frac{c}{x} = \frac{1}{a}.$$

$$\begin{aligned} 11. \quad & x + y + 3z + 3u = 16 \\ & 3x + 3y + z + u = 24, \\ & x - 6y - 13z + u = -67, \\ & x - 2y + 7z + u = -2, \end{aligned}$$

$$\begin{aligned} 12 \quad & 2x - 4y + 3z - 5u = 0, \\ & 4x - 2y + 2z - u = 14, \\ & \frac{x-y}{3} = \frac{u+z}{5} = 1. \end{aligned}$$

198. Theorem

$$\text{If } ax + by + cz = 0, \dots (1)$$

$$\text{and } a'x + b'y + c'z = 0, \dots (2)$$

$$\text{then will } \frac{a}{c' - b'c} = \frac{y}{a' - c'a} = \frac{z}{ab' - a'b}.$$

This theorem is generally referred to as the **Rule of Cross Multiplication**

Multiply (1) by c' , then $ac'x + bc'y + cc'z = 0$; (3)

„ (2) by c , then $a'cx + b'cy + c'cz = 0$. (4)

Subtract (4) from (3), thus $(ac' - a'c)x + (bc' - b'c)y = 0$;

$$\therefore (c' - b'c)y = -'ac' - a'c)x = (a' - c'a)z;$$

$$\text{dividing each by } (b' - b'c)(a' - c'a), \quad \frac{y}{a' - c'a} = \frac{z}{ab' - a'b} \quad (5)$$

Similarly $\frac{y}{a' - c'a} = \frac{z}{ab' - a'b}$. Hence the theorem.

Rule To find the divisors of x , y and z , write down the coefficients of y and z , z and x , x and y in order, placing the corresponding coefficients in the two equations one under another, and enclosing the different sets by vertical lines. In each set, multiply the quantities diagonally, beginning with the first, and then subtract the second product from the first. The three remainders thus obtained are the divisors of x , y and z in order.

We shall clear up the rule still further.

The student should now note the order of the symbols, viz., y and z , z and x , x and y

Write down the coefficients of y and z , z and x , x and y as just directed, with vertical lines to enclose the different sets. Thus we have

First set

$$\left| \begin{array}{cc} b & c \\ b' & c' \end{array} \right|$$

2nd set

$$\left| \begin{array}{cc} c & a \\ c' & a' \end{array} \right|$$

3rd set.

$$\left| \begin{array}{cc} a & b \\ a' & b' \end{array} \right|$$

Now, in the first set multiply the diagonally opposite quantities; *i.e.*, *b* by *c'*, and *c* by *b'*. We then get *bc'* and *b'c*. Subtracting the second product from the first, we get, *bc' - b'c*, which we see is the denominator of the fraction $\left(\frac{x}{bc' - b'c}\right)$ of which *x* is the numerator.

The second set similarly treated gives us *a'c - c'a*, which is the divisor of *y* in $\frac{y}{a'c - c'a}$.

The third set treated in the same manner gives us *ab' - a'b*, which is the divisor of *z* in $\frac{z}{ab' - a'b}$.

N. B. It is practice rather than any rule that enables the student to make a correct use of the theorem. Its application is so very wide that it is extremely necessary to acquire the utmost facility in its use. We like here simply to mention that any order of symbols, say three in number, in which we pass from any one of them to a second, next from the second to the third, and lastly from the third to the first, is called a **cyclic order**. The reason for this nomenclature, is obvious from the consideration that if *a*, *b* and *c* be three points in order on the circumference of a circle, then in going round it starting from *a*, we pass from *a* to *b*, next from *b* to *c*, and then from *c* to *a* back.

$$\begin{array}{l} \text{Ex. 1. Solve } x - 2y + z = 0, \quad (1) \\ \qquad \qquad 9x - 8y + 3z = 0, \quad (2) \\ \qquad \qquad 2x + 3y + 5z = 36 \quad (3) \end{array} \quad \left. \vphantom{\begin{array}{l} (1) \\ (2) \\ (3) \end{array}} \right\} \text{C. U 1887.}$$

The first two equations are

$$1x - 2y + 1z = 0,$$

$$9x - 8y + 3z = 0.$$

Arrange the coefficients of *y* and *z*, *z* and *x*, and *x* and *y* thus :

$$\left| \begin{array}{cc} -2, & 1 \\ -8, & 3 \end{array} \right| \quad \left| \begin{array}{cc} 1, & 1 \\ 3, & 9 \end{array} \right| \quad \left| \begin{array}{cc} 1, & -2 \\ 9, & -8 \end{array} \right|$$

$$\text{Hence } -\frac{x}{2 \times 3 - (-8) \times 1} = \frac{y}{1 \times 9 - 1 \times 3} = \frac{z}{1(-8) - (-2)9};$$

$$\therefore \frac{x}{-6+8} = \frac{y}{9-3} = \frac{z}{-8+18};$$

(In practice this result is written down at once by performing the preceding operations mentally.)

$$i. e., \quad \frac{x}{2} = \frac{y}{6} = \frac{z}{10} = k, \text{ suppose } \dots\dots\dots(4)$$

$$\left. \begin{array}{l} \text{Now since by (4), } \frac{x}{20} = k, \therefore x = 2k; \\ \text{,, ,, } \frac{y}{6} = k, \therefore y = 6k; \\ \text{,, ,, } \frac{z}{10} = k, \therefore z = 10k. \end{array} \right\} \dots\dots\dots(5)$$

Substitute these values of x, y and z in (3) (which we have not as yet touched); then

$$\begin{aligned} 4k + 18k + 50k &= 36; \\ \text{i.e., } 72k &= 36; \\ \therefore k &= \frac{1}{2}. \\ \therefore \text{ from (5) } \left. \begin{array}{l} x = 2 \times \frac{1}{2} = 1, \\ y = 6 \times \frac{1}{2} = 3, \\ z = 10 \times \frac{1}{2} = 5. \end{array} \right\} \text{Ans.} \end{aligned}$$

Ex. 2. Solve immediately by Cross Multiplication

$$\left. \begin{array}{l} ax + by + c = 0, \\ a_1x + b_1y + c_1 = 0 \end{array} \right\}$$

We have

$$ax + by + c \times 1 = 0,$$

and

$$a_1x + b_1y + c_1 \times 1 = 0.$$

$$\therefore \left| \begin{array}{cc} x & y \\ b, c & c, a \\ b_1, c_1 & c_1, a_1 \end{array} \right| = \left| \begin{array}{cc} 1 & \\ a, b & \\ a_1, b_1 & \end{array} \right|$$

$$\text{i.e., } \frac{x}{b_1c - b_1c} = \frac{y}{c_1a - c_1a} = \frac{1}{ab_1 - a_1b};$$

$$\therefore x = \frac{bc_1 - b_1c}{ab_1 - a_1b}, \quad y = \frac{ca_1 - c_1a}{ab_1 - a_1b}. \quad \text{Ans.}$$

N. B. The above is the readiest Method of solving Simple Simultaneous Equations in two unknowns. Consult the other methods. Art. 190.

$$\left. \begin{array}{l} \text{Ex. 3. Solve } ax + by + cz = a + b, \\ bx + cy + az = b + c, \\ cx + ay + bz = c + a. \end{array} \right\} \text{M. U. 1879.}$$

By transposition the first equation becomes

$$ax + by + cz - a - b = 0;$$

$$\text{i.e., } a(x - 1) + b(y - 1) + cz = 0 \dots\dots\dots(1)$$

The 2nd equation similarly treated gives

$$b(x-1) + c'y-1 + az = 0 \dots\dots\dots(2)$$

Let $x-1 = x', y-1 = y'$

Then (1) and (2) reduce to

$$ax' + by' + cz = 0 \dots\dots\dots(3)$$

$$bx' + cy' + az = 0 \dots\dots\dots(4)$$

From (3) and (4),
$$\begin{vmatrix} x' & y' \\ b, c & c, a \end{vmatrix} = \begin{vmatrix} y' & z \\ c, a & a, b \end{vmatrix} = \begin{vmatrix} z & x \\ a, b & b, c \end{vmatrix}$$

i. e.,
$$\frac{x'}{ab - c^2} = \frac{y'}{bc - a^2} = \frac{z}{ac - b^2} = k, \text{ suppose.}$$

Then
$$\begin{aligned} x' &= k(ab - c^2), \\ y' &= k(bc - a^2), \\ z &= k(ac - b^2). \end{aligned}$$

But $\left. \begin{aligned} x' &= x-1 \\ y' &= y-1 \end{aligned} \right\} \therefore \left. \begin{aligned} x-1 &= k(ab - c^2), \\ y-1 &= k(bc - a^2), \\ z &= k(ac - b^2). \end{aligned} \right\} \dots\dots\dots(5)$

Substitute these values of x, y and z in the third of the given equations, viz., $cx + ay + bz = c + a$.

Then $c\{1 + k(ab - c^2)\} + a\{1 + k(bc - a^2)\} + b\{k(ac - b^2)\} = c + a$;

$\therefore c + k(ab - c^2) + a + k(bc - a^2) + k(ab - b^2) = c + a$;

$\therefore k(3abc - a^3 - b^3 - c^3) = 0$;

$\therefore k = 0$;

\therefore from (5), $x=1, y=1, z=0$. Ans.

N. B. Although **Ex. 1** gives the typical set of equations to which the Rule of Cross Multiplication is applicable, **Ex. 3** is meant to show how some other sets, apparently not amenable to the *Rule*, may sometimes be brought under it on their reduction by simple inspection to the form of **Ex. 1**.

EXAMPLES 111

Solve the following equations :

1. $x + y + z = 0,$

2. $2x + 4y + 5z = 0,$

$2x + 3y + 4z = 0,$

$7x + 5y + 4z = 0,$

$7x + 2y + 8z = 11.$

$9x + 3y + 4z = 8.$

$$\begin{aligned} 3. \quad & 4x - 3y + 5z = 0, \\ & 2x + y + 3z = 0, \\ & x + y + z + 6 = 0. \end{aligned}$$

$$\begin{aligned} 5. \quad & 2x + 4y + z = 0, \\ & 2y - 3x + 6z = 0, \\ & x + y + 2z = 39. \end{aligned}$$

$$\begin{aligned} 7. \quad & 11x + 9y = 0, \\ & 4x + y + 5z = 0, \\ & x + 2y + 3z = 35. \end{aligned}$$

(Put 1st eqn : $11x + 9y + 0z = 0$).

$$\begin{aligned} 4. \quad & 2x - y - z = 0, \\ & x - 2y - 3z = 0, \\ & 11x + 9y + 7z = 25. \end{aligned}$$

$$\begin{aligned} 6. \quad & 9x - 9y + z = 0, \\ & 3z - 13x + 8y = 0, \\ & (2x + 1)(3z + 1) = (3x + 7)(2z - 3) \end{aligned}$$

$$\begin{aligned} 8. \quad & 3y + 5z = 0, \\ & 8x + y - 9z = 0, \\ & x - \frac{5}{y} + \frac{1}{2} = 0 \end{aligned}$$

$$\begin{aligned} 9. \quad & x + y - z = 0, \\ & 2x + 3y = 5, \\ & x + y + 2z = 6 \end{aligned}$$

$$\begin{aligned} 10. \quad & x + y + z = 0, \\ & x + y = \frac{1}{2}(z + x), \\ & 2x + 3y + 4z = 9 \end{aligned}$$

$$\begin{aligned} 11. \quad & \frac{x}{2} + \frac{y}{4} = \frac{2z}{5}, \\ & \frac{x}{4} + \frac{y}{5} = \frac{3z}{100}, \\ & \frac{x}{5} + \frac{y}{6} = 1 + \frac{4z}{75}. \end{aligned}$$

$$\begin{aligned} 12. \quad & \frac{1}{2} + \frac{y}{5} - \frac{z}{6} = 0, \\ & \frac{6}{1} + \frac{1}{y} - \frac{3}{z} = \frac{1}{10}, \\ & \frac{y}{4} + \frac{4z}{5} - \frac{1}{2} = \frac{2}{15}z + \frac{3}{16}x. \end{aligned}$$

$$\begin{aligned} 13. \quad & \frac{2}{1} - \frac{3}{y} + \frac{7}{z} = 0, \\ & \frac{5}{x} - \frac{6}{y} - \frac{1}{2z} = 0, \\ & \frac{x}{4} + \frac{y}{3} + \frac{z}{6} = 3. \end{aligned}$$

$$\begin{aligned} 14. \quad & \frac{1}{2z} - \frac{1}{y} + \frac{3}{5z} = 0, \\ & \frac{1}{3x} + \frac{1}{4y} - \frac{1}{z} = \frac{4y - 3z}{12yz}, \\ & \frac{x + 2y + 5z}{11 + 5z + 1} = \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} 15. \quad & 2x + 3y + 1 = 0, \\ & 5x + 11y + 6 = 0. \end{aligned}$$

$$\begin{aligned} 16. \quad & 4x + 9y = 26, \\ & 7x + 8y = 30. \end{aligned}$$

$$\begin{aligned} 17. \quad & ax + by + cz = 0, \\ & x + y + z = 0, \\ & \frac{x}{b-c} + \frac{y}{a-c} + \frac{z}{a-b} = 1. \end{aligned}$$

$$\begin{aligned} 18. \quad & ax + by + cz = 0, \\ & (b+c)x + (c+a)y + (a+b)z = 0, \\ & a^2x + b^2y + c^2z + (b-c)(c-a)(a-b) = 0. \end{aligned}$$

19. $ax + by + cz = 0$,
 $x + y + z = (a-b)(b-c)(a-c)$,
 $a^2x + b^2y + c^2z = 0$.
20. $x + y + z = 0$,
 $ax + by + cz = 2abc - a^3 - b^3 - c^3$,
 $bx + cy + az = ca + ay + bz$.
21. $x + ay + a^2z = 0$,
 $y + bz + b^2x = 0$,
 $z + cx + c^2y = d$.
22. $ax + by + cz = 0$,
 $a_1x + b_1y + c_1z = 0$,
 $a_2x + b_2y + c_2z = 0$.
23. $\frac{b-c}{a}x + \frac{c-a}{b}y + \frac{a-b}{c}z = 0$,
 $\frac{r}{c+a} + \frac{y}{a+b} + \frac{z}{b+c}$
 $= \frac{x}{a+b} + \frac{y}{b+c} + \frac{z}{c+a}$,
 $\frac{r}{a} + \frac{y}{b} + \frac{z}{c} = \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}$.
24. $\frac{a}{x} + \frac{b}{y} - \frac{c}{z} = 0$,
 $\frac{x-y}{a+b} + \frac{y+z}{b+c} - \frac{z-x}{a-c}$
 $= \frac{2(a+b+c)}{(a+b)(b+c)}$,
 $\frac{a^3}{x} - \frac{b^3}{y} - \frac{c^3}{z} = c$.
25. $x + y + z = a + b + c$,
 $ax + by + cz = a^3 + b^3 + c^3$,
 $a^2x + b^2y + c^2z = a^3 + b^3 + c^3$.
26. $x + y + z = 6$,
 $ax + by + cz = a + 2b + 3c$,
 $(a-b)x + (b-c)y + (c-a)z = b + c - 2a$.
27. $x + y + z = a + b + c$,
 $(b-c)x + (c-a)y + (a-b)z = 0$,
 $(b^2-c^2)x + (c^2-a^2)y + (a^2-b^2)z = (b-c)(c-a)(a-b)$.
- [Since $a(b-c) + b(c-a) + c(a-b) = 0$, the second equation is the same as $(b-c)(x-a) + (c-a)(y-b) + (a-b)(z-c) = 0$.]
28. $ax + by + cz = 2a + 3b + 4c$,
 $mx + ny + rz = 2m + 3n + 4r$,
 $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$.
29. $a^2x + b^2y + c^2z = 3$,
 $a^3x + b^3y + c^3z = a + b + c$,
 $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}$.
30. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$, $\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, and
 $\left(\frac{1}{b} - \frac{1}{c}\right)x + \left(\frac{1}{c} - \frac{1}{a}\right)y + \left(\frac{1}{a} - \frac{1}{b}\right)z + abc\left(\frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{c} - \frac{1}{a}\right)\left(\frac{1}{a} - \frac{1}{b}\right) = 0$.

197. **The method of Indeterminate Multipliers.**
Suppose we are given the following equations :

$$\left. \begin{aligned} a_1x + b_1y + c_1z &= d_1, \dots\dots\dots(1) \\ a_2x + b_2y + c_2z &= d_2, \dots\dots\dots(2) \\ a_3x + b_3y + c_3z &= d_3, \dots\dots\dots(3) \end{aligned} \right\}$$

Multiply (1) by l ; then $a_1lx + b_1ly + c_1lz = d_1l$;(4)

„ (2) by m ; „ $a_2mx + b_2my + c_2mz = d_2m$ (5)

Add (4) and (5) to (3) ; thus we have

$$\begin{aligned} (a_1l + a_2m + a_3)x + (b_1l + b_2m + b_3)y + (c_1l + c_2m + c_3)z \\ = d_1l + d_2m + d_3, \dots\dots\dots(6) \end{aligned}$$

To find x determine l and m such that the coefficients of y and z in (6) may be zero.

$$\text{We have } \left. \begin{aligned} b_1l + b_2m + b_3 &= 0, \\ c_1l + c_2m + c_3 &= 0 ; \end{aligned} \right\} \dots\dots\dots(7)$$

and (6) reduces to $(a_1l + a_2m + a_3)x = d_1l + d_2m + d_3$;

$$\text{i.e., } x = \frac{d_1l + d_2m + d_3}{a_1l + a_2m + a_3} \dots\dots\dots(8)$$

$$\text{From (7), } \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \begin{vmatrix} l \\ m \end{vmatrix} = \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} \begin{vmatrix} l \\ m \end{vmatrix} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \begin{vmatrix} l \\ m \end{vmatrix}$$

$$\text{i.e., } \frac{l}{b_2c_3 - b_3c_2} = \frac{m}{b_3c_1 - b_1c_3} = \frac{1}{b_1c_2 - b_2c_1} ;$$

$$\therefore l = \frac{b_2c_3 - b_3c_2}{b_1c_2 - b_2c_1}, \quad m = \frac{b_3c_1 - b_1c_3}{b_1c_2 - b_2c_1}.$$

Substitute the values of l and m in (8) ; thus

$$\begin{aligned} x &= \frac{d_1 \frac{b_2c_3 - b_3c_2}{b_1c_2 - b_2c_1} + d_2 \frac{b_3c_1 - b_1c_3}{b_1c_2 - b_2c_1} + d_3}{a_1 \frac{b_2c_3 - b_3c_2}{b_1c_2 - b_2c_1} + a_2 \frac{b_3c_1 - b_1c_3}{b_1c_2 - b_2c_1} + a_3} \\ &= \frac{d_1(b_2c_3 - b_3c_2) + d_2(b_3c_1 - b_1c_3) + d_3(b_1c_2 - b_2c_1)}{a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)} \end{aligned}$$

The values of y and z can be similarly obtained. It is usual now to write down the values of y and z by comparison. For this purpose examine the denominator in the value of x . Observe that a_1, a_2 and a_3 are the coefficients of x in the given equations, and the first terms of *their multipliers* are respectively b_2c_3, b_3c_1 and b_1c_2 , so that the corresponding products are $a_1b_2c_3, a_2b_3c_1$ and

$a_3b_1c_2$. The order of the suffixes in these products is—1, 2, 3 ; 2, 3, 1 ; and 3, 1, 2. Further, if in the *denominator* of the value of x we *replace* a_1, a_2 and a_3 by d_1, d_2 and d_3 respectively, we get the *numerator*.

Let us now find y . We should now start with b_1, b_2 and b_3 , which are the coefficients of y . The products similar to those mentioned above are $b_1c_2a_3, b_2c_3a_1$ and $b_3c_1a_2$, the order of the suffixes being *cyclic*, as already explained. Note also that the symbols b, c, a , are conveniently taken in cyclic order. Hence the denominator in the present case is

$$b_1(c_2a_3 - c_3a_2) + b_2(c_3a_1 - c_1a_3) + b_3(c_1a_2 - c_2a_1).$$

To find the numerator replace b_1, b_2, b_3 by d_1, d_2, d_3 .

$$\text{Thus } y = \frac{d_1(c_2a_3 - c_3a_2) + d_2(c_3a_1 - c_1a_3) + d_3(c_1a_2 - c_2a_1)}{b_1(c_2a_3 - c_3a_2) + b_2(c_3a_1 - c_1a_3) + b_3(c_1a_2 - c_2a_1)};$$

$$\text{and } z = \frac{d_1(a_2b_3 - a_3b_2) + d_2(a_3b_1 - a_1b_3) + d_3(a_1b_2 - a_2b_1)}{c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)}.$$

EXAMPLES 112.

Determine x only in the following equations :

- | | |
|---------------------------|-------------------------|
| 1. $2x + 3y + 4z = 20,$ | 2. $4x + 9y + 3z = 26,$ |
| $4x + 5y + 6z = 32,$ | $2x + y + 2z = 13,$ |
| $6x + y + z = 11.$ | $x + 9y + z = 20.$ |
| 3. $x + 3y + z - 38 = 0,$ | 4. $9x + y + z = 18,$ |
| $x + 8y - 2z = 64,$ | $3x - y + 4z = 4,$ |
| $x - 3y + 3z = 0.$ | $y - z = 9 - 4x.$ |

Determine y only in the following equations :

- | | |
|-------------------------------|---------------------------|
| 5. $9y + 16z = 9 - 4x,$ | 6. $y + ax + a^2x = a^3,$ |
| $3y + 4z = 3 - 2x,$ | $y + bx + b^2x = b^3,$ |
| $2x + 2y + 3z = \frac{3}{2}.$ | $y + cx + c^2x = c^3.$ |

Solve the following equations :

- | | |
|---|--------------------------------|
| 7. $2x + 3y - z = 13,$ | 8. $z - x + 2y = 2,$ |
| $3x - 4y + 3z = 8,$ | $x - 4y + z + 4 = 0,$ |
| $x - y + 2z = 9.$ | $6x + 5y - 9z = 19.$ |
| 9. $x + y + z = \frac{2}{3},$ | 10. $8x + 9y + 100z = 12,$ |
| $2x - \frac{1}{3}y + 3z + 1 = 0,$ | $4x + 5y + 10z = \frac{1}{8},$ |
| $\frac{1}{2}x + \frac{1}{3}y + 9z = \frac{1}{4}.$ | $16x + 18y + 30z = 7.$ |

198. Independence of Equations. Take the equations $2x - 3y = 1$, and $8x - 12y = 4$. The first equation is easily seen to be satisfied when x and y are equal respectively to 2 and 1, or 3 and $\frac{5}{3}$, or 5 and 3, &c. The second equation also is readily seen to be satisfied by the same values of x and y . Thus there are no *two determinate values* of the unknowns satisfying the given equations, but innumerable pairs of values satisfying them both. This is due to the fact that *the second equation is not essentially different from the first, but results from the latter by only multiplying it by a constant, viz., 4.*

Hence the above equations are not *independent* of each other.

Now let us take the equations

$$2x - 5y + 3z = -1, \dots\dots\dots(1)$$

$$\text{and } 3x + 2y - 6z = 7, \dots\dots\dots(2)$$

Multiply (1) by 3 and (2) by 2, and add the resulting equations; thus

$$3(2x - 5y + 3z) + 2(3x + 2y - 6z) = 3 \times (-1) + 2 \times 7;$$

$$\text{simplifying, } 12x - 11y - 3z = 11, \dots\dots\dots(3)$$

It will be found that all the equations (1), (2) and (3) are satisfied by any of the following sets of values of x , y and z : 3, 2, 1; 11, 9, $7\frac{2}{3}$; &c. Thus there are no *three determinate values* of x , y and z from (1), (2) and (3), as is generally the case. This is because the third equation results from simply *multiplying* (1) and (2) by *certain constants*, viz., 3 and 2, *and then adding up the equations thus obtained*. Hence the equations (1), (2) and (3) do not form an *independent system*.

Ex. 1. Examine if the following equations form an independent system:

$$2x - 3y + 4z = 11, \dots\dots(1)$$

$$3x + 5y - 2z = 13, \dots\dots(2)$$

$$18x + 11y + 4z = 85, \dots\dots(3)$$

We shall examine if any of the given equations can be obtained from the other two by multiplying the latter by some constants, and then adding up the resulting equations.

Multiply (1) by l , and (2) by m , and add up the resulting equations. Thus $(2l + 3m)x + (5m - 3l)y + (4l - 2m)z = 11l + 13m \dots (4)$

Now find l and m so that the coefficients of x and y in (4) may be the same as in (3); thus

$$\left. \begin{array}{l} 2l + 3m = 18, \\ 5m - 3l = 11; \end{array} \right\} \text{whence we find } \left. \begin{array}{l} l = 3, \\ m = 4 \end{array} \right\}$$

\therefore by substitution, (4) becomes $18x + 11y + 4z = 11 \times 3 + 13 \times 4 = 85$, which is identical with (3).

Hence (3) results from multiplying (1) and (2) respectively by 3 and 4, and then adding up the equations so obtained. Therefore (1), (2) and (3) are *not an independent system*.

199. Consistency of Equations. Let us examine the equations $x+1=3$, and $3x+2=8$. Each is satisfied by $x=2$. Hence the two equations are *consistent*, i.e., *true for the same value of the unknown*.

Again, take the equations $ax+by=b$, $bx+ay=a$, and $x+y=1$. Evidently each is satisfied by $x=0$, and $y=1$. Hence the three equations are *consistent*.

Ex. Examine if the following equations are consistent :

$$\left. \begin{aligned} x+y+z &= 9, \dots (1) \\ 2x+3y+4z &= 29, \dots (2) \\ 3x+2y+5z &= 32, \dots (3) \\ 4x+3y+2z &= 25, \dots (4) \end{aligned} \right\}$$

First find x , y and z from *any three* of the four equations, and then examine if the *fourth* is satisfied by the values obtained.

From (1), (2) and (3), we get by the usual method $x=2$, $y=3$, $z=4$.

By substitution, (4) becomes $4 \times 2 + 3 \times 3 + 2 \times 4 = 25$, which is evidently true.

Hence the given equations are consistent. *Ans.*

Otherwise thus : Multiply (1), (2) and (3) by l , m and n respectively, and add the resulting equations ; thus

$$(l+2m+3n)x + (l+3m+2n)y + (l+4m+5n)z = 9l+29m+32n. (5)$$

Now find l , m and n so that the co-efficients of x , y and z in (5) may be the same as in (4). Thus

$$\left. \begin{aligned} l+2m+3n &= 4, \\ l+3m+2n &= 3, \\ \text{and } l+4m+5n &= 2; \end{aligned} \right\} \quad \text{whence } \left. \begin{aligned} l &= 6, \\ m &= -1, \\ n &= 0. \end{aligned} \right\}$$

By substitution, (5) becomes $4x+3y+2z=9 \times 6 + 29 \times (-1) + 32 \times 0 = 25$, which is *identical* with (4).

Hence the given equations are consistent.

EXAMPLES 113.

Do the following equations form independent systems ?

- | | |
|-------------------------|------------------------|
| 1. $2x - 6y = 11,$ | 2. $3x + 5y - z = 26,$ |
| $3x + y + z = 32,$ | $x - 3y + 9z = 20,$ |
| $7x - 17y + z = 54.$ | $2x + y + 4z = 23.$ |
| 3. $7x - 4y + 9z = 14,$ | 4. $x + y + z = 53,$ |
| $5x - 2y + 3z = 6,$ | $x + 2y + 3z = 105,$ |
| $x - y + 3z = 4.$ | $x + 3y + 4z = 134.$ |
| 5. $5x + 8y = 72,$ | 6. $ax + by = a + b,$ |
| $12y - 5z = 93,$ | $bx + cz = b + c,$ |
| $3x + 2z = 6.$ | $x + y + z = 3.$ |

Are the following equations consistent ?

- | | |
|----------------------|----------------------|
| 7. $x - 2y + z = 0,$ | 8. $x + y + z = 2,$ |
| $5x + 4y - 5z = 0,$ | $2x + y + 3z = 9,$ |
| $2x + y - z = 4,$ | $4x - y + 2z = 12,$ |
| $3x + 5y + 2z = 48.$ | $7x + 5y + 4z = 13.$ |

For what value of a are the following equations consistent

- | | |
|----------------------|--------------------------|
| 9. $x - y - z = 0,$ | 10. $4x - 2y + 5z = 18,$ |
| $4x - 3y + z = 0,$ | $2x + 4y - 3z = 22,$ |
| $3x + 2y + 4z = 16,$ | $6x + 7y - z = 63,$ |
| $ax - 4y + 6z = 4.$ | $5x - 3y + z = a.$ |

CHAPTER XXX.

SOME HIGHER SIMULTANEOUS EQUATIONS.

200. Mode of solution. Some equations of higher degree are easily reduced to simpler ones, and then solved.

Ex. 1. Solve
$$\left. \begin{aligned} ax + by &= mxy, \\ bx + ay &= nxy. \end{aligned} \right\}$$

Divide each of the given equations by xy ; thus

$$\frac{a}{y} + \frac{b}{x} = m \dots \dots \dots (1)$$

$$\frac{b}{y} + \frac{a}{x} = n \dots \dots \dots (2)$$

Multiply (1) by a ; then $\frac{a^2}{y} + \frac{ab}{x} = ma$;

„ (2) „ b „ $\frac{b^2}{y} + \frac{ab}{x} = nb$,

∴ by subtraction, $\frac{a^2 - b^2}{y} = ma - nb$.

$$\therefore \frac{1}{y} = \frac{ma - nb}{a^2 - b^2};$$

inverting, $y = \frac{a^2 - b^2}{ma - nb}$.

To find x :

multiply (2) by a ; then $\frac{ab}{y} + \frac{a^2}{x} = na$,

„ (1) by b ; „ $\frac{ab}{y} + \frac{b^2}{x} = mb$;

∴ by subtraction, $\frac{a^2 - b^2}{x} = na - mb$;

inverting, $\frac{1}{a^2 - b^2} = \frac{1}{na - mb}$;

$$\therefore \frac{a^2 - b^2}{na - mb}$$

$$\therefore x = \frac{a^2 - b^2}{na - mb}, \quad y = \frac{a^2 - b^2}{ma - nb}. \quad \text{Ans.}$$

Ex. 2. Solve $xyz = xy + xz - yz = 4(yz + xy - xz) = 6(xz + yz - xy)$

M U 1864.

We are given the equations,
$$\left. \begin{aligned} xy + xz - yz &= xyz, \dots\dots\dots(1) \\ 4(yz + xy - xz) &= xyz, \dots\dots\dots(2) \\ 6(xz + yz - xy) &= xyz, \dots\dots\dots(3) \end{aligned} \right\}$$

Dividing (1) by xyz , we get $\frac{1}{z} + \frac{1}{y} - \frac{1}{x} = 1 \dots\dots\dots(4)$

Dividing (2) by $4xyz$, we get $\frac{1}{x} + \frac{1}{z} - \frac{1}{y} = \frac{1}{4} \dots\dots\dots(5)$

Dividing (3) by $6xyz$, we get $\frac{1}{y} + \frac{1}{x} - \frac{1}{z} = \frac{1}{6} \dots\dots\dots(6)$

Now solve (4), (3) and (6).

$$\left. \begin{aligned} \text{Adding (4) and (5), } \frac{2}{z} = \frac{5}{4}, \quad \therefore \frac{z}{2} = \frac{4}{5}, \text{ and } z = \frac{8}{5}; \\ \text{,, (5) and (6), } \frac{2}{x} = \frac{5}{12}, \quad \therefore \frac{x}{2} = \frac{12}{5}, \text{ ,, } x = \frac{24}{5}; \\ \text{,, (6) and (4), } \frac{2}{y} = \frac{7}{6}, \quad \therefore \frac{y}{2} = \frac{6}{7}, \text{ ,, } y = \frac{12}{7}; \end{aligned} \right\}$$

$$\therefore x = \frac{24}{5}, y = \frac{12}{7}, z = \frac{8}{5}. \text{ Ans.}$$

Ex. 3. Solve $xy = 6, \dots (1)$
 $yz = 12, \dots (2)$
 $zx = 8, \dots (3)$

Multiply (1), (2) and (3) together; then

$$x^2 y^2 z^2 = 6 \times 12 \times 8 = 6 \times 6 \times 2 \times 8 = 6^2 \times 4^2;$$

extract the square root; thus

$$\text{either } xyz = 6 \times 4, \text{ or } xyz = -6 \times 4.$$

First let $xyz = 6 \times 4, \dots (4)$

Divide (4) by (1), (2) and (3) respectively;

$$\text{thus } z = 4, x = 2, y = 3.$$

2ndly, if $xyz = -6 \times 4 = -24, \dots (5)$

divide (5) by each of (1), (2) and (3); thus

$$z = -4, x = -2, y = -3.$$

Thus we have found two sets of values,

$$\left. \begin{aligned} \text{Either } x = 2, \\ y = 3, \\ z = 4; \end{aligned} \right\} \quad \text{or} \quad \left. \begin{aligned} x = -2, \\ y = -3, \\ z = -4. \end{aligned} \right\} \quad (\text{Verify}).$$

EXAMPLES 114.

1. $x + y = 6xy,$

$$2x + 3xy = 13xy.$$

2. $\frac{4}{x} - \frac{5}{y} = \frac{x+y}{xy} + 1\frac{2}{3},$

$$xy = \frac{35}{4}(y-x).$$

3. $xy + xz - yz = 2xyz,$

$$xz + yz - xy = 3xyz,$$

$$yz + xy - xz = 4xyz.$$

4. $x(y + 2z) = yz(3 + x),$

$$y(2x + 3z) = xz(4 + y),$$

$$z(3x + 4y) = xy(5 + z).$$

5. $\frac{xy + yz - xz}{5} = \frac{yz + xz - xy}{7} = \frac{zx + xy - yz}{1} = \frac{xyz}{12}.$

6. $ay + bx = 3xy$,
 $bx + cy = 5yz$,
 $cx + az = 4zx$.
7. $xy = 5$,
 $yz = 4$,
 $zx = 20$.
8. $x(y+z) = 22$,
 $y(z+x) = 40$,
 $z(x+y) = 42$.
9. $y+z = \frac{14}{x}$, $z+x = \frac{18}{y}$, $x+y = \frac{20}{z}$.
10. $x^3 - y^3 = a^3$, $x - y = b$.

201. We conclude the present chapter with a few examples as an additional exercise upon this as well as the preceding chapter. The student is required to find out the most convenient method in each case.

EXAMPLES 115.

1. $\frac{p}{x} + \frac{q}{y} = 0$,
 $px + qy = r$.
2. $\frac{x+6}{y} = \frac{3}{4}$, $\frac{x}{y-2} = \frac{1}{2}$.
3. $x+y+z=6$,
 $3x-y+2z=7$,
 $4x+3y-z=7$.
4. $\frac{2}{x} - \frac{3}{y} + \frac{4}{z} - \frac{73}{60} = 0$, $\frac{1}{x} + \frac{1}{y} - \frac{1}{z} + \frac{37}{60} = 0$,
 $\frac{5}{x} - \frac{2}{y} + \frac{3}{z} - \frac{4}{15} = 0$.
5. $\frac{2x+3y}{xy} = \frac{4}{x} - \frac{5}{y} + 20$,
 $3x+2y = 11xy$.
6. $x+y+z = 2a+3b+4c$,
 $ax+by+cz = 2a^3+3b^3+4c^3$,
 $(a+b)x+ay+bz = 2a^2+5ab+4bc$.
7. $x(by+cz) = a(b^3+c^3)$,
 $y(cz+ax) = b(c^3+a^3)$,
 $z(ax+by) = c(a^3+b^3)$.
8. $x+y+z = 0$,
 $y-8x-2z = 0$,
 $x-3y-4z = 7$.
9. $x^2 - y^2 = 12$, $x+y = 6$.
10. $x-y = 5 = \sqrt{x} + \sqrt{y}$.
11. $\frac{1}{2(x+1)} + 3y = 3\frac{1}{2}$, $2y(x+1) = 3x-1$.
12. $xyz = \frac{xy+yz-zx}{4} = \frac{2xy+3yz-7zx}{5} = \frac{3xy-4yz+5zx}{6}$.
13. $xyz = \frac{y+z}{b^3+c^3} = \frac{z+x}{c^3+a^3} = \frac{x+y}{a^3+b^3}$.

14. $\frac{2}{x+y} + \frac{3}{x-y} = 7$,
 $\frac{-4}{x-y} + \frac{5}{x+y} = 11$.
15. $\frac{1}{2} \left(\frac{3x}{x+y} - y \right) - \frac{1}{8} \left(\frac{7x}{x+y} - 3y \right)$
 $= \frac{x}{12(x+y)} - \frac{y}{30} = \frac{1}{10}$.
16. $\frac{x}{a+b} + \frac{y}{b+c} + \frac{z}{c+a} = 1$,
 $(a-b)x + (b-c)y + (a-c)z = 0$,
 $cx + ay + bz = 2ac$.
17. $\frac{2x-3}{5} + \frac{4x-5y}{x+y-3} = \frac{1+4x}{10}$,
 $\frac{y-1}{8} + \frac{5y-1}{x+y+3} = \frac{y+11}{8}$.
18. $(x+1)(2x+y+7)$,
 $= (x+1)(2x+3) + (y+1)(x+2)$,
 $x+y+6=0$.
19. $2^x 2^{x-2y} = 4$,
 $2^{2x} = 8 \cdot 2^{2y}$,
 $2^{x+y+2} = 64$.
20. $3xy - 3yz + 2zx = 0$, $5yz - 4xy - 4zx = 0$, $x+y=7$.
21. $\frac{a}{x} + \frac{2b}{y} - \frac{2c}{z} = 0$, $\frac{2x+3y}{3x+2y} = \frac{6(a+b)}{9a+4b}$,
 $\left(\frac{1}{a} - \frac{1}{b} \right) \frac{1}{4z} + \left(\frac{1}{c} - \frac{1}{a} \right) \frac{1}{3y} + \left(\frac{1}{b} - \frac{1}{c} \right) \frac{1}{2x} = 0$.
22. $\frac{\{x + \sqrt{(x^2 - y^2)}\}^{\frac{3}{2}}}{\{x - \sqrt{(x^2 - y^2)}\}^{\frac{3}{2}}} = 8$, $\frac{x(y+z)}{yz} = 16\frac{1}{4}$,
 $\frac{y-z}{\sqrt{(x+y)} + \sqrt{(x+z)}} = \frac{11}{3\sqrt{3+4}}$.

CHAPTER XXXI.

PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS INVOLVING
TWO OR MORE UNKNOWN QUANTITIES.

202. **Mode of Solution.** Express symbolically *all the conditions* of the problem, and solve the resulting equations.

Ex. 1. Four times a certain number added to five times another equal 50, and five times the first diminished by four times the second equal 1, find the numbers.

Let x and y denote the numbers.

Then by the first condition of the problem, $4x + 5y = 50$, (1)
 and " " second " " " " " " $5x - 4y = 1$. (2)

N. B. Note that $x + \frac{y}{2}$ is different from $x \times \frac{y}{2}$, which means $x \times \frac{y}{2}$.

Ex. 3. A pound of tea and three pounds of sugar cost 6 shillings, but if sugar were to rise 50 per cent., and tea 10 per cent. in price, they would cost 7s. Find the price of tea and sugar.
B. U. 1866.

Let the price of tea = x shillings per lb,
and " " " " sugar = y shillings per lb.

Then the price in shillings of 1 lb. of tea + that of 3 lbs. of sugar
= $(x + 3y)$;

\therefore by the question, $x + 3y = 6$(1).

The prices being supposed to rise 10 per cent. and 50 per cent.

respectively, the price in shillings of 1 lb. of tea = $x(1 + \frac{10}{100}) = \frac{11x}{10}$,

and that of 3 lbs. of sugar = $3y(1 + \frac{50}{100}) = \frac{9y}{2}$;

\therefore the total price = $(\frac{11x}{10} + \frac{9y}{2})$ shillings

\therefore by the question $\frac{11x}{10} + \frac{9y}{2} = 7$(2)

Multiply (1) by 3 ; thus $3x + 9y = 18$.

Multiply (2) by 2 ; thus $\frac{11x}{5} + 9y = 14$ }
5

\therefore by subtraction, $3x - \frac{11x}{5}$, i.e., $\frac{4x}{5} = 4$;

$\therefore x = 5$.

\therefore substituting 5 for x in (1), $5 + 3y = 6$; $\therefore y = \frac{1}{3}$.

\therefore the required price of tea = 5s. per lb
and that of sugar = $\frac{1}{3}$ s. or 4d. per lb. } *Ans.*

Ex. 4 Two passengers have together 5 cwt. of luggage, and are charged for the excess above the weight allowed 5s. 2d. and 9s. 10d. respectively ; but if the luggage had all belonged to one of them, he would have been charged 19s. 2d. How much luggage is each passenger allowed to carry free of charge, and how much luggage had each passenger ?
C. U. 1877.

Let x cwt. = the luggage each is allowed to carry free of charge,
and y cwt. = the whole luggage with the 1st man ;

then $(5 - y)$ cwt. = " " " " 2nd man.

The 1st man pays 5s. 2d. for $(y-x)$ cwt.,

$$\therefore \text{the charge per cwt.} = \frac{62}{y-x}d.$$

The 2nd pays 9s. 10d. for $(5-y-x)$ cwt.,

$$\therefore \text{the charge per cwt.} = \frac{118}{5-y-x}d.$$

Had the luggage all belonged to one man, he would have had to pay for $(5-x)$ cwt., and the total charge in this case being 19s 2d,

$$\text{the charge per cwt} = \frac{19s \ 2d}{5-x} = \frac{230}{5-x}d.$$

Since the charge per cwt. is the same in every case,

$$\frac{62}{y-x} = \frac{118}{5-y-x} = \frac{230}{5-x} \dots \dots \dots (1)$$

$$\text{Since } \frac{62}{y-x} = \frac{118}{5-y-x},$$

$$\text{each fraction} = \frac{\text{sum of numrs.}}{\text{sum of denrs.}}$$

$$= \frac{62 + 118}{y-x + 5-y-x} = \frac{180}{5-2x} \dots \dots \dots (2)$$

$$\therefore \text{from (1) and (2), } \frac{180}{5-2x} = \frac{230}{5-x}, \text{ i.e., } \frac{18}{5-2x} = \frac{23}{5-x};$$

$$\therefore 18(5-x) = 23(5-2x);$$

$$\therefore 90 - 18x = 115 - 46x.$$

$$\therefore 28x = 25,$$

$$\therefore x = \frac{25}{28}$$

$$\text{Since from (1), } \frac{62}{y-x} = \frac{230}{5-x},$$

$$\text{by substitution for } x, \frac{62}{y - \frac{25}{28}} = \frac{230}{5 - \frac{25}{28}} = \frac{230}{\frac{119}{28}} = \frac{230 \times 28}{119} = 56;$$

$$\therefore 56(y - \frac{25}{28}) = 62;$$

$$\text{i.e., } 56y - 50 = 62;$$

$$\therefore 56y = 112, \therefore y = 2, \text{ and } 5-y = 3.$$

\therefore the luggage allowed to each = $\frac{25}{28}$ cwt. or 100 lbs, } Ans.
and one has 2 cwt., and the other 3 cwt., of luggage.

Ex. 5. *A* and *B* went out to shoot. *A* shot 3 pheasants for every 5 partridges, and *B* 5 pheasants for every 9 partridges. *A* shot 4 birds to *B*'s 5; how many pheasants, and how many partridges had they brought down when they had shot 126 birds?

M. U. 1866.

Let $3x$ = the number of pheasants brought down by *A*,

and $5y$ = " " " " " " " " *B*.

Now, *A* shot 5 partridges for every 3 pheasants;

\therefore " " $5x$ " " " $3x$ pheasants;

\therefore the number of partridges shot by *A* = $5x$,

and the total number shot by him = $5x + 3x = 8x$.

For a similar reason, while *B* shot $5y$ pheasants, he brought down $9y$ partridges, so that the total number shot by him = $5y + 9y = 14y$.

Since, the two together shot 126 birds, $8x + 14y = 126$(1)

Since on the whole *A* shot 4 birds to *B*'s 5,

$8x$ and $14y$ must be in the ratio of 4 to 5, so that $\frac{8x}{14y} = \frac{4}{5}$(2)

From (2), $56y = 40x$, $\therefore y = \frac{5x}{7}$(3)

\therefore by substitution for y in (1), $8x + 14 \times \frac{5x}{7} = 126$,

$\therefore 18x = 126$, and $x = 7$.

\therefore from (3), $y = \frac{5}{7} \times 7 = 5$.

Hence the number of pheasants shot down by *A* = $3x = 21$, }

and " " of partridges " " " $A = 5x = 35$; }

the number of pheasants " " " $B = 5y = 25$, }

and " " of partridges " " " $B = 9y = 45$. }

\therefore the total no. of pheasants = 46, and that of partridges = 80. *Ans.*

Ex. 6. Two men *A* and *B* are employed on a piece of work which has to be finished in 14 days. In 3 days they do $\frac{1}{4}$ th of the work, and then *A*'s place is taken by *C*. *B* and *C* work for one day, and do $\frac{1}{10}$ th of the whole work, and then *B*'s place is taken by *A*. *A* and *C* finish the work a day before the appointed time. Find the time in which the work could have been done by each working separately. M. U. 1886.

Let x = the number of days in which *A* can do the work,

y = " " " " " " *B* " " " " " ,

and z = " " " " " " *C* " " " " " .

Let w = the whole work.

$$\left. \begin{aligned} \therefore \text{ the daily work of } A &= \frac{w}{x}, \\ \text{,, ,, ,, of } B &= \frac{w}{y}, \\ \text{and ,, ,, ,, of } C &= \frac{w}{z}. \end{aligned} \right\}$$

Since A and B do $\frac{1}{2}$ th of the work in 3 days, $3\left(\frac{w}{x} + \frac{w}{y}\right) = \frac{w}{5}$. (1)

Since B and C do $\frac{1}{20}$ th of the work in 1 day, $\frac{w}{y} + \frac{w}{z} = \frac{w}{20}$. (2)

Since the work is finished a day earlier, i.e., in 13 days,

C and A do $w - \left(\frac{w}{5} + \frac{w}{20}\right)$ in $(13 - 3 - 1)$ or 9 days

$$\therefore 9\left(\frac{w}{x} + \frac{w}{z}\right) = w - \left(\frac{w}{5} + \frac{w}{20}\right) = \frac{7w}{4} \dots \dots \dots (3).$$

Divide (1) by $3w$, then $\frac{1}{x} + \frac{1}{y} = \frac{1}{15}$(4).

Divide (2) by w ; then $\frac{1}{y} + \frac{1}{z} = \frac{1}{20}$(5).

Divide (3) by $9w$; then $\frac{1}{x} + \frac{1}{z} = \frac{1}{12}$(6).

Add (4), (5) and (6), and take half the sum on each side

$$\begin{aligned} \text{Thus } \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{1}{2} \left(\frac{1}{15} + \frac{1}{20} + \frac{1}{12} \right) \\ &= \frac{4+3+5}{2 \times 60} = \frac{1}{10} \dots \dots \dots (7) \end{aligned}$$

From (7) subtract each of (4), (5) and (6), then

$$\frac{1}{z} = \frac{1}{10} - \frac{1}{15} = \frac{1}{30}; \quad \frac{1}{x} = \frac{1}{10} - \frac{1}{20} = \frac{1}{20}; \quad \frac{1}{y} = \frac{1}{10} - \frac{1}{12} = \frac{1}{60};$$

$$\therefore z = 30, \quad x = 20, \quad y = 60.$$

$\therefore A, B$ and C can do the work in 20, 60 and 30 days respectively.

Ans

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*N. B. It is evident that we might have begun by representing w by 1.

Ex. 7 A train running from A to B meets with an accident 50 miles from A , after which it moves with $\frac{2}{3}$ ths of its original velocity and arrives at B 3 hours late. Had the accident happened

50 miles further on, it would have been only 2 hours late. Find the distance from A to B , and the original speed of the train.
M. U. 1857.

Let the distance from A to $B = x$ miles,

and the original velocity of the train $= y$ miles per hour.

\therefore the usual time (without any accident) $= \frac{x}{y}$ hours.

The accident occurring 50 miles from A , the train runs the first 50 miles at the rate of y miles per hour, and the remainder, $(x - 50)$ miles, at the rate of $\frac{2}{3}y$ miles per hour, and the time taken $= \left(\frac{50}{y} + \frac{x - 50}{\frac{2}{3}y} \right)$ hours; and since the train is in this case 3 hours late, the time just found $=$ the usual time $+ 3$ hours $= \left(\frac{x}{y} + 3 \right)$ hours.

$$\therefore \frac{50}{y} + \frac{x - 50}{\frac{2}{3}y} = \frac{x}{y} + 3 \dots \dots \dots (1).$$

Supposing the accident to have occurred 100 miles from A , and the train to have been 2 hours late, we shall get the equation,

$$\frac{100}{y} + \frac{x - 100}{\frac{2}{3}y} = \frac{x}{y} + 2 \dots \dots \dots (2).$$

Subtracting (2) from (1), we have $\frac{-50}{y} + \frac{50}{\frac{2}{3}y} = 1$.

Multiply both sides by y ; then $y = -50 + \frac{50}{\frac{2}{3}} = 50(-1 + \frac{3}{2}) = 1\frac{1}{2} \times 50 = 37\frac{1}{2}$.

Multiply (1) by y ; then $50 + \frac{2}{3}(x - 50) = x + 3y$;

by substitution for y , $50 + \frac{2}{3}(x - 50) = x + 3 \times 37\frac{1}{2} = x + 112\frac{1}{2}$

\therefore by transposition, $\frac{2}{3}x - x = 112\frac{1}{2} - 50 + \frac{2}{3} \times 50 = 1\frac{1}{2} \times 50$.

$$\therefore \frac{2}{3}x - x = 112\frac{1}{2} - 50 + \frac{2}{3} \times 50 = 1\frac{1}{2} \times 50$$

$$\therefore x = 200. \text{ [Verify the results.]}$$

$\therefore AB = 200$ miles, and the required rate $= 37\frac{1}{2}$ miles per hour.

Ans.

Ex. 8. A criminal having escaped from prison, travelled 10 hours before his escape was known. He was pursued so as to be gained upon 2 miles an hour. After his pursuers had travelled 3 hours, they met an express going at the same rate as themselves, who met the criminal 2 hours 24 minutes before. In what time after the commencement of the pursuit will they overtake him? B. U. 1883.

Let the pursuit last for x hours.

Suppose the criminal to go at the rate of y miles per hour ; then the pursuers go at the rate of $(y+3)$ miles per hour, as also the express.

Since in x hours the pursuers go as far as the criminal in $(x+10)$ hours, $x(y+3)=(x+10)y$(1). [Distance = time \times rate.]

A ————— B ————— C

Suppose A to be the starting point, B the point where the pursuers meet the express, and C the point where the express met the criminal. When the express arrives at B , the pursuers have been travelling for 3 hours, and the criminal has been travelling for $(10+3)$ or 13 hours ; and since the criminal was met by the express $2\frac{2}{3}$ hours before the time of its arrival at B ,

$$AC = \text{the distance passed over by the criminal in } 13 - 2\frac{2}{3} \text{ hours} \\ = (13 - 2\frac{2}{3})y, \text{ i. e., } \frac{35}{3}y \text{ miles.}$$

$$AB = 3 \text{ hour's run of the pursuers} = 3(y+3) \text{ miles,}$$

$$\text{and } BC = 2\frac{2}{3} \text{ " " " " express} = \frac{10}{3}(y+3) \text{ miles.}$$

$$\text{Since } AC = AB + BC, \quad \frac{35}{3}y = 3(y+3) + \frac{10}{3}(y+3) = \frac{27}{3}y + \frac{46}{3};$$

$$\text{multiplying by 3,} \quad 35y = 27y + 46;$$

$$\therefore 8y = 46;$$

$$\therefore y = \frac{23}{4};$$

$$\text{from (1), cancelling } xy, \quad 3x = 10y = 10 \times \frac{23}{4} = \frac{5 \times 81}{13};$$

$$\therefore x = \frac{5 \times 81}{3 \times 13} = \frac{5 \times 27}{13} = 10\frac{5}{13}.$$

\therefore the required time = $10\frac{5}{13}$ hours. *Ans.*

Ex. 9. A challenged B to ride a bicycle race of 1040 yds ; he first gave B a start of 120 yds. and lost by 5 seconds ; he then gave B 5 second's start and won by 120 ft. How long does each take to ride the distance ? C. U. 1881.

Let A take x seconds and B , y seconds to ride the whole distance.

In the first race A rode 1040 yds, and B $(1040 - 120)$ or 920 yds, and since A lost by 5 sec, and therefore took a longer time,

$$A's \text{ time of riding } 1040 \text{ yds.} = B's \text{ time of riding } 920 \text{ yds.} + 5 \text{ sec.}$$

$$\therefore x = \frac{920}{1040}y + 5 = \frac{23}{26}y + 5 \dots\dots\dots(1).$$

In the second race A rode 1040 yds. and B $1040 \text{ yds.} - 120 \text{ ft.}$ or 1000 yds., and since B had been given a start of 5 sec, he was out 5 sec. longer.

$\therefore A$'s time of riding 1040 yds. + 5 sec. = B 's time of riding 1000 yds.

$$\therefore x + 5 = \frac{1040}{1040}y = \frac{2}{5}y \dots \dots (2)$$

Subtracting (1) from (2), we have $5 = \frac{2}{26}y - 5 = \frac{y}{13} - 5$.

$$\therefore y = 130$$

$$\therefore \text{from (1), } x = \frac{2}{26} \times 130 + 5 = 115 + 5 = 120.$$

$\therefore A$ takes 120 sec. or 2 min, B 130 sec. or 2 min 10 sec. *Ans*

Ex. 10. A boat goes up-stream 30 miles and down-stream 44 miles in 10 hours; it also goes up stream 40 miles and down-stream 55 miles in 13 hours: find the rate of the stream and of the boat C. U. 1880.

Let the rate of the stream = x miles per hour,

and that of the boat, i. e., the rate at which it can be rowed on still water = y miles per hour.

\therefore the rate of motion up stream = $(y - x)$ miles per hour,

and " " " " down-stream = $(y + x)$ " " " ;

\therefore the time taken to go 30 miles up-stream = $\frac{30}{y-x}$ hours,

and " " " " 44 " down-stream = $\frac{44}{y+x}$ " ;

and since the total time = 10 hours, $\frac{30}{y-x} + \frac{44}{y+x} = 10 \dots \dots (1)$

In like manner, we get $\frac{40}{y-x} + \frac{55}{y+x} = 13 \dots \dots (2)$

Subtract 3 times (2) from 4 times (1), thus $\frac{11}{y+x} = 1$, $\therefore y+x = 11$. (3)

Subtract 5 times (1) from 4 times (2); thus $\frac{10}{y-x} = 2$, $\therefore y-x = 5$. (4)

Add (4) to (3), and take (4) from (3); thus $2y = 16$, $2x = 6$;

$$\therefore y = 8, \quad x = 3.$$

\therefore the rate of the stream = 3 miles per hour.
and that of the boat = 8 miles per hour. } *Ans.*

EXAMPLES 110.

✓ 1. Find two numbers such that twice one of them plus thrice the other will yield 54, and thrice the first plus twice the second number will yield 51.

2. 4 lbs. of tea and 3 lbs. of coffee together cost 19s. 6d, and 3 lbs of tea and 2 lbs of coffee together cost 13s. 6d; find the prices per lb. of tea and coffee.

3. A number is divided into two parts such that thrice one part exceeds four times the other by 7, and if each part be increased by 1, the larger divided by the smaller gives $\frac{5}{2}$ as quotient; find the number.

4. A says to B: Two-fifths of my salary is $\frac{4}{5}$ of yours, and the difference between our salaries is Rs 600. What is A's salary?

5. What fraction is that which, if 3 be added to each of the numerator and denominator, equals $\frac{1}{5}$, and if 3 be subtracted from each of them, equals $\frac{1}{4}$? (Let x/y be the fraction).

6. What fraction is that which is equal to $\frac{3}{5}$, and of which the numerator exceeds 20 per cent of the denominator by 6?

7. A and B together own 50 horses in unequal numbers. If A has 10 more horses than B, how many has each?

8. If 7 cows and 11 sheep be worth Rs. 162, and 5 cows and 7 sheep be worth Rs. 114, find the value of each kind of animal.

9. A man paid a bill of Rs. 20 with 8-anna and 4-anna pieces and used in all 66 coins; how many coins of each sort did he give?

10. Two boys have Rs. 12. 8 as between them; one spends $\frac{4}{5}$ of his money, and the other $\frac{2}{3}$ of his own, and then they have only Rs. 3. 2 as; how much has each at first?

11. A farmer sold to one person 9 horses and 7 cows for Rs. 3000, and to another 6 horses and 13 cows at the same prices and for the same sum; what was the price of a horse and of a cow?

12. A lump composed of gold and silver measures 6 cubic inches and weighs 100 oz; if a cubic inch of gold weighs 20 oz., and an equal bulk of silver 12 oz, find the weight of each metal in the mixture.

13. In a company of 100 people of whom some are rich and some are poor, the rich subscribe and give 1a 3p. to each poor man; this costs the rich men 7as. 1p. each; how many rich and how many poor men are there?

14. If A were to give B one-third of his money, the latter could pay off a debt of Rs 63; the same debt could be paid off by A, if B assisted him with three-eighths of his own money. How much had each?

15. A fruiterer spent Rs. 11. 8as. in buying mangoes, some at Rs. 10 and the rest at Rs. 15 per hundred; he sold them for

Rs. 17. 2as. and gained one anna per mango ; how many of each kind did he buy ?

16. The expenses of a family when rice is 12 seers for a rupee are Rs. 50 a month ; when rice is 14 seers for a rupee the expenses are Rs. 48 a month, other expenses remaining unalterable ; what will they be when rice is 16 seers for a rupee ?

17. Divide Rs. 750 among A , B and C , so that if A were to give Rs. 100 to each of B and C , his share would be half as much as B 's, or one-third as much as C 's share.

18. Divide £226 among A , B and C , so that as often as A receives £3, B will receive £5, and as often as B receives £6, C will receive £7.

19. 8 men, 10 women and 12 children together earn Rs. 11. 2as ; each man earns 1a. more than a child and a woman together, and the men altogether earn as much as 8 women and 10 children together. How much does each man earn ?

20. A gentleman paid off his workmen engaged for a definite sum of money ; had there been 4 more workmen, each would have received 1 anna less than he did ; if there had been 4 fewer men, each would have received $1\frac{3}{4}$ as more. Find the number of men and the amount for which they were engaged.

21. A person bought a number of apples for a certain sum ; had the price apiece been 8 annas higher a dozen, he would have got 2 less for the same sum ; but if the price fell by the same amount, he would have got 6 more. Find the number of apples bought and the amount spent.

22. A man sold a horse and a carriage together for Rs. 562, the horse at a profit of 10 per cent., and the carriage at a profit of 16 per cent. ; if he had sold each at a profit of 15 per cent., he would have got Rs. 13 more. Find the prime cost of each.

23. a lbs. of inferior tea are mixed with b lbs. of superior tea, and by selling the mixture at c shillings per lb. the grocer cleared $33\frac{1}{3}$ per cent. on his outlay ; had the proportion of the different sorts of tea in the same total quantity of the mixture been reversed, he would have lost 25 per cent. Find the prime cost per lb. of each sort.

24. There are three numbers such that the sum of the first and second equals $\frac{7}{4}$ of *their* product ; the sum of the second and third equals $\frac{9}{10}$ of *their* product ; and the sum of the first and third equals $\frac{1}{8}$ of *their* product. Find the numbers.

25. The expression $ax - 5b$ is equal to 3 when x is 7, and to 27 when x is 13 ; what is its value when x is 9, and for what value of x is it zero ?

26. A man allowed a pauper 3s. 6d. per week until the number of shillings left was the same as the number of weeks he had been paying; he then increased the allowance to 4s. per week; had he continued his former rate, his money would have lasted a week longer. Find how much he gave away.

27. A person taking two tickets (a first and a second class) from Norwich to Stonemarket receives 7s. 6d. change out of a sovereign; how much had he to pay for each ticket separately, supposing that the first and second class fares from Norwich to Diss are 3s. 6d. and 2s. 9d. respectively? Of course the fares throughout are supposed proportional to the distances.

28. Two vessels *A* and *B* contain different mixtures of wine and water, the one in the proportion of 2 : 5, and the other in that of 5 : 9. What quantity must be taken from each to form a mixture which shall contain 5 quarts of wine and 12 of water?

29. In a mixed number the fractional part equals $\frac{1}{4}$ when the denominator is increased by 1; the integral part exceeds the numerator of the fractional part by 3; and if the integral part and the numerator be interchanged, the entire number is diminished by $2\frac{1}{4}$. Find the number.

30. Being asked to find the value of the *algebraical fraction* x^y_z by ascribing to *x*, *y* and *z* certain given values, a boy interprets it like $3\frac{2}{5}$ and gives a value exceeding the correct one by $4\frac{3}{5}$; the given value of *z* exceeds that of *y* by 2, and the sum of the values of *x*, *y* and *z* is 18. What is the correct answer wanted from the boy?

31. A man was 24 years old when his first child was born; some years hence the wife's age will be double that of the child, and in 4 years more the sum of the ages of the man and the wife will be four times the age of the child. Find the age of each at the last time.

32. One-third the sum of the ages of some children exceeds one-fourth of their number by 2, and 2 years hence the united ages of the children will be 17. Find the number of children.

[If *x* = number of children, and *y* years = their united ages now, their united ages 2 years hence will = $y + 2x$ (2 years being added for each child)].

33. The united ages of a man and his wife are six times the united ages of their children. Two years ago their united ages were ten times the united ages of the children, and six years hence their united ages will be three times the united ages of the children. How many children have they?

34. The dimensions of a rectangular court are such that if the length were increased by 3 yds., and the breadth diminished by the same, its area would be diminished by 18 sq. yds.; and if its length were increased by 3 yds., and its breadth increased by the same, its area would be increased by 60 sq. yds.; find the dimensions.

35. If a rectangular plot of land were 1 ft. longer and 2 ft. less broad, its area would have been 13 sq. ft. less; if it were a square of the same perimeter, its area would have been 1 sq. ft. greater. Find the area.

36. A room of which the floor is rectangular is such that the addition of a foot to the height will increase the area of the walls as much as the addition of a foot to both the length and breadth, the increase in each case being 60 sq. ft.; and if the floor be made square, the perimeter remaining the same as before, its area will be increased by 9 sq. ft. Find the length, breadth and height of the room. M. U. 1868.

37. An officer travels by a train with an assistant and a servant, himself in a first class carriage, the assistant in a second class, and the servant in a third class carriage. They have together 20 cwt. of luggage. The weight the third class passenger is allowed to carry free of charge = half the weight allowed to the second class passenger = one-third the weight allowed to the first class passenger. The officer's luggage falling short of the weight allowed, he is charged nothing for luggage, but the assistant and the servant have to pay 2s. each. Had the whole luggage belonged to the servant, he would have been required to pay 16s., and had it belonged to the assistant, he would have been charged 12s. How much luggage has each?

38. *A* and *B* play four games of chance, of which *A* wins the first and last, and *B* the other two. The amount which each stakes for the first game is half the whole sum of money possessed by both together, and for the other games half the money possessed by the loser of the preceding game. At the end of the fourth game *A* finds that he has 18s. less than he would have had, if he had won them all, and *B* finds that he has 9s. less than he had at starting. Find the amount of money possessed by each at first.

39. There are 1000 men in a regiment, who are arranged into two hollow squares, 2 and 4 deep respectively, one square being posted by the side of the other so as to present an unbroken front line. On 500 more men coming in, the whole party is rearranged into a hollow square, 5 deep, having a front line of the same length as before. Find the number of men in each of the first arrangements.

40. A sets about a piece of work, and after 3 days is joined by B ; they work together for 5 days, at the end of which the work is found to have been half done, and in 7 days more the work is finished. How long will each take to do it alone?

41. A piece of work can be done by a men and b women in c days, and also by b men and a women in d days. In what time can a man and a woman do it together?

42. A , B and C can do a piece of work in $5\frac{1}{2}$ days. A works on it for 2 days, after which he is joined by B ; A and B then work together for 4 days, at the end of which $\frac{3}{4}$ of the work is found to have been accomplished: if A and B worked together from the first, they could have done $\frac{2}{3}$ of the work in 6 days. How long before the completion of the work should C join A and B in order that the work may be finished in $7\frac{1}{2}$ days altogether?

43. There are two equal pipes to fill a cistern, having a leak in the bottom; if all are open for 20 minutes, the cistern is filled; if the leak be stopped, it takes 15 minutes in filling. How long will it take in filling, if only one pipe be stopped?

44. A letter-carrier has to go daily from P to Q in a prescribed time. If he goes a mile an hour faster than his ordinary rate, he arrives at Q half an hour before the time. But if he goes a mile an hour slower, he arrives three-quarters of an hour too late. Find his ordinary rate and the distance from P to Q .

45. A ship bound for a distant port sails at a uniform rate to the next coaling station, where it is detained for 3 days. The remainder of the voyage, which is $\frac{1}{4}$ of the whole, is accomplished at $\frac{1}{2}$ of the former rate, and the whole time taken from starting is 13 days. Had the coaling station been 100 miles further on, the ship might have stayed there 2 days longer and yet reached the end of the voyage in the same time. Find the length of the voyage.

46. A man performs a journey from A to B , part of which is up-hill and part down-hill, in a hours, and returns from B to A in b hours: if the up-hill and down-hill rates of motion be respectively c and d miles per hour, find the lengths of the two parts of the road.

47. A and B are two places 12 miles apart. Two persons starting at the same time from A and B towards each other meet in 4 hours; had they walked in the same direction, they would have met in 6 hours; find the rate of each.

48. A starts from P towards Q at 10 A. M., and B starts from Q towards P at 10-30 A. M.; they pass each other at 11-45 A. M. B on reaching P turns back and overtakes A , who continues walking in the same direction, at 1-30 P. M. Find the rates of walking of A and B , the distance PQ being $7\frac{1}{2}$ miles.

49. A starts from a place, and 5 hours later B starts from the same place in the same direction, walking a mile faster per hour than A ; after walking for 3 hours B meets C , who walks towards, and as fast as, B , and who met A 2 hours before. Find the time that B takes to catch A .

50. A bicyclist going from A to B has a bad fall 8 miles from A , on account of which he rides thence-forward with $\frac{3}{4}$ of his original speed, and arrives at B 35 minutes late; had the fall occurred 8 miles further on, he would have been only 25 minutes late. Find his original speed and the distance from A to B .

51. Two persons start from two places A and B respectively at the same time, and continue running between those places, distant 21 miles, without stopping; they pass each other first at the 9th mile-stone from A , and next meet again after 9 hours from starting, while going in opposite directions. Find the speed of each.

52. If in the preceding problem the men start from A , find the speed of each

53. A train passes a post in 3 minutes, and a bridge in 4 minutes, running uniformly at the rate of 10 ft. per sec; how long is the bridge?

54. A train running at the rate of 30 miles per hour, passes a man walking in the same direction with itself in $1\frac{3}{4}$ seconds; on its return the train passes the same man in $1\frac{1}{2}$ seconds. How long is the train, and how fast does the man walk?

55. A train and a tram-car, 44 and 4 yards long respectively take 4 seconds to pass each other when moving in opposite directions, and 6 seconds when moving in the same direction. Find the rate of each.

56. A boat takes 5 hours to go up $10\frac{1}{2}$ miles and back, and it takes 1 hour and 12 minutes to go down $8\frac{1}{2}$ miles; how far will a log wood carried down by the current go in 1 hour?

57. A steam-launch takes 6 hours to go up and down a certain distance on a river, where the current flows at the rate of 3 miles per hour; had the rate of the launch on still water been half as great again, it would have taken 2 hours to go up the same distance. Required this distance.

58. A boat's crew rowed $3\frac{1}{2}$ miles down a river and up again in 100 minutes. Had the stream been half as strong again, they would have taken $31\frac{1}{2}$ minutes longer. Find the rate of the stream. B. U. 1860.

59. A man rowing against a stream meets a log of wood which is being carried down by the current. He continues rowing

in the same direction for a quarter of an hour longer and then turns and rows down the stream, overtaking the log $1\frac{1}{2}$ miles lower down than the place where he first met it. Find the rate at which the current flows. Can you find the rate of the boat on still water? If not, why?

60. In a race of 600 yds. A gives B a start of 25 yds. and wins by 3 seconds; when he gives B a start of $4\frac{1}{2}$ seconds, he loses by 15 yds.; find the time that each takes to run the whole distance.

61. In a race of 150 yds. A can give B a start of 30 yds.; in a race of 400 yds. A gives B a start of 100 yds. and loses by $1\frac{1}{2}$ seconds. Find the rate of each.

62. A and B run a race, and A wins by 5 seconds; A then gives B a start of 10 yds. and wins by 3 sec.; A next gives B a start of $6\frac{1}{2}$ sec., but loses by 10 yds., how long is the run and what is the rate of each?

203. Digits. Take the number 23. The digit in the place of units is 3, and that in the place of tens is 2, and the number $23 = 10 \times 2 + 3$. Thus a *number of two digits* = $10 \times$ the digit in the tens' place + the digit in the units' place = $10x + y$, if x and y be the digits in the places of tens and units respectively. The last expression depends upon the local values (i.e., values in virtue of place) of x and y . In the case of 23 the local value of 2 is 20 or 10×2 ; in the case of 243 the local value of 2 is 200 = 100×2 , and so on. Again, $243 = 100 \times 2 + 10 \times 4 + 3$. If instead of 2, 4 and 3 we put x , y and z , we see that the *number of three digits* having x , y and z for the digits in the hundreds', tens' and units' place respectively = $100x + 10y + z = 100 \times$ the digit in the hundreds' place + $10 \times$ the digit in the tens' place + the digit in the units' place.

Ex. 1. Find the number of three digits which is the same when reversed, and the sum of whose digits is 16 and the difference 2. C. U. 1883.

Since the number is the same when reversed, the extreme digits must be the same (If we reverse 434, the reversed number is still 434)

Let x denote any of the extreme digits,

and y denote the digit in the place of tens.

Then the number = $100x + 10y + x$.

Since the sum of the digits is 16, $x + y + x = 16 \dots (1)$.

Since the difference of the digits is 2, $x - y = 2 \dots (2)$.

Add (1) and (2); thus $3x = 18$, $\therefore x = 6$. $\therefore y = 4$, by (2).
 \therefore the required number = **646**. *Ans.*

In (2) we have taken $x - y = 2$ instead of $y - x = 2$.

If $y - x = 2$, we get from this equation and from (1) $x = \frac{14}{3}$, $y = \frac{20}{3}$; but these fractional values of x and y are inadmissible, because x and y , being digits of a number, must be themselves *whole numbers*.

EXAMPLES 117.

1. A number consisting of two digits equals $3\frac{1}{2}$ times the sum of the digits, and double the left-hand digit is short of three times the right-hand digit by 7; find the number.

2. A certain number consists of two digits; the sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits; and the number when inverted bears to the original number the same ratio as 3 does to 8. Find the number.

3. There is a number of two digits such that if 9 be added to it, or if 11 times the digit in the place of tens be added to twice the digit in the place of units, the number will be inverted. What is the number?

4. Reverse the digits of a number, and it will become five-sixths of what it was before; also the difference between the two digits is 1. Find the number. C. U. 1883.

5. A number consists of three digits of which the sum is 6. The difference between the digits in the place of tens and units equals 2, and if 99 be added to the number, it will be inverted. Find the number.

6. A number consisting of three digits is the same when inverted; the sum of its digits is 10, and the greatest difference between a pair is 2. Find the number.

7. A number consists of three digits, the right-hand one being zero. If the left-hand and middle digits be interchanged, the number is diminished by 180; if the left-hand digit be halved, and the middle and right-hand digits be interchanged, the number is diminished by 336. Find the number. B. U. 1887.

8. Two boys are asked to find the numerical value of the product ab by assuming a given value for each of a and b ; one of them adopts the form ab , and the other the form ba , but each gives the required value by merely putting in these forms the numerical values of a and b , so that their answers are too great by half and thirteen-eighths of the correct one. Find the correct value wanted.

CHAPTER XXXI.

RATIO AND PROPORTION.

204. Definition. The **ratio** between two quantities is the value of one relatively to the other, and is determined by finding what multiple, part or parts one is of the other. Hence the ratio between two quantities can be expressed by a fraction. Thus the ratio of a to $b = \frac{a}{b}$.

The quantities compared must be of the same kind, for we cannot compare quantities of different kinds, we can compare 2 inches with 3 inches, but not 2 inches with 3 maunds.

The ratio of a to b is written as $a : b$, where a and b are called the **Terms** of the ratio; the first term (*i.e.*, a) is called the **Antecedent**, and the second term the **Consequent** of the ratio.

The ratio between two quantities cannot always be expressed by the ratio between two integers. For, take a rectangular court yard, 6 ft. long and 4 ft. broad. Then the diagonal of the court yard = $\sqrt{(4^2 + 6^2)}$ ft. = $\sqrt{(52)}$ ft. Eucl. I. 47.

Hence length : breadth = $6 : 4 = 3 : 2$;

but diagonal : breadth = $\sqrt{(52)} : 4 = \sqrt{(13)} : 4 = \sqrt{(15)} : 2$

Since we cannot fully extract the square root of 13, we cannot express the ratio between the diagonal and the breadth by the ratio between two integers. When the ratio between two quantities can be expressed as that between two integers, the quantities are called **commensurable** quantities, otherwise they are called **incommensurable** quantities. The diagonal and a side of a square are incommensurable, for the ratio of the one to the other is easily found to be equal to $\sqrt{2} : 1$.

A **Ratio of Equality** is one in which the antecedent is equal to the consequent

A Ratio is called one of **greater or less inequality** according as the antecedent is greater or less than the consequent.

Thus $5 : 5$ is a ratio of equality;

$6 : 5$ „ „ of greater inequality;

$5 : 6$ „ „ of less inequality

Hence if $a : b$ be a ratio of greater inequality, $\frac{a}{b} > 1$; if the ratio be one of less inequality, $\frac{a}{b} < 1$.

N. B. A ratio of greater inequality is sometimes called a ratio of **majority**, and that of less inequality is called a ratio of **minority**.

205. Theorem. *A ratio of greater inequality is diminished and one of less inequality increased by adding the same positive quantity to both of its terms.* ✓

Let $\frac{a}{b}$ be the given ratio, and x any positive quantity.

Required to shew that $\frac{a+x}{b+x} < \frac{a}{b}$, if $\frac{a}{b}$ be a ratio of greater inequality, and that $\frac{a+x}{b+x} > \frac{a}{b}$, if $\frac{a}{b}$ be a ratio of less inequality.

Proof. $\frac{a+x}{b+x} > = \text{or} < \frac{a}{b}$ (1)

according as $b(a+x) > = \text{or} < a(b+x)$,

„ as $ab+bx > = \text{or} < ab+ax$,

„ as $bx > = \text{or} < ax$,

„ as $b > = \text{or} < a$, x being positive.

i.e., according as $a : b$ is a ratio of less inequality, equality, or greater inequality.

Thus $\frac{a+x}{b+x} > \frac{a}{b}$, when $\frac{a}{b}$ is a ratio of less inequality,

and $\frac{a+x}{b+x} < \frac{a}{b}$, „ „ of greater inequality.

For example, $\frac{3}{4}$ is a ratio of less inequality, $\therefore \frac{3+4}{4+4}$ or $\frac{7}{8} > \frac{3}{4}$,

and $\frac{4}{3}$ of greater „ „, $\therefore \frac{4+4}{3+4}$ or $\frac{8}{7} < \frac{4}{3}$.

206. Definition. The **Duplicate**, **TriPLICATE**, **Subduplicate** and **Subtriplicate** ratios between two quantities are respectively the ratios between the squares, cubes, square roots and cube roots of those quantities.

Thus $a^2 : b^2$ is the duplicate ratio of $a : b$.

$a^3 : b^3$ „ „ triplicate ratio of $a : b$.

„ $\sqrt{a} : \sqrt{b}$ „ „ subduplicate ratio of $a : b$.

„ $\sqrt[3]{a} : \sqrt[3]{b}$ „ „ subtriplicate ratio of $a : b$.

The **sesquiplicate** ratio between two quantities is the ratio between the square roots of their cubes.

Thus $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is the sesquiplicate ratio of $a : b$.

Ex. 1. If $ax + by = cx + dy$, find the ratio $x : y$.

Given $ax + by = cx + dy$;

transposing, $(a - c)x = (d - b)y$;

dividing by $(a - c)y$, $\frac{x}{y} = \frac{d - b}{a - c}$.

i. e., $x : y = d - b : a - c$. *Ans.*

Ex. 2. If $4x - 3y = 5$, and $5x - 6y = 4$, find the ratio

$$x^2 + 1 : y^2 + 1.$$

Solving the equations $4x - 3y = 5$, and $5x - 6y = 4$, we easily get

$$x = 2, y = 1.$$

$\therefore x^2 + 1 : y^2 + 1 = 2^2 + 1 : 1^2 + 1 = 5 : 2$. *Ans.*

Ex. 3. If $\frac{x+a}{x+b} = \frac{x+a+c}{x+b+d}$, find the ratio $x+a : x+b$.

$\frac{x+a}{x+b} = \frac{x+a+c}{x+b+d}$ dif. of num's. = $\frac{c}{d}$, Art. 166, page 240.
 $\frac{x+a}{x+b} = \frac{x+a+c}{x+b+d}$ „ „ den's = $\frac{c}{d}$.

i. e., $x+a : x+b = c : d$. *Ans.*

EXAMPLES 118.

1. If $2x - 3y = 4y - 5x$, find the ratio $x : y$.

2. If $ax + by = bx + ay$, find the ratio $x : y$.

3. If $ax - 5y = 1$, and $3x - 4y = 5$, find the ratio $x : y$.

4. If $\frac{x}{a} + \frac{y}{b} = b + c$, and $x + y = b(c + a)$, find the ratios

$$x^2 : y^2 \text{ and } x - 2y : x + 3y$$

5. If $2x + 3y - 4z = 0$, and $3x - 6y + z = 0$, find the ratios $x : y : z$.

6. If $x + y + z = 12$, $x + 2y + 3z = 28$, and $3x + 2y + 2z = 26$, find the ratios $x : y : z$, and $x + y : y + z : z + x$.

7. If $\frac{a+1}{b+1} = \frac{a+2}{b+3}$, find the ratio $a+1 : b+1$.

8. Write down the subduplicate and subtriplicate ratios of $\frac{64}{27}$.

9. Two regiments composed of 1000 and 1500 men respectively are each re-inforced by 400 men; in favour of which regiment is the increase?

10. Prove that a ratio of greater inequality is increased and of less inequality diminished by taking the same number from

both its terms, the number subtracted being less than either of the terms.

207. Definition. Four quantities are said to be **proportionals** when the ratio of the first to the second is the same as that of the third to the fourth. Hence **proportion** is the equality of two ratios. Thus if a, b, c, d be proportionals, $a : b = c : d$, which is written as $a : b :: c : d$. $a : b :: c : d$ is read 'a is to b as c is to d.' The terms a and d are called the **extremes**, and b and c are called the **means**. d is called the **fourth proportional** to a, b and c .

Four quantities are said to be **inversely proportional** when the first is to the second as the reciprocal of the third is to the reciprocal of the fourth, i.e., as the fourth is to the third.

Thus, a, b, c and d are inversely proportional,

when $a : b :: \frac{1}{c} : \frac{1}{d}$.

Now, $\frac{1}{c} : \frac{1}{d} = \frac{\frac{1}{c}}{\frac{1}{d}} = \frac{d}{c} = d : c$; therefore it follows in this

case that $a : b :: d : c$.

208. Proposition I. If $a : b = c : d$, then $ad = bc$, and conversely, if $ad = bc$, then $a : b = c : d$.

That is, if four quantities be proportionals, then the product of the extremes is equal to the product of the means, and conversely.

For the proof of the first part, see Art. 160, page 230.

Proof of converse : Let $ad = bc$;

Dividing both sides by bd , $\frac{ad}{bd} = \frac{bc}{bd}$;

reducing, $\frac{a}{b} = \frac{c}{d}$; i.e., $a : b = c : d$.

209. Proposition II. If $a : b = c : d$, then $b : a = d : c$.

That is, if four quantities be proportionals, they are also proportionals when taken inversely. (*Invertendo*).

Proof : Given $\frac{a}{b} = \frac{c}{d}$; $\therefore 1 \div \frac{a}{b} = 1 \div \frac{c}{d}$,

whence $\frac{b}{a} = \frac{d}{c}$, i.e., $b : a = d : c$.

210. Proposition III. If $a : b = c : d$, then $a : c = b : d$.

That is, if four quantities be proportionals, they are also proportionals when taken alternately (*Alternando*.) See Art. 163, page 234.

211. Proposition IV. If $a : b = c : d$, then will

$$a + b : b = c + d : d. \text{ (Componendo).}$$

For the proof of this proposition and the two following, the student is referred back to Art. 174, page 254.

212. Proposition V. If $a : b = c : d$, then will

$$a - b : b = c - d : d. \text{ (Dividendo)}$$

213. Proposition VI. If $a : b = c : d$, then will

$$a + b : a - b = c + d : c - d \\ \text{(Componendo and Dividendo).}$$

214. Proposition VII. If $a : b = c : d$, then will

$$a : a - b = c : c - d \text{ (Convertendo).}$$

$$\text{For, let } a : b = c : d;$$

$$\text{inverting, } \frac{b}{a} = \frac{d}{c},$$

$$\text{subtracting each from 1, } 1 - \frac{b}{a} = 1 - \frac{d}{c};$$

$$\text{i.e., } \frac{c-b}{a} = \frac{c-d}{c};$$

$$\text{inverting again, } \frac{a}{c-b} = \frac{c}{c-d}$$

Ex. 1 If $(2a + 3b + 2c + 3d)(2a - 3b - 2c + 3d)$
 $= (2a + 3b - 2c - 3d)(2a - 3b + 2c - 3d)$, then will
 $a : b = c : d$.

$$\text{By prop. I, } \frac{2a + 3b + 2c + 3d}{2a + 3b - 2c - 3d} = \frac{2a - 3b + 2c - 3d}{2a - 3b - 2c + 3d};$$

\therefore by Componendo and Dividendo,

$$\frac{2a + 3b}{2c + 3d} = \frac{2a - 3b}{2c - 3d};$$

\therefore by Alternando, $\frac{2a+3b}{2a-3b} = \frac{2c+3d}{2c-3d}$; Prop. III.

again by Comp. and Div., $\frac{2a}{3b} = \frac{2c}{3d}$.

Multiply both by $\frac{3}{2}$; thus $\frac{a}{b} = \frac{c}{d}$; i.e., $a : b = c : d$.

Or thus : we have $\{(2a+3d)+(3b+2c)\}\{(2a+3d)-(3b+2c)\}$
 $\bullet = \{(2a-3d)+(3b-2c)\}\{(2a-3d)-(3b-2c)\}$;

$$\therefore (2a+3d)^2 - (3b+2c)^2 = (2a-3d)^2 - (3b-2c)^2.$$

\therefore by transposition, $(2a+3d)^2 - (2a-3d)^2 = (3b+2c)^2 - (3b-2c)^2$;

factorize each side; then $4a \times 6d = 6b \times 4c$; Art. 75.

$$\therefore ad = bc;$$

$$\therefore a : b = c : d \text{ Prop. I.}$$

Ex. 2. If $a : b :: c : d$, and if a be the greatest of the four quantities, then will $a > b+c$, a being positive.

Since a is greater than b as well as c , $a-b$ and $a-c$ are each positive:

$\therefore (a-b)(a-c)$ is positive.

i.e., $a^2 - a(b+c) + bc$ is positive. $\therefore a^2 + bc > a(b+c)$ (1)

Now, since $a : b = c : d$, $ad = bc$.

\therefore putting ad for bc in (1), we have $a^2 + ad > a(b+c)$.

Divide each side by a ; thus $a+d > b+c$, $\therefore a$ is positive.

EXAMPLES 119.

If $a : b :: c : d$, prove that

1. $a^2 - bc = a(a-d)$.

2. $a(b+d) = b(c+a)$.

3. $(a-b)(c+d) = ac - bd$.

4. $(ax+by)(cx-dy) = acx^2 - bdy^2$.

5. $a^2 - b^2 - c^2 + d^2 = (a+b+c+d)(a-b-c+d)$.

Prove that $a : b :: c : d$, being given that

6. $b^2 - ad = b(b-c)$.

7. $(a+b)(c-d) = (a-b)(c+d)$.

8. $\left(\frac{b}{c} - \frac{c}{b}\right)\left(\frac{1}{b+c} + \frac{1}{b-c}\right) = \frac{2b}{ad}$.

9. $\frac{1}{a} - \frac{1}{b} - \frac{1}{c} + \frac{1}{d} = \frac{(a-b)(a-c)}{abc}$.

10. If $a : b = c : d$, and if a be the greatest of the four quantities, prove that $a^2 + d^2 > b^2 + c^2$, and $a^n + d^n > b^n + c^n$, when n is positive.

11. If $3a-2b : 3a+2b = 2 : 3$, find the value of $5a+3b : 5a-3b$.

12. If $11a+13b : 7a+9b = 5a+13b : 3a+9b$, find the value of $2a+3b : 3a+2b$.

13. If $a^2 + ab + b^2 : a^2 - ab + b^2 = c^2 + cd + d^2 : c^2 - cd + d^2$, then will
 $(ad - bc)(ac - bd) = 0$

14. If $a : c = a - b : b - c$, then will $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$.

215. **Continued proportion.** Quantities are said to be in **continued proportion** when the first : the second = the second : the third = the third : the fourth, and so on. Thus a, b, c, d, e, \dots &c. are said to be in continued proportion, when $a : b = b : c : c : d : d : e = \dots$ &c. If a, b, c be in continued proportion, b is said to be the **mean proportional** between a and c , and c is said to be the **third proportional** to a and b .

216. **Theorem.** If a, b, c be in continued proportion, $b^2 = ac$, and conversely

$$\text{For, } \frac{a}{b} = \frac{b}{c};$$

$$\therefore \frac{a}{b} \times bc = \frac{b}{c} \times bc;$$

$$\text{i.e., } ac = b^2.$$

Conversely, if $b^2 = ac$, then a, b and c will be in continued proportion.

$$\text{For, let } ac = b^2;$$

$$\text{dividing each by } bc, \frac{ac}{bc} = \frac{b^2}{bc},$$

$$\therefore \frac{a}{b} = \frac{b}{c};$$

i.e., a, b and c are in continued proportion. [Cf. Art. 208]

Ex. 1. If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$, shew that $\frac{a}{d} = \frac{a^3}{b^3}$. M. U. 1887.

$$\text{Given } \frac{a}{b} = \frac{b}{c} = \frac{c}{d};$$

$$\therefore \left(\frac{a}{b}\right)^3 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{d}$$

$$\text{i.e., } \frac{a^3}{b^3} = \frac{a}{d}$$

Ex. 2. If a, b, c be in continued proportion, prove that

$$\frac{a^{2n} + b^{2n} + c^{2n}}{a^n + b^n + c^n} = a^n - b^n + c^n.$$

Since a, b, c are in continued proportion, $b^2 = ac$;

$$\begin{aligned} \therefore a^{2n} + b^{2n} + c^{2n} &= a^{2n} + 2b^{2n} + c^{2n} - b^{2n} \\ &= a^{2n} + 2(b^2)^n + c^{2n} - (b^2)^n \\ &= a^{2n} + 2(ac)^n + c^{2n} - (b^2)^n, \quad \because b^2 = ac, \\ &= (a^n + c^n)^2 - (b^n)^2 \\ &= (a^n - b^n + c^n)(a^n + b^n + c^n); \end{aligned}$$

$$\therefore \frac{a^{2n} + b^{2n} + c^{2n}}{a^n + b^n + c^n} = a^n - b^n + c^n.$$

Ex. 3. If a, b, c, d, \dots, x be in continued proportion,* show that $b, c, d, \&c$ may be expressed as the product of a by some power of a constant factor.

Given $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \&c. ;$

inverting, $\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \dots = k$, suppose.

Since $\frac{b}{a} = k$, $b = ka$;

„ $\frac{c}{b} = k$, $c = kb = k \cdot ka$, ($\because b = ka$, proved,) $= k^2 a$;

„ $\frac{d}{c} = k$, $d = kc = k \cdot k^2 a$, ($\because c = k^2 a$, proved,) $= k^3 a$;

and so on.

Hence, $b, c, d, e \&c.$ are respectively equivalent to

$$ka, k^2 a, k^3 a, k^4 a, \&c. \quad \text{Ans.}$$

EXAMPLES 120.

If a, b, c, d be in continued proportion, prove that

1. $(a - b)(b + c) = b(a - c)$. 2. $(a + b)(b + c) - 2ac = b(a + c)$.

3. $(a + b)(b + d) = b(a + b + c + d)$. 4. $(a^n + b^n)(b^n - c^n) = b^n(a^n - c^n)$.

5. $\frac{a^3 + ab + b^3}{b^3 + bc + c^3} = \frac{a}{c} = \frac{c^2}{a^2}$. 6. $\frac{4b^3}{a^3 - c^3} = \frac{a + c}{a - c} - \frac{a - c}{a + c}$.

Show that a, b and c are in continued proportion, if

7. $c(a - b) = b(b - c)$. 8. $(a + b)(a - b) = a(a - c)$.

9. $b(a + b + c) = bc + ca + ab$. 10. $(a + b + c)(a - b + c) = a^2 + b^2 + c^2$.

11. If $x + 1, x + 3, x + 6$ be in continued proportion, find x .

12. Find the mean proportional between 9 and 16.

13. Find the third proportional to 8 and 12.
 14. Find two numbers between 1 and 64 so as to be in continued proportion with them
 15. Find three numbers between 1 and 1296 so as to be in continued proportion with them
 16. Find the third proportional to $(a+b)^2$ and a^3+b^3 .

217. Important theorem If $a : b = c : d = e : f = \&c.$,

Ans.

then each ratio $= \left(\frac{pa^n + qc^n + re^n + \dots}{pb^n + qd^n + rf^n + \dots} \right)^{\frac{1}{n}}$.

Now, $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots$

Let each fraction $= k$.

Then, $\therefore \frac{a}{b} = k, a = kb, \therefore a^n = k^n b^n$, and $pa^n = k^n pb^n; \dots (1)$

$\therefore \frac{c}{d} = k, c = kd, \therefore c^n = k^n d^n, \therefore qc^n = k^n qd^n; \dots (2)$

$\therefore \frac{e}{f} = k, e = kf, \therefore e^n = k^n f^n, \therefore re^n = k^n rf^n; \dots (3)$

and so on

\therefore by adding (1), (2), (3), we get

$pa^n + qc^n + re^n + \dots = k^n (pb^n + qd^n + rf^n + \dots); (4)$

$\therefore \frac{pa^n + qc^n + re^n + \dots}{pb^n + qd^n + rf^n + \dots} = k^n;$

$\left(\frac{pa^n + qc^n + re^n + \dots}{pb^n + qd^n + rf^n + \dots} \right)^{\frac{1}{n}} = k = \frac{a}{b} = \frac{c}{d}, \&c.$

Hence the theorem

Corollaries. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots$, then

(1) each fraction $= \frac{pa + qc + re + \dots}{pb + qd + rf + \dots},$

this follows from the above theorem on putting $n=1$;

(2) each fraction $= \frac{a+c+e+\dots}{b+d+f+\dots} = \frac{\text{sum of numrs.}}{\text{sum of denms.}};$

this follows from Cor. 1 on putting $p=q=r=\dots=1$;

(3) each fraction $= \frac{a-c+e+\dots}{b-d+f+\dots};$

this follows from Cor. 1 on putting $p=1, q=-1, \&c.$;

$$(4) \text{ each fraction } = \frac{a-c}{b^2-d} = \frac{c-e}{d-f} = \frac{a-e}{b-f} = \dots$$

$$= \frac{\text{dif. of any two numrs.}}{\text{dif. of corresponding denrs.}}$$

Ex. 1. If a, b, c, d be proportionals, prove that
 $a^2d - b^2c + b^2d : a^2c + b^2d :: d : c+d$.

Since a, b, c, d are in proportion, $\frac{a}{b} = \frac{c}{d}$.

Let $\frac{a}{b} = \frac{c}{d} = k$. Then $a = bk, c = dk$.

$$\therefore \frac{a^2d - b^2c + b^2d}{a^2c + b^2d} = \frac{b^2k^2d - b^2dk + b^2d}{b^2k^2dk + b^2d}$$

$$= \frac{b^2d(k^2 - k + 1)}{b^2d(k^2 + 1)} = \frac{1}{k+1} \dots \dots \dots (1)$$

$$\frac{d}{c+d} = \frac{d}{dk+d} = \frac{d}{d(k+1)} = \frac{1}{k+1} \dots \dots \dots (2)$$

$$\frac{a^2d - b^2c + b^2d}{a^2c + b^2d} = \frac{d}{c+d}; \text{ whence the above proportion.}$$

Ex. 2. a, b, c, d be in continued proportion, prove that
 $(ab+bc+cd)^2 = (a^2+b^2+c^2)(b^2+c^2+d^2) \dots \dots \dots$ M. U. 1887.

We have $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k$, suppose.

Then evidently $k = \frac{a^2}{ab} = \frac{b^2}{bc} = \frac{c^2}{cd}$, $\therefore \frac{a}{b} = \frac{a \times a}{b \times a} = \frac{a^2}{ab}$, &c.

$$= \frac{\text{sum of numerators}}{\text{sum of denominators}}$$

$$= \frac{a^2 + b^2 + c^2}{ab + bc + cd} \dots \dots \dots (1)$$

Again $k = \frac{ab}{b^2} = \frac{bc}{c^2} = \frac{cd}{d^2}$, $\therefore \frac{ab}{b^2} = \frac{a}{b}$, $\frac{bc}{c^2} = \frac{b}{c}$, &c.

$$= \frac{\text{sum of numerators}}{\text{sum of denominators}}$$

$$= \frac{ab + bc + cd}{b^2 + c^2 + d^2} \dots \dots \dots (2)$$

\therefore from (1) and (2), $\frac{a^2 + b^2 + c^2}{ab + bc + cd} = \frac{ab + bc + cd}{b^2 + c^2 + d^2}$.

$\therefore (a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$. Art. 160.

Or thus : Let $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = k$.

Then $\therefore \frac{c}{d} = k, \therefore c = dk$;

„ $\frac{b}{c} = k, \therefore b = ck = dk \times k = dk^2$;

„ $\frac{a}{b} = k, \therefore a = bk = dk^2 \times k = dk^3$.

$$\begin{aligned} \therefore (ab + bc + cd)^2 &= (dk^3 \times dk^2 + dk^2 \times dk + dk \times d)^2 \\ &= (d^2k^5 + d^2k^3 + d^2k)^2 \\ &= \{kd^2(k^4 + k^2 + 1)\}^2 \\ &= k^2d^4(k^4 + k^2 + 1)^2 \quad \dots \dots \dots (1) \end{aligned}$$

$$\begin{aligned} (a^2 + b^2 + c^2)(b^2 + c^2 + d^2) &= (d^2k^6 + d^2k^4 + d^2k^2)(d^2k^4 + d^2k^2 + d^2) \\ &= d^2k^2(k^4 + k^2 + 1) \cdot d^2(k^4 + k^2 + 1) \\ &= k^2d^4(k^4 + k^2 + 1)^2 \quad \dots \dots \dots (2) \end{aligned}$$

\therefore from (1) and (2), $(ab + bc + cd)^2 = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$.

$$3. \text{ Solve } \frac{x+a}{b+c} = \frac{y+b}{c+a} = \frac{z+c}{a+b} \dots \dots \dots (1)$$

$$x + y + z = a + b + c. \quad \dots \dots \dots (2)$$

$$\begin{aligned} \text{Each member of (1)} &= \frac{\text{sum of numerators}}{\text{sum of denominators}} \\ &= \frac{x+y+z+a+b+c}{2a+2b+2c} \\ &= \frac{a+b+c+a+b+c}{2(a+b+c)}, \text{ from (2),} \\ &= 1. \end{aligned}$$

$$\therefore \left. \begin{aligned} x+a &= b+c, \\ y+b &= c+a, \\ z+c &= a+b. \end{aligned} \right\} \quad \therefore \left. \begin{aligned} x &= b+c-a, \\ y &= c+a-b, \\ z &= a+b-c. \end{aligned} \right\} \quad \text{Ans.}$$

4. A vessel contains a mixture of wine and water. Had there been a gallon more wine and a gallon less water, the ratio of wine to water would have been as 7 : 8. but had there been a gallon more water and a gallon less wine, this ratio would have been as 2 : 3. Of how many gallons does the mixture consist?

Let x denote the number of gallons of wine in the mixture,
and y „ „ „ „ „ of water „ „ „

Then, by the question,

$$x+1 : y-1 = 7 : 8, \quad \therefore 8(x+1) = 7(y-1), \quad (1)$$

$$\text{and } x-1 : y+1 = 2 : 3, \quad \therefore 3(x-1) = 2(y+1). \quad (2)$$

$$\therefore \text{From (1), } 8x - 7y = -15,$$

$$\text{and from (2), } 3x - 2y = 5$$

Multiplying the first of these equations by 2 and the second by 7, we get

$$16x - 14y = -30,$$

$$21x - 14y = 35;$$

$$\therefore \text{by subtraction, } -5x = -65;$$

$$\therefore x = 13;$$

$$\therefore \text{from (1), } 8 \times 13 = 7(y-1),$$

$$\therefore y-1 = 16;$$

$$\therefore y = 17.$$

$$\therefore \text{the total number of gallons required} = x + y = 30. \quad \text{Ans.}$$

Ex. 5. Two silver balls of diameters a and b inches respectively are melted down into a single ball; find its diameter, being given that the volume of a ball is always proportional to the cube of its diameter.

Let x denote the diameter of the new ball in inches,

and let V denote its volume in cubic inches.

Let V_1 and V_2 cubic inches be the volumes of the old balls.

Since the volume is proportional to the cube of the diameter,

$$\frac{V_1}{a^3} = \frac{V_2}{b^3} = \frac{V}{x^3} \dots \dots \dots (1)$$

Since the new ball contains the same quantity of silver as the two original balls, $V_1 + V_2 = V \dots \dots \dots (2)$

$$\text{From (1), } \frac{V}{x^3} = \frac{\text{sum of first two numrs.}}{\text{denrs.}}$$

$$= \frac{V_1 + V_2}{a^3 + b^3}$$

$$= \frac{V}{a^3 + b^3}, \text{ by (2);}$$

$$x^3 = a^3 + b^3.$$

$$x = \sqrt[3]{a^3 + b^3}. \quad \text{Ans.}$$

EXAMPLES 121.

If $a : b :: c : d$, show that

1. $a+b : a-b :: c+d : c-d$.
2. $ma+nc : mb+nd :: pa+qc : pb+qd$.
3. $a-c : 2a+c :: b-d : 2b+d$.
4. $ma+nb : pa+qb :: mc+nd : pc+qd$.
5. $17a+13c : 11a+9c = 17b+13d : 11b+9d$.
6. $a^2+b^2 : a^2-b^2 = c^2+d^2 : c^2-d^2 = ac+bd : ac-bd$.
7. $a^2+c^2 : c^2 = a^2+b^2+c^2+d^2 : c^2+d^2$.
8. $a^3 : c^3 = (a+b)^3 : (c+d)^3$. $\theta \quad (ac+bd)d = b(bd+c^2)$.
10. $a^4-a^2b^2+b^4 : c^4-c^2d^2+d^4 = (a^2+b^2)^2 : (c^2+d^2)^2$.
11. $\sqrt{(a^2+b^2)} : \sqrt{(c^2+d^2)} = \sqrt{(a^3-a^2b+b^3)} : \sqrt{(c^3-c^2d+d^3)}$.
12. $a + \sqrt{(a^2+c^2)} : a - \sqrt{(a^2+c^2)} = b + \sqrt{(b^2+d^2)} : b - \sqrt{(b^2+d^2)}$.
13. $la+mb : lc+md = \sqrt{(pa^2+qb^2)} : \sqrt{(pc^2+qd^2)}$.
14. $a-c : b-d :: a+c + \sqrt{(a^2+c^2)} : b+d + \sqrt{(b^2+d^2)}$.
15. $(la+mb)^3 : (lc+md)^3 :: pa^3l^2+qb^3d^2 : pc^3l^2+qd^3d^2$.
16. $(la^3+mb^3)\left(\frac{l}{a^3}+\frac{m}{b^3}\right) = (ld^3+md^3)\left(\frac{l}{d^3}+\frac{m}{c^3}\right)$.

If $a : b = c : d = e : f = g : h = \dots$, prove that

17. $a : b = 2a+3c-e : 2b+3d-f$.
18. $a+c+e+g : a-c+e-g = b+d+f+h : b-d+f-h$.
19. $a+c : b+d = 4 : \frac{b}{a} + \frac{d}{c} + \frac{f}{e} + \frac{h}{g}$.
20. $pa+qc : pb+qd = \frac{1}{b} + \frac{1}{d} + \frac{1}{f} + \dots : \frac{1}{a} + \frac{1}{c} + \frac{1}{e} + \dots$.
21. $a+c : b+d = ac+bc+e : bc+bd+f$.
22. $e : f = a+c + \sqrt{(e^2+g^2)} : b+d + \sqrt{(f^2+h^2)}$.
23. $a^2-e^2 : ab-ef = a-c + \sqrt{(c^2+e^2)} : b-d + \sqrt{(d^2+f^2)}$.
24. $(ae+c^2)^{\frac{3}{2}} : (be+cd)^{\frac{3}{2}} = (a^2+e^2)^{\frac{3}{2}} + ace : (b^2+f^2)^{\frac{3}{2}} + bdf$.
25. $(c^3-e^3)^{\frac{2}{3}} : (d^3-f^3)^{\frac{2}{3}} = (ac+ce+eg)^{\frac{2}{3}} : (bd+df+fh)^{\frac{2}{3}}$.
26. $a^2+c^2+e^2 : b^2+d^2+f^2 = u(a + \sqrt{c^2+e^2}) : v(b + \sqrt{d^2+f^2})$.
27. $ac+ce+eg : bd+df+fh = (a^2+c^2+e^2)^{\frac{3}{2}} : (b^2+d^2+f^2)^{\frac{3}{2}}$.
28. $(a^3+c^3+e^3+g^3+\dots)(b^3+d^3+f^3+h^3+\dots) = (ab+cd+ef+gh+\dots)^3$.
29. $\left(\frac{a}{b}\right)^3 + \left(\frac{c}{d}\right)^3 + \left(\frac{e}{f}\right)^3 = \frac{ac}{bd} + \frac{ce}{df} + \frac{ef}{fh}$.

$$30. \quad g(c+d)(cd-f^2)(g-b)=h(a+b)(c^2-ef)(c-d)$$

If a, b, c, d be in continued proportion, prove that

$$31. \quad a^2+b^2 : a^2-b^2 = a+c : a-c$$

$$32. \quad a(a-1) : b^2-c-b^2-a : c(c-1)$$

$$33. \quad a : d = a^2+b^2+abc : b^2+c^2+bcd.$$

$$34. \quad a : c = a^2-b^2+c^2 : b^2-c^2+d^2$$

$$35. \quad ac+bd : ac-bd = a^2+2b^2+2c^2+d^2 : a^2-d^2.$$

$$36. \quad (a+d)^2+(b-c)^2=(b+c)^2+(a-d)^2.$$

$$37. \quad a^4+a^2c^2+c^4=b^2\left(\frac{b^2}{c^2}-1+\frac{b^2}{a^2}\right)(a^2+b^2+c^2).$$

$$38. \quad bd(b^2+4d^2)=c(ad-2c^2+2cd)(b+2c+2d).$$

$$39. \quad a-b, b-c, c-d \text{ are in continued proportion.}$$

Prove that a, b, c, d will be in proportion, if

$$40. \quad a(b+c) : c^2+ad = b : d.$$

$$41. \quad pa+rc : pb+rd = a+c : b+d$$

$$42. \quad (a+b+c+d)(a-b-c+d)=(a+b-c-d)(a-b+c-d)$$

$$43. \quad (la+mc)(mb-lc)=(ma-lc)(lb+md).$$

$$44. \quad (a+b)(a-d)+b(c+d)=a(a+b).$$

$$45. \quad a(c+a)+b(a-c) : c(c+a)+d(a-c) = a+b : c+d.$$

$$46. \quad a(b+c)-c(b-c) : b(b+c)-d(b-c) = a-c : b-d.$$

$$47. \quad \frac{a+d}{a-d} \cdot \frac{a-d}{a+d} : \frac{b+c}{b-c} \cdot \frac{b-c}{b+c} = b^2-c^2 : a^2-d^2.$$

Show that b is the mean proportional between a and c , if

$$48. \quad 2a+3b : 5a-4b = 2b+3c : 5b-4c$$

$$49. \quad ab+c^2 : b+c = a-b+c : 1. \quad 50. \quad a^2+b^2 : ab = a+c : b.$$

$$51. \quad b^2 : ca = a^2+b^2+c^2 : a^2+ac+c^2.$$

$$52. \quad (b+c)(a^2+ab+b^2) : (b-c)(a^2-ab+b^2) \\ = (a+b)(b^2+bc+c^2) : (a-b)(b^2-bc+c^2).$$

If a, b, c, d, e be in continued proportion, prove that

$$53. \quad \frac{a}{e} - \frac{ab}{de} = \left(\frac{a}{c} - \frac{b}{e}\right)\left(\frac{b}{d} + \frac{a}{e}\right) \quad 54. \quad \frac{ab}{de} - \frac{ab}{cd} = \left(\frac{a}{d} - \frac{b}{e}\right)\left(\frac{b}{e} + \frac{a}{c}\right).$$

$$55. \quad d : e = \sqrt[5]{(a^5+b^5+c^5)} : \sqrt[5]{(b^5+c^5+a^5)}.$$

$$56. \quad \text{If } x : y = a : a+b, \text{ prove that } x^2-xy+y^2 : a^2+ab+b^2 \text{ is the duplicate ratio of } x : a$$

$$57. \quad \text{If } a : b = c : d, \text{ shew that } (a^2+b^2+c^2+d^2)\sqrt{abcd} \text{ is a mean proportional between } (ac+bd)^2 \text{ and } (ab+cd)^2.$$

Solve

$$\sqrt{58.} \quad 5-3x : 4x - \frac{2+x}{3} :: 2 : 3.$$

$$59. \quad 2x-3a : 3x-4b :: 4x-c : 6x-d.$$

$$60. \quad x^2+2x+3 : 2x^2+3x+3 = x+2 : 2x+3.$$

$$61. \quad x+y : a+b-c=y+z : b+c-a = z+x : a-b+c, \\ x+y+z = 2(a+b+c).$$

$$62. \quad ax+by-cz : c(a-b) = ax-by+cz : b(c-a) \\ = by+cz-ax : a'b-c, \quad a^2x+b^2y+c^2z+(a-b)(b-c)(c-a)=0.$$

$$63. \quad \frac{x+y-z}{c} = \frac{x-y+z}{b} = \frac{z+y-x}{a} = \frac{1}{(b+c)(c+a)(a+b)}.$$

$$64. \quad \frac{ay+bx}{c} = \frac{cx+az}{b} = \frac{bz+cy}{a} = \frac{abc}{a^2+b^2+c^2}.$$

65. In what ratio should tea at 2s. per lb. be mixed with tea at 3s. per lb. in order that the mixture may be mixed 2s. 8d. per lb.?

66. Two trains start from two stations, 50 miles distant, to meet each other. Their speeds are as 2 : 3. Where do they meet?

67. The numerical strength of one army is to that of another as 3 : 5, and if the former were reinforced by 1000 men and the latter by 1400 men, the strength of the one would have been to that of the other as 5 : 8; find the number of men in each army.

68. What number must be added to 3, 5, 7, 10 each in order to get four numbers in proportion?

69. Find four consecutive numbers so that 1, 2, 3 and 5 being added to them in order, they may give four numbers in proportion.

70. The ratio of the sum of the ages of two persons to 5 is equal to the ratio of the difference of their ages to 3, and is also equal to the ratio of 10 to 1; find their ages.

71. In a boat race on still water the speed of one boat is to that of another as 4 : 7, and the number of rowers in each boat being increased by 5, their rates of rowing are found to be as 13 : 19. Assuming that the rate of rowing is proportional to the number of rowers, find the original number of rowers in each boat.

72. Three spheres of diameters 3, 4 and 6 inches respectively are formed into a single sphere; find its diameter, supposing the volume of a sphere to be proportional to the cube of its diameter.

73. The length and breadth of a room are as 7 : 5, and the carpet leaves an uncovered strip of breadth $\frac{3}{4}$ ft. all round the room. If the area of the floor exceeds the area of the carpet by $42\frac{3}{4}$ sq. ft., find the dimensions of the room.

74. Rs. 360 are divided among some men, women and children, numbering 60 in all. The total shares of the men, women and children are respectively as 5, 4 and 3, while the shares of each man, woman and child are respectively as 3 : 2 : 1. Find the number of men, women and children separately, and the respective shares of each man, woman and child.

75. A and B have £45 between them, and make a bet; their stakes are proportional to the respective sums they own. If A loses, the amounts with A and B respectively will be as 1 : 2, and if B loses the same will be as 11 : 4; find the amount with each at first.

CHAPTER XXXIII.

HARDER WORK IN RATIO AND PROPORTION.

218. We work out below a few examples requiring a good deal of facility in the application of the theorems and artifices of the last chapter.

Ex. 1. If $\frac{x}{b+c-a} = \frac{y}{c+a-b} = \frac{z}{a+b-c}$, find the value of $(b-c)x + (c-a)y + (a-b)z$. C. U. 1878.

$$\text{Let } \frac{x}{b+c-a} = \frac{y}{c+a-b} = \frac{z}{a+b-c} = k.$$

$$\therefore \begin{cases} x = k(b+c-a) \\ y = k(c+a-b) \\ z = k(a+b-c) \end{cases}$$

$$\begin{aligned} \therefore (b-c)x + (c-a)y + (a-b)z &= k\{(b-c)(b+c-a) + (c-a)(c+a-b) \\ &\quad + (a-b)(a+b-c)\} \\ &= k\{b^2 - c^2 - a(b-c) + \{c^2 - a^2 - b(c-a)\} + \{a^2 - b^2 - c(a-b)\}\} \\ &= k\{b^2 - c^2 + c^2 - a^2 + a^2 - b^2 - \{a(b-c) + b(c-a) + c(a-b)\}\} \\ &= k \times 0 = 0. \quad \text{Art. 73, Formulæ.} \end{aligned}$$

Ex. 2. If $a+c : c = 2a : b$, shew that

$$\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right) = \frac{4}{ac} - \frac{3}{b^2}.$$

$$\begin{aligned} \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right) &= \left\{\frac{1}{b} + \left(\frac{1}{a} - \frac{1}{c}\right)\right\} \left\{\frac{1}{b} - \left(\frac{1}{a} - \frac{1}{c}\right)\right\} \\ &= \frac{1}{b^2} - \left(\frac{1}{a} - \frac{1}{c}\right)^2. \end{aligned}$$

Hence the proposed result must be true,

$$\text{if } \frac{1}{b^2} - \left(\frac{1}{c} - \frac{1}{a}\right)^2 = \frac{4}{ac} - \frac{3}{b^2},$$

$$\begin{aligned} \text{2. e., if } \quad \frac{4}{b^2} &= \frac{4}{ac} + \left(\frac{1}{c} - \frac{1}{a}\right)^2 \text{ [transposing]} \\ &= \frac{4}{ac} + \left(\frac{1}{c^2} - \frac{2}{ac} + \frac{1}{a^2}\right) \\ &= \left(\frac{1}{c} + \frac{1}{a}\right)^2, \text{ simplifying,} \\ &= \frac{(a+c)^2}{a^2c^2}; \end{aligned}$$

$$\text{1. e., if } \quad 4a^2c^2 = b^2(a+c)^2 \dots\dots\dots (1)$$

$$\text{Now, } \because a+c : c = 2a : b,$$

$$\therefore 2ac = b(a+c);$$

$$\text{squaring, } 4a^2c^2 = b^2(a+c)^2.$$

\therefore (1) is true, and hence the proposed result

Ex. 3 If $a(y+z) = b(z+x) = c(x+y)$, prove that

$$\frac{a-b}{x^2-y^2} = \frac{b-c}{y^2-z^2} = \frac{c-a}{z^2-x^2}$$

$$\text{We have } \frac{a}{y+z} = \frac{b}{z+x} = \frac{c}{x+y}$$

$$\therefore \text{ each } = \frac{a-b}{\frac{1}{y+z} - \frac{1}{z+x}} = \frac{b-c}{\frac{1}{z+x} - \frac{1}{x+y}} = \frac{c-a}{\frac{1}{x+y} - \frac{1}{y+z}}, \text{ Art. 217,}$$

$$\text{i.e., } \frac{a-b}{\frac{a-b}{(y+z)(z+x)}} = \frac{b-c}{\frac{b-c}{(z+x)(x+y)}} = \frac{c-a}{\frac{c-a}{(x+y)(y+z)}};$$

$$\begin{aligned} \therefore \frac{(y+z)(z+x)(a-b)}{x-y} &= \frac{(z+x)(x+y)(b-c)}{y-z} \\ &= \frac{(x+y)(y+z)(c-a)}{z-x}; \end{aligned}$$

divide each by $(y+z)(z+x)(x+y)$; thus

$$\frac{a-b}{x^2-y^2} = \frac{b-c}{y^2-z^2} = \frac{c-a}{z^2-x^2}$$

Ex. 4. If $\frac{bz-cy}{a} = \frac{cx-az}{b} = \frac{ay-bx}{c}$, then will $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

$$\text{We have } \frac{bc\left(\frac{z}{c} - \frac{y}{b}\right)}{a} = \frac{ca\left(\frac{x}{a} - \frac{z}{c}\right)}{b} = \frac{ab\left(\frac{y}{b} - \frac{x}{a}\right)}{c};$$

$$\text{divide each by } abc; \text{ then } \frac{\frac{z}{c} - \frac{y}{b}}{a^2} = \frac{\frac{x}{a} - \frac{z}{c}}{b^2} = \frac{\frac{y}{b} - \frac{x}{a}}{c^2};$$

$$\therefore \text{ each} = \frac{\text{sum of numerators}}{\text{sum of denominators}} \\ = \frac{0}{a^2 + b^2 + c^2} = 0;$$

$$\therefore \frac{z}{c} - \frac{y}{b} = \frac{x}{a} - \frac{z}{c} = \frac{y}{b} - \frac{x}{a} = 0; \text{ whence } \frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

Ex. 5. If $\frac{x-3y}{a-2b} = \frac{y-3z}{b-2c} = \frac{z-3x}{c-2a}$, prove that

$$\frac{x+y+z}{a+b+c} = \frac{7y+5z}{8b+6c-a}.$$

$$\text{Let } \frac{x-3y}{a-2b} = \frac{y-3z}{b-2c} = \frac{z-3x}{c-2a} = k;$$

$$\text{then } k = \frac{\text{sum of numerators}}{\text{sum of denominators}} = \frac{-2x-2y-2z}{-a-b-c} \\ = \frac{2(x+y+z)}{a+b+c}; \dots (1)$$

$$\text{also, } k = \frac{p(x-3y) + q(y-3z) + r(z-3x)}{p(a-2b) + q(b-2c) + r(c-2a)}, \dots (2)$$

where p, q and r are any quantities. COR. 1, Art 217.

Now, choose p, q and r so that $p(x-3y) + q(y-3z) + r(z-3x)$,
i.e., $(p-3r)x + (q-3p)y + (r-3q)z$ may identically $= 7y+5z$; that is,

$$\left. \begin{aligned} p-3r &= 0, \\ q-3p &= 7, \\ r-3q &= 5; \end{aligned} \right\} \text{ whence, } \begin{aligned} p &= -3, \\ q &= -2, \\ r &= -1. \end{aligned}$$

$$\therefore \text{ from (2), } k = \frac{7y+5z}{-3(a-2b) - 2(b-2c) - (c-2a)} \\ = \frac{7y+5z}{4b+3c-a}, \text{ simplifying } \dots (3)$$

By (1) and (3), $\frac{2(x+y+z)}{a+b+c} = \frac{7y+5z}{4b+3c-a}$;

dividing by 2, $\frac{x+y+z}{a+b+c} = \frac{7y+5z}{2(4b+3c-a)} = \frac{7y+5z}{8b+6c-2a}$.

EXAMPLES 122.

1. If $x : b-c = y : c-a = z : a-b$, shew that
 $(b+c)x + (c+a)y + (a+b)z = 0$.

2. If $x : b-c = y : c-a = z : a-b$, then will each ratio

$$= \sqrt[3]{\left(\frac{-xyz}{a^2x + b^2y + c^2z} \right)}.$$

3. If $\frac{x}{a^2(b-c)} = \frac{y}{b^2(c-a)} = \frac{z}{c^2(a-b)}$, then will $\frac{x+y+z}{a+b+c} = \frac{x}{a} + \frac{y}{b} + \frac{z}{c}$.

4. If $\frac{x}{a+2b} = \frac{y}{b+2c} = \frac{z}{c+2a}$, prove that
 $(ax+by+cz) \div (x+y+z) = \frac{1}{3}(a+b+c)$.

5. If $x+y : a^2-bc = y+z : b^2-ca = z+x : c^2-ab$, then will

(1) $(c+a)x + (a+b)y + (b+c)z = 2(a+b+c)(x+y+z)$;

(2) $(b+c)x + (c+a)y + (a+b)z = (a+b)x + (b+c)y + (c+a)z = 0$.

6. If $x : a+b = y : b+c = z : c+a$, then will

(1) $x+y : a+2b+c = y+z : b+2c+a = z+x : c+2a+b$;

(2) $2x+3y : 2a+5b+3c = 2y+3z : 2b+5c+3a = 2z+3x : 2c+5a+3b$.

7. If $x+y : a+b-c = y+z : b+c-a = z+x : c+a-b$, then will

(1) $\frac{x-y}{a-b} = \frac{y-z}{b-c} = \frac{z-x}{c-a}$.

(2) $\frac{x^2-y^2}{(a^2+bc)-(b^2+ca)} = \frac{y^2-z^2}{(b^2+ca)-(c^2+ab)} = \frac{z^2-x^2}{(c^2+ab)-(a^2+bc)}$;

(3) $\frac{x}{a-b-c} = \frac{y}{b-c-a} = \frac{z}{c-a-b}$.

8. If $\frac{x-y}{b-a} = \frac{y-z}{c-b} = \frac{x+y+z}{2(a+b+c)}$, then will $\frac{x}{b+c} = \frac{y}{c+a} = \frac{z}{a+b}$.

9. If $x+y-z : a+b = y+z-x : b+c = z+x-y : c+a$, then will

(1) $\frac{x}{2a+b+c} = \frac{y}{2b+c+a} = \frac{z}{2c+a+b}$;

(2) $\frac{x-y}{a-b} = \frac{y-z}{b-c} = \frac{z-x}{c-a}$;

(3) $\frac{x^2-y^2}{(a-b)(3a+3b+c)} = \frac{y^2-z^2}{(b-c)(3b+3c+a)} = \frac{z^2-x^2}{(c-a)(3c+3a+b)}$.

10. If $\frac{2x+3y-z}{7b+5c} = \frac{2y+3z-x}{7c+5a} = \frac{2z+3x-y}{7a+5b}$, shew that

each fraction $= \frac{x}{a+2b} = \frac{y}{b+2c} = \frac{z}{c+2a}$, and that

$$(a+2b)(cx+ay+bz) = 3x(ab+bc+ca)$$

11. If $\frac{x+2y}{a-4c} = \frac{y+2z}{b-4a} = \frac{z+2x}{c-4b}$, shew that

$$\frac{x+y+z}{a+b+c} = \frac{2x+3y+4z}{b+2c+6a}$$

12. If $\frac{2x+3y+4z}{6a+5b+7c} = \frac{x-2y+3z}{4a-b+c} = \frac{3x-y+2z}{5a+2b+c}$, prove that

$$x : a+b = y : b+c = z : c+a.$$

13. If $\frac{4x+5y-3z}{8a+15b-12c} = \frac{2x-y+z}{4a-3b+4c} = \frac{x-4y+z}{2(a-6b+2c)}$, then will

$$(4a+3b+4c)(x+2y+3z) = (a+5b+6c)(4x+2y+2z).$$

14. If $\frac{x(by+cz)}{a(b^2+c^2)} = \frac{y(cz+ax)}{a(b^2+c^2)} = \frac{z(ax+by)}{c(a^2+b^2)}$, then will

$$(1) \frac{x}{a} = \frac{y}{b} = \frac{z}{c}; \quad (2) \frac{a(by+cz)}{x(y^2+z^2)} = \frac{b(cz+ax)}{y(z^2+x^2)} = \frac{c(ax+by)}{z(x^2+y^2)}.$$

15. If $(b+c)x = (c+a)y = (a+b)z$, then will

$$(1) \frac{x-y}{a^2-b^2} = \frac{y-z}{b^2-c^2} = \frac{z-x}{c^2-a^2};$$

$$(2) a-b : z(x-y) = b-c : x(y-z) = c-a : y(z-x).$$

16. If $(a+b)(y+z-x) = (b+c)(z+x-y) = (c+a)(x+y-z)$, then will

$$\frac{x-y}{c^2-a^2} = \frac{y-z}{a^2-b^2} = \frac{z-x}{b^2-c^2}.$$

17. If $a(y+z) = b(z+x) = c(x+y)$, then will

$$\frac{x}{ab+ac-bc} = \frac{y}{ab+bc-ac} = \frac{z}{bc+ac-ab}.$$

18. If $\frac{x-a}{a-a_1} = \frac{y-b}{b-b_1} = \frac{z-c}{c-c_1}$, then each of these ratios

$$= \frac{bx-ay}{ab_1-a_1b} = \frac{cy-bz}{bc_1-b_1c} = \frac{az-cx}{ca_1-c_1a} = \frac{ax+by+cz-(a^2+b^2+c^2)}{a^2+b^2+c^2-(aa_1+bb_1+cc_1)}.$$

19. If $a : b : c = 2y+2z-3x : 2z+2x-3y : 2x+2y-3z$, prove that $x : y : z = a+2b+2c : b+2c+2a : c+2a+2b$.

20. If $a_1 : b_1 = a_2 : b_2 = a_3 : b_3 = \dots = a_n : b_n$, shew that

$$\frac{a_1}{b_1} = \frac{\{a_1 a_2 \dots a_r b_{r+1} b_{r+2} \dots b_n\}^{\frac{1}{2r-n}}}{\{b_1 b_2 \dots b_r a_{r+1} a_{r+2} \dots a_n\}}.$$

CHAPTER XXXIV.

ELIMINATION.

219. Definition. The **elimination** of one or more algebraical symbols from given equations is the operation of obtaining from those equations a new equation *free from those symbols and involving only the other symbols*; the equation obtained is called the **eliminant**.

Suppose we have the equations $ax + b = 0$, and $a_1x + b_1 = 0$. From the first equation $x = -\frac{b}{a}$, and from the second $x = -\frac{b_1}{a_1}$. These values of x are *generally* different, and the two equations in x are true for the same value of x only when $\frac{b}{a} = \frac{b_1}{a_1}$, *i.e.*, when $a_1b = ab_1$. Hence in the present case $a_1b = ab_1$ is the eliminant.

220. Process. Elimination is an operation for which no particular rules can be laid down, but which calls for a command over algebraic principles and artifices generally. We shall begin with simple cases

Ex. 1. Eliminate x from the equations

$$\left. \begin{array}{l} ax + b = 0, \quad (1) \\ x^2 + 1 = x. \quad (2) \end{array} \right\}$$

Find x from the simplest of the equations, and substitute its value in the other.

From (1), $x = -\frac{b}{a};$

substituting this value of x in (2), we have

$$\frac{b^2}{a^2} + 1 = -\frac{b}{a}, \text{ whence } a^2 + ab + b^2 = 0. \quad \text{Ans.}$$

N. B. We have here two equations in order to eliminate one symbol, x . This is necessary; for from one of the equations we find x , and substitute the value obtained in the other so as to get an equation free from x . Similarly to eliminate two symbols, three equations are necessary, and so on. In general, one more equation is necessary than the number of symbols to be eliminated. In such equations as $x - y = a$, $y - z = b$,

and $z - x = c$, we at once get by addition $a + b + c = 0$, and thus eliminate three symbols from three equations, but this is only possible for the special relation that holds among the quantities $x - y$, $y - z$ and $z - x$.

Ex. 2. Eliminate x and y from the equations

$$\left. \begin{aligned} x + y &= 2a, \dots\dots (1) \\ ax + by &= a^2 + b^2, \dots (2) \\ x^{\frac{1}{3}} + y^{\frac{1}{3}} &= c^{\frac{1}{3}}, \dots\dots (3) \end{aligned} \right\}$$

Solving (1) and (2), we easily get $x = a - b$, $y = a + b$;
substituting these values in (3), we have

$$(a - b)^{\frac{1}{3}} + (a + b)^{\frac{1}{3}} = c^{\frac{1}{3}}. \text{ Ans.}$$

N. B. Notice that we are given three equations to eliminate two symbols.

Ex. 3. Eliminate x and y from the equations

$$\left. \begin{aligned} ax + by &= c, \dots\dots (1) \\ bx - ay &= d, \dots\dots (2) \\ x^2 + y^2 &= l^2, \dots\dots (3). \end{aligned} \right\}$$

We may proceed exactly as in Ex. 2, but the forms of the given equations render the following nicer method possible.

Squaring (1) and (2), and adding the results, we have

$$(a^2x^2 + 2abxy + b^2y^2) + (b^2x^2 - 2abxy + a^2y^2) = c^2 + d^2 ;$$

simplifying,

$$(a^2 + b^2)x^2 + (a^2 + b^2)y^2 = c^2 + d^2 ;$$

$$\text{i. e., } (a^2 + b^2)(x^2 + y^2) = c^2 + d^2 ;$$

substituting for $x^2 + y^2$ from (3), $(a^2 + b^2)l^2 = c^2 + d^2. \text{ Ans.}$

Ex. 4. Eliminate x , y and z from the equations

$$\left. \begin{aligned} a_1x + b_1y + c_1z &= 0, \dots\dots (1) \\ a_2x + b_2y + c_2z &= 0, \dots\dots (2) \\ a_3x + b_3y + c_3z &= 0, \dots\dots (3) \end{aligned} \right\}$$

From (1) and (2) by Cross Multiplication, we have

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} = k, \text{ suppose ;}$$

thus $x = k(b_1c_2 - b_2c_1)$, $y = k(c_1a_2 - c_2a_1)$, $z = k(a_1b_2 - a_2b_1)$;

substituting these values in (3), we obtain

$$a_3k(b_1c_2 - b_2c_1) + b_3k(c_1a_2 - c_2a_1) + c_3k(a_1b_2 - a_2b_1) = 0 ;$$

dividing out k , $a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0. \text{ Ans.}$

N. B. We have here apparently eliminated three symbols from three equations. But the first equation is equivalent to $a_1\frac{x}{a_1} + b_1\frac{y}{b_1} + c_1\frac{z}{c_1} = 0$,

i.e., to $a_1l + b_1m + c_1 = 0$, where $l = \frac{x}{z}$ and $m = \frac{y}{z}$. On similarly transposing the other equations, the independent symbols are found to be only two, *viz.*, l and m , and we thus really eliminate two symbols from three equations.

Ex. 5. Eliminate x from the following equations :

$$\left. \begin{aligned} ax^m + bx^n + c &= 0, \\ a_1x^m + b_1x^n + c_1 &= 0. \end{aligned} \right\}$$

By Cross Multiplication, $\frac{x^m}{b_1c_1 - b_1c} = \frac{x^n}{ca_1 - c_1a} = \frac{1}{ab_1 - a_1b}$.

$$\therefore x^m = \frac{bc_1 - b_1c}{ab_1 - a_1b} \dots \dots \dots (1)$$

$$\text{and } x^n = \frac{ca_1 - c_1a}{ab_1 - a_1b} \dots \dots \dots (2)$$

Raising (1) to the n th power, $x^{mn} = \frac{(bc_1 - b_1c)^n}{(ab_1 - a_1b)^n}$.
 " (2) " " m th " " , $x^{mn} = \frac{(ca_1 - c_1a)^m}{(ab_1 - a_1b)^m}$;

$$\therefore \frac{(ca_1 - c_1a)^m}{(ab_1 - a_1b)^m} = \frac{(bc_1 - b_1c)^n}{(ab_1 - a_1b)^n},$$

whence $\frac{(ca_1 - c_1a)^m}{(b_1c_1 - b_1c)^n} = (ab_1 - a_1b)^{m-n}$. *Ans.*

Ex. 6 Shew that $\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d} = 1$, being given

$$\left. \begin{aligned} x &= by + cz + du, \dots \dots (1) \\ y &= ax + cz + du, \dots \dots (2) \\ z &= ax + by + du, \dots \dots (3) \\ u &= ax + by + cz \dots \dots (4) \end{aligned} \right\}$$

Adding ax to (1), by to (2), cz to (3), and du to (4), we get
 $(1+a)x = (1+b)y = (1+c)z = (1+d)u = ax + by + cz + du = k$, suppose ;

$\therefore x = \frac{k}{1+a}$, $y = \frac{k}{1+b}$, &c. Substituting these values in the equation $ax + by + cz + du = k$, we get

$$\frac{ka}{1+a} + \frac{kb}{1+b} + \frac{kc}{1+c} + \frac{kd}{1+d} = k ;$$

dividing out k , $\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d} = 1$.

Ex. 7. Find the relation between a , b and c , being given

$$\left. \begin{aligned} x^3 + 3xy^2 &= a^3, \dots\dots (1) \\ y^3 + 3x^2y &= b^3, \dots\dots (2) \\ xy &= c^3, \dots\dots (3) \end{aligned} \right\} .$$

Adding (1) and (2), $(x+y)^3 = a^3 + b^3$, whence $x+y = (a^3 + b^3)^{\frac{1}{3}}$.

Subtracting (2) from (1), $(x-y)^3 = a^3 - b^3$, whence $x-y = (a^3 - b^3)^{\frac{1}{3}}$.

From these equations, by addition and subtraction,

$$2x = (a^3 + b^3)^{\frac{1}{3}} + (a^3 - b^3)^{\frac{1}{3}},$$

$$\text{and} \quad 2y = (a^3 + b^3)^{\frac{1}{3}} - (a^3 - b^3)^{\frac{1}{3}};$$

$$\text{multiplying,} \quad 4xy = (a^3 + b^3)^{\frac{2}{3}} - (a^3 - b^3)^{\frac{2}{3}};$$

$$\text{using (3),} \quad 4c^2 = (a^3 + b^3)^{\frac{2}{3}} - (a^3 - b^3)^{\frac{2}{3}}. \quad \text{Ans.}$$

Ex. 8. Obtain an equation independent of x , being given

$$\left. \begin{aligned} x^5 + 10x + \frac{5}{x^5} &= a, \dots\dots (1) \\ 5x^3 + \frac{10}{x} + \frac{1}{x^5} &= b, \dots\dots (2) \end{aligned} \right\}$$

We know that $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

From this formula, putting successively $y = \frac{1}{x}$ and $y = -\frac{1}{x}$,

$$\text{we get} \quad \left(x + \frac{1}{x}\right)^5 = x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5},$$

$$\text{and} \quad \left(x - \frac{1}{x}\right)^5 = x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}.$$

Hence, from (1) and (2), by addition and subtraction,

$$\left(x + \frac{1}{x}\right)^5 = a + b, \text{ and } \left(x - \frac{1}{x}\right)^5 = a - b;$$

$$\text{whence} \quad x + \frac{1}{x} = (a+b)^{\frac{1}{5}}, \text{ and } x - \frac{1}{x} = (a-b)^{\frac{1}{5}}. \quad (3)$$

$$\text{Now, } \left(x + \frac{1}{x}\right)^2 - \left(x - \frac{1}{x}\right)^2 = \left(x^2 + 2 + \frac{1}{x^2}\right) - \left(x^2 - 2 + \frac{1}{x^2}\right) = 4;$$

hence, by substitution from (3) in the last identity,

$$(a+b)^{\frac{2}{5}} - (a-b)^{\frac{2}{5}} = 4. \quad \text{Ans.}$$

Ex. 9. Eliminate x and y from the equations,

$$\left. \begin{aligned} x+y &= a, \dots\dots(1) \\ x^3+y^3 &= b^3, \dots\dots(2) \\ x^5+y^5 &= c^5, \dots\dots(3) \end{aligned} \right\}$$

Cubing (1), $x^3+y^3+3xy(x+y)=a^3$;

subtracting (2), $3xy(x+y)=a^3-b^3$;

using (1) again, $3axy=a^3-b^3, \dots\dots\dots(4)$

Raising (1) to the 5th power, we have

$$x^5+5xy(x^3+2x^2y+2xy^2+y^3)+y^5=a^5 ;$$

subtracting (3), $5xy(x^3+2x^2y+2xy^2+y^3)=a^5-c^5$;

$$i.e., \quad 5xy\{x^3+y^3+2xy(x+y)\}=a^5-c^5 ;$$

using (1) and (2), $5xy(b^3+2axy)=a^5-c^5$;

substituting for xy from (4),

$$5 \frac{a^3-b^3}{3a} \{b^3+\frac{2}{3}(a^3-b^3)\}=a^5-c^5 ;$$

whence

$$5(a^3-b^3)(2a^3+b^3)=9a(a^5-c^5) ;$$

reducing,

$$a(a^5+9c^5)=5b^3(a^3+b^3). \quad \text{Ans.}$$

Ex. 10. Eliminate x , y and z from the equations

$$\left. \begin{aligned} x+y+z &= a, \dots\dots\dots(1) \\ x^2+y^2+z^2 &= b^2, \dots\dots\dots(2) \\ x^3+y^3+z^3 &= c^3, \dots\dots\dots(3) \\ xyz &= a^3, \dots\dots\dots(4) \end{aligned} \right\}$$

Squaring (1), and subtracting (2) from the result, we have

$$2(xy+yz+zx)=a^2-b^2 ;$$

$$\therefore xy+yz+zx=\frac{1}{2}(a^2-b^2), \dots\dots(5)$$

From (3) and (4), $c^3-3a^3=x^3+y^3+z^3-3xyz$

$$=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

$$=a\{b^2-\frac{1}{2}(a^2-b^2)\}, \text{ by (1), (2), (5),}$$

$$=a \times \frac{1}{2}(3b^2-a^2) ;$$

$$2(c^3-3a^3)=a(3b^2-a^2). \quad \text{Ans.}$$

Otherwise thus : $(x+y+z)^3=x^3+y^3+z^3+3(x+y)(y+z)(z+x)$

$$=x^3+y^3+z^3+3\{(x+y+z)(xy+yz+zx)$$

$$-xyz\} \text{ as can be readily seen ;}$$

\therefore by substitution, $a^3-c^3+3\{a \times \frac{1}{2}(a^2-b^2)-a^3\}$, which simplified gives the same result as before.

Ex. 11. Eliminate x, y and z from the equations

$$\left. \begin{aligned} x^2(y-z) &= a, \dots\dots (1) \\ y^2(z-x) &= b, \dots\dots (2) \\ z^2(x-y) &= c, \dots\dots (3) \\ xyz &= d, \dots\dots (4) \end{aligned} \right\}$$

Multiplying up (1), (2), (3), $abc = x^2y^2z^2(y-z)(z-x)(x-y)$

$$= d^2(y-z)(z-x)(x-y), \text{ by (4),}$$

$$[\text{by Formula, Art. 73}] = -d^2\{x^2(y-z) + y^2(z-x) + z^2(x-y)\}$$

$$= -d^2(a+b+c), \text{ by (1), (2), (3).}$$

$$\therefore abc + d^2(a+b+c) = 0. \quad \text{Ans.}$$

Ex. 12. Find the equation connecting a, b and c only, being given

$$\left. \begin{aligned} \frac{y}{z} + \frac{z}{y} &= a, \dots\dots (1) \\ \frac{z}{x} + \frac{x}{z} &= b, \dots\dots (2) \\ \frac{x}{y} + \frac{y}{x} &= c, \dots\dots (3) \end{aligned} \right\}$$

We know that $\left(\frac{y}{z} + \frac{z}{y}\right)^2 + \left(\frac{z}{x} + \frac{x}{z}\right)^2 + \left(\frac{x}{y} + \frac{y}{x}\right)^2$

$$= 4 + \left(\frac{y}{z} + \frac{z}{y}\right)\left(\frac{z}{x} + \frac{x}{z}\right)\left(\frac{x}{y} + \frac{y}{x}\right); [\text{Page 156}]$$

\therefore by substitution,

$$a^2 + b^2 + c^2 = 4 + abc. \quad \text{Ans.}$$

EXAMPLES 123.

Eliminate x from the following equations :

1. $ax + b = 0,$

2. $ax^2 - b^2 = 0,$

3. $ax + bx^{-1} = c,$

$ax^2 + 2bx + c = 0.$

$cx^2 - a^2 = 0.$

$ax - bx^{-1} = d.$

4. $x + x^{-1} = a,$

5. $a_1x^2 + b_1x + c_1 = 0,$

6. $ax^4 + bx^3 + c = 0,$

$x^3 + x^{-2} = b^3.$

$a_2x^2 + b_2x + c_2 = 0.$

$a_1x^4 + b_1x^3 + c_1 = 0$

Eliminate x and y from the following equations

7. $a_1x + b_1y + c_1 = 0,$

8. $lx + my = ma - lb,$

$a_2x + b_2y + c_2 = 0,$

$mx - ly = la + mb,$

$a_3x + b_3y + c_3 = 0.$

$x^2 + y^2 = 1.$

$$\begin{aligned} 9. \quad x+y &= a, \\ x^2+y^2 &= b^2, \\ x^3+y^3 &= c^3. \end{aligned}$$

$$\begin{aligned} 10. \quad x(x^2+3y^2) &= 4(a^3+b^3), \\ y(y^2+3x^2) &= 4(a^3-b^3), \\ x^3+y^3 &= c^3. \end{aligned}$$

$$\begin{aligned} 11. \quad \frac{x}{y} + \frac{y}{x} &= a, \\ \frac{1}{x} + \frac{1}{y} &= b, \\ x+y &= c. \end{aligned}$$

$$\begin{aligned} 12. \quad x + \frac{1}{x} &= a, \\ y + \frac{1}{y} &= b, \\ xy + \frac{1}{xy} &= c \end{aligned}$$

Eliminate x , y and z from the following equations :

$$\begin{aligned} 13. \quad ax+by+cz &= 0, \\ bx+cy+az &= 0, \\ cx+ay+bz &= 0. \end{aligned}$$

$$\begin{aligned} 14. \quad x+y &= l(x-y), \\ y+z &= m(y-z), \\ z+x &= n(z-x). \end{aligned}$$

$$\begin{aligned} 15. \quad x+y+z &= l, \\ xy+yz+zx &= m^2, \\ x^2+y^2+z^2 &= n^2, \\ xyz &= p^3. \end{aligned}$$

$$\begin{aligned} 16. \quad x^2(y+z) &= a, \\ y^2(z+x) &= b, \\ z^2(x+y) &= c, \\ xyz &= d. \end{aligned}$$

$$\begin{aligned} 17. \quad (y+z)^2 &= ayz, \\ (z+x)^2 &= bzx, \\ (x+y)^2 &= cxy. \end{aligned}$$

$$\begin{aligned} 18. \quad x+y+z &= a, \\ x^2+y^2+z^2 &= b^2, \\ (x+y)(y+z)(z+x) &= c^3 \end{aligned}$$

$$19. \quad \frac{x}{y+z} = a, \quad \frac{y}{z+x} = b, \text{ and } \frac{z}{x+y} = c.$$

$$20. \quad x=by+cz, y=cz+ax, \text{ and } z=ax+by.$$

$$21. \quad \text{If } y-z=ax, z-x=by, \text{ and } x-y=cz, \text{ prove that}$$

$$abc + a + b + c = 0.$$

CHAPTER XXXV.

MISCELLANEOUS THEOREMS AND ARTIFICES.

221. Artifices. The following examples are intended to illustrate how by suitable algebraic operations nice results may be easily deduced from given conditions.

Ex. 1. If $x + \frac{1}{x} = p$, prove that $x^3 + \frac{1}{x^3} = p^3 - 3p$. B. U. 1864.

$$\text{Since } x + \frac{1}{x} = p, \quad \left(x + \frac{1}{x}\right)^3 = p^3.$$

\therefore expanding the left-hand side, $x^3 + \frac{1}{x^3} + 3x \cdot \frac{1}{x} \left(x + \frac{1}{x}\right) = p^3$;

$$\text{i.e., } x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = p^3.$$

$$\text{But } x + \frac{1}{x} = p, \text{ given.}$$

$$\therefore \text{ by substitution, } x^3 + \frac{1}{x^3} + 3p = p^3.$$

$$\therefore \text{ by transposition, } x^3 + \frac{1}{x^3} = p^3 - 3p.$$

Ex. 2. If $\frac{a}{b} + \frac{c}{d} = \frac{b}{a} + \frac{d}{c}$, prove that $\frac{a^3}{b^3} + \frac{c^3}{d^3} = \frac{b^3}{a^3} + \frac{d^3}{c^3}$. M. U. 1866.

$$\text{Given } \frac{a}{b} + \frac{c}{d} = \frac{b}{a} + \frac{d}{c};$$

$$\therefore \text{ transposing, } \frac{a}{b} - \frac{b}{a} = \frac{d}{c} - \frac{c}{d}; \dots\dots\dots(1)$$

$$\text{cubing both sides, } \frac{a^3}{b^3} - \frac{b^3}{a^3} - 3\left(\frac{a}{b} - \frac{b}{a}\right) = \frac{d^3}{c^3} - \frac{c^3}{d^3} - 3\left(\frac{d}{c} - \frac{c}{d}\right) \dots(2)$$

$$\text{Add (2) to 3 times (1); thus } \frac{a^3}{b^3} - \frac{b^3}{a^3} = \frac{d^3}{c^3} - \frac{c^3}{d^3}.$$

$$\therefore \text{ by transposition, } \frac{a^3}{b^3} + \frac{c^3}{d^3} = \frac{b^3}{a^3} + \frac{d^3}{c^3}.$$

Ex. 3. If $a + b + c = 0$, shew that $\frac{a^2 + b^2 + c^2}{a^3 + b^3 + c^3} + \frac{2}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = 0$.

M. U. 1872.

$$\text{Given } a + b + c = 0, \quad \therefore a + b = -c.$$

$$\therefore \text{ cubing both sides, } a^3 + b^3 + 3ab(a + b) = -c^3.$$

$$\begin{aligned} \therefore \text{ by transposition, } a^3 + b^3 + c^3 &= -3ab(a + b) \\ &= -3ab(-c), \therefore a + b = -c, \\ &= 3abc. \end{aligned}$$

$$\begin{aligned}
 \therefore \text{ the given expression} &= \frac{a^3 + b^3 + c^3}{3abc} + \frac{2}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \\
 &= \frac{a^3 + b^3 + c^3 + 2(ab + bc + ca)}{3abc} \\
 &= \frac{(a + b + c)^3}{3abc} \\
 &= 0, \quad \because a + b + c = 0.
 \end{aligned}$$

Ex. 4. If $xy = ab(a + b)$, and $x^3 - xy + y^3 = a^3 + b^3$, shew that

$$\left(\frac{x}{a} - \frac{y}{b} \right) \left(\frac{x}{b} - \frac{y}{a} \right) = 0. \quad \text{B. U. 1872.}$$

$$\begin{aligned}
 \text{Multiplying out, } \left(\frac{x}{a} - \frac{y}{b} \right) \left(\frac{x}{b} - \frac{y}{a} \right) &= \frac{x^2}{ab} - xy \left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \frac{y^2}{ab} \\
 &= \frac{x^2 + y^2}{ab} - xy \cdot \frac{a^2 + b^2}{a^2 b^2} + \frac{xy}{ab} - \frac{xy}{ab} \\
 &= \frac{x^2 + y^2 - xy}{ab} - xy \cdot \frac{a^2 + b^2 - ab}{a^2 b^2} \\
 &= \frac{a^3 + b^3}{ab} - \frac{a^3 + b^3}{ab} = 0.
 \end{aligned}$$

Ex 5. If $x + y + z = xyz$, then will

$$\frac{x+y}{1-xy} + \frac{y+z}{1-yz} + \frac{z+x}{1-zx} = \frac{x+y}{1-xy} \cdot \frac{y+z}{1-yz} \cdot \frac{z+x}{1-zx}.$$

Given $x + y + z = xyz$;

transposing, $x + y = xyz - z = -z(1 - xy)$;

dividing by $1 - xy$, $\frac{x+y}{1-xy} = -z$; (1)

similarly, we obtain $\frac{y+z}{1-yz} = -x$, (2)

and $\frac{z+x}{1-zx} = -y$; (3)

$$\begin{aligned}
 \text{adding up (1), (2), (3), } \frac{x+y}{1-xy} + \frac{y+z}{1-yz} + \frac{z+x}{1-zx} &= -(x+y+z) \\
 &= -xyz \dots \dots \dots H, p. \\
 &= (-z) \cdot (-x) \cdot (-y) \\
 &= \frac{x+y}{1-xy} \cdot \frac{y+z}{1-yz} \cdot \frac{z+x}{1-zx}.
 \end{aligned}$$

[using (1), (2), (3)]

Ex. 6. If $ab+ac+bc=1$, prove that $\left(1-\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}-\frac{c^2}{1+c^2}\right)^2$

$$=\frac{4a^2b^2c^2}{(1+a^2)(1+b^2)(1+c^2)}. \quad \text{M. U. 1872.}$$

Given $ab+ac+bc=1$

Add a^2 ; thus $a^2+ab+ac+bc$ or $(a+b)(a+c)=1+a^2$.

Similarly $(b+a)(b+c)=1+b^2$, $(c+a)(c+b)=1+c^2$.

$$\begin{aligned} \therefore 1-\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}-\frac{c^2}{1+c^2} &= 1-\frac{a^2}{(a+b)(a+c)}-\frac{b^2}{(b+a)(b+c)}-\frac{c^2}{(c+a)(c+b)} \\ &= \frac{(a+b)(b+c)(c+a)-a^2(b+c)-b^2(c+a)-c^2(a+b)}{(a+b)(b+c)(c+a)} \\ &= \frac{2abc}{(a+b)(b+c)(c+a)}; \dots\dots\dots \text{Formula, Art. 80.} \end{aligned}$$

$$\begin{aligned} \therefore \text{squaring, } \left(1-\frac{a^2}{1+a^2}-\frac{b^2}{1+b^2}-\frac{c^2}{1+c^2}\right)^2 &= \frac{4a^2b^2c^2}{(a+b)^2(b+c)^2(c+a)^2} \\ &= \frac{4a^2b^2c^2}{\{(a+b)(a+c)\}\{(b+a)(b+c)\}\{(c+a)(c+b)\}} \\ &= \frac{4a^2b^2c^2}{(1+a^2)(1+b^2)(1+c^2)}, \because 1+a^2=(a+b)(a+c), \&c. \end{aligned}$$

Ex. 7. If ax^3+bx^2+cx+d be a perfect cube, shew that

$$bc=9ad, \text{ and } ac^3=b^3d$$

Let $ax^3+bx^2+cx+d=(lx+m)^3$, *identically*,

$$=l^3x^3+3l^2mx^2+3lm^2x+m^3;$$

comparing coefficients, $a=l^3$, $b=3l^2m$, $c=3lm^2$, $d=m^3$;

$$\left. \begin{aligned} \therefore bc &= 9l^3m^3 = 9ad, \\ \text{and } ac^3 &= 27l^6m^6 = b^3d \end{aligned} \right\}$$

EXAMPLES 124.

1. If $x-\frac{1}{x}=1$, prove that $x^2+\frac{1}{x^2}=3$, and $x^3-\frac{1}{x^3}=4$.

2. If $x=md+\frac{b}{m}$, and $y=ab$, prove that $m^3a^3+\frac{b^3}{m^3}=x^3-3xy$.

3. If $\frac{a}{b}+\frac{b}{a}=\frac{c}{d}+\frac{d}{c}$, then will $\frac{a^3}{b^3}+\frac{b^3}{a^3}=\frac{c^3}{d^3}+\frac{d^3}{c^3}$, and

$$c^2d^2(a^4+b^4)=a^2b^2(c^4+d^4).$$

4. If $a^2+b^2=1=c^2+d^2$, prove that $(ac+bd)(ac-bd)=(a+d)(a-d)$.

5. If $a=y-\frac{1}{y}$, and $b=y^3-\frac{1}{y^3}$, then will $a^2(a^2+4)=b^2$.
6. If $\left(a+\frac{1}{a}\right)^3=3$, prove that $a^2+\frac{1}{a^2}=0$ B U. 1876.
7. If $(p^2+q^2)^2=3p^2q^2$, then will $p^6+q^6=0$.
8. If $a(1-b^2)^{\frac{1}{2}}+b(1-a^2)^{\frac{1}{2}}=1$, prove that $a^2+b^2=1$.
9. If $a+b+c=0$, then will $\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=3$
10. If $a+b+c=0$, then $bc-a^2=ca-b^2=ab-c^2=(ab+ac+bc)$.
11. If $a+b+c=0$, then $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$.
12. If $x+y+z=xyz$, prove that
- $$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}-1=(1+z)(1+y)(1+x)$$
13. If $x+y+z+xyz=0$, prove that
- $$\frac{x+y}{1+xy}+\frac{y+z}{1+yz}+\frac{z+x}{1+zx}+\frac{(1+x)(1+y)(1+z)}{(1+xy)(1+yz)(1+zx)}=0$$
14. If $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$, then each side $=\frac{1}{b-a}+\frac{1}{b-c}$.
15. If $x : z = x-y : y-z$, then $\frac{y+x}{y-x}+\frac{y+z}{y-z}=2$.
16. If $ac=b^2$, then will $\frac{a}{a^2+b^2}=\frac{c}{b^2+c^2}$.
17. If $2ac=b(a+c)$, then $(b+c-a)^2+(a+b-c)^2=2(a+c-b)^2$.
18. If $x^2+y^2=1$, then will
- $$(2x^2-1)^2=(1-2xy)(1+2xy); (4x^3-3x)^2+(4y^3-3y)^2=1.$$
19. If $x^2+y^2=1$, then will $x\left(1+\frac{1}{y}\right)+y\left(1+\frac{y}{x}\right)=\frac{1}{x}+\frac{1}{y}$.
20. If $\frac{a^2-1}{b^2-1}=\frac{b(x-a)}{a(x-b)}$, then will $\frac{1-x}{1+x}=\frac{1-a}{1+a}\cdot\frac{1-b}{1+b}$.
21. If $x=by=cx$, and $1-x^2=(1-y)(1-z)$, then will
- $$\frac{1-x}{1+x}=\frac{1-b}{1+b}\cdot\frac{1-c}{1+c}$$
22. If $\frac{1-\frac{b}{a}}{1+\frac{b}{a}}+\frac{c^2(1+a^2)}{a^2(1+c^2)}=1$, then $c^2=ab$.
23. If $x+y=1$, prove that $(x^2-y^2)^2+xy=x^2+y^2$.

24. If $x^2 = yz$, prove that

$$y(x^2 - xy)^2 + x(y^2 - zx)^2 + 2x(y^2 - zx)(x^2 - xy) = 0.$$

25. If $2a^2 = (a-1)(a+2)$, prove that $a^5 + a = 6$.

If $a+b+c=0$, prove that

$$26. \quad 2a^2 + bc = (a-b)(a-c). \quad 27. \quad 2b^2 + ca = (b-a)(b-c).$$

$$28. \quad \frac{a}{b^2c^2} + \frac{b}{c^2a^2} + \frac{c}{a^2b^2} = \frac{3}{abc}. \quad 29. \quad \frac{a^2}{2a^2+bc} + \frac{b^2}{2b^2+ca} + \frac{c^2}{2c^2+ab} = 1.$$

$$30. \quad a(a^2 - bc) + b(b^2 - ca) + c(c^2 - ab) = 0.$$

$$31. \quad (a^2 - bc)^2 = a^2b^2 + b^2c^2 + c^2a^2, \text{ and } -(ab + bc + ca) = \frac{1}{2}(a^2 + b^2 + c^2).$$

$$32. \quad a(b-c)^2 + b(c-a)^2 + c(a-b)^2 = 0.$$

$$33. \quad \left(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c} \right) \left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} \right) = 9.$$

$$34. \quad \frac{1}{a^2+b^2-c^2} + \frac{1}{b^2+c^2-a^2} + \frac{1}{c^2+a^2-b^2} = 0.$$

$$35. \quad \text{If } a+b+c=1, \quad ab+bc+ca=\frac{1}{3}, \text{ and } abc=\frac{1}{27},$$

$$\text{then will } \frac{1}{a+bc} + \frac{1}{b+ca} + \frac{1}{c+ab} = \frac{27}{4}.$$

36. If $x/a + y/b + z/c = xyz/abc$, prove that

$$\frac{a}{a+x} + \frac{b}{b+y} + \frac{c}{c+z} - 1 = \frac{2abc}{(a+x)(b+y)(c+z)}.$$

$$37. \quad \text{If } \frac{1}{y} + \frac{1}{z} = \frac{4}{x}, \text{ then } (x-y+z)^2 + (x+y-z)^2 + 2(x+y-z)^2 = 2(y+z)^2.$$

$$38. \quad \text{If } b = \frac{2a}{1-a^2}, \quad c = \frac{2b}{1-b^2}, \quad d = \frac{2c}{1-c^2}, \text{ then will}$$

$$a + 2b + 4c + 8d = a^{-1}.$$

$$\text{If } \frac{a+b}{1-ab} + \frac{c+d}{1-cd} = 0, \text{ prove that}$$

$$39. \quad \frac{a+c}{1-ac} + \frac{b+d}{1-bd} = \frac{b+c}{1-bc} + \frac{a+d}{1-ad} = 0.$$

$$40. \quad \frac{(1+a^2)(1+b^2)}{(1-ab)^2} = \frac{(1+c^2)(1+d^2)}{(1-cd)^2}.$$

222. Meaning of a/∞ . We readily see that $\frac{5}{10} = \frac{1}{2}$, $\frac{5}{10^2} = \frac{1}{20}$,

$\frac{5}{10^3} = \frac{1}{2000}$, $\frac{5}{10^4} = \frac{1}{20000}$, &c. Thus the numerator remaining

constant, the value of a fraction diminishes continually as its denominator increases, and can be made as small as we please by increasing the denominator indefinitely. In other words, *the quotient of a finite quantity by an infinitely large quantity is infinitely small*. This is usually put as

$$\frac{a}{\infty} = 0, a \text{ being any finite quantity.}$$

223. Meaning of $a/0$ We find that $2 \div \frac{1}{5} = 10$, $2 \div \frac{1}{10} = 20$, $2 \div \frac{1}{100} = 200$, &c. That is for the same dividend, the quotient increases continually as the divisor diminishes, and can be made as large as we please by diminishing the divisor indefinitely. In other words, *the quotient of a finite quantity by an infinitely small quantity is infinitely large*.

Symbolically, $\frac{a}{0} = \infty$, a being any finite quantity.

224. Theorem. *If the sum of the squares of any number of real quantities be zero, then each of those quantities must be equal to zero.*

Let $a^2 + b^2 = 0$. Now the square of a real quantity is always positive, whether that quantity be positive or negative, for instance, $(+a)^2 = +a^2$, as also $(-a)^2 = +a^2$.

Hence if a and b be not each zero, $a^2 + b^2$ can never vanish, for the latter will then be the sum of two positive quantities. Thus must $a = 0$, and $b = 0$, simultaneously.

Similarly, if $a^2 + b^2 + c^2 = 0$, then must $a = 0$, $b = 0$, and also $c = 0$; and so on.

N. B. The student should pay attention to the condition that each of a , b , c , &c., must be real. On this condition a^2 , b^2 , &c., are positive. Notice that $(\sqrt{-3})^2$ is negative, $\sqrt{-3}$ being imaginary.

Ex. 1. If $a^2 + b^2 + c^2 = ab + bc + ca$, and a , b and c be all real, then $a = b = c$.

Multiplying by 2, $2a^2 + 2b^2 + 2c^2 = 2ab + 2bc + 2ca$;

transposing and re-arranging terms,

$$(a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2) = 0;$$

$$\text{i.e., } (a-b)^2 + (b-c)^2 + (c-a)^2 = 0;$$

therefore, since a , b and c , and consequently $a-b$, $b-c$ and $c-a$ are all real, $a-b=0$, $b-c=0$, and $c-a=0$; i.e., $a=b=c$.

Ex. 2. If $(x^2+y^2+z^2)(a^2+b^2+c^2)=(ax+by+cz)^2$, prove that

$$x : y : z = a : b : c.$$

Multiplying out each side of the given equation, and cancelling

$$a^2x^2+b^2y^2+c^2z^2, \text{ we have}$$

$$a^2(y^2+z^2)+b^2(z^2+x^2)+c^2(x^2+y^2)=2abxy+2bcyz+2caxz;$$

transposing and re-arranging terms, we get

$$(a^2y^2-2abxy+b^2x^2)+(b^2z^2-2bcyz+c^2y^2)+(c^2x^2-2caxz+a^2z^2)=0;$$

$$\text{i.e., } (ay-bx)^2+(bz-cy)^2+(cx-az)^2=0;$$

since the left-hand side is the sum of three squares,

$$ay-bx=0, \quad bz-cy=0, \quad \text{and} \quad cx-az=0;$$

$$\left. \begin{array}{l} \text{i.e., } ay=bx, \text{ whence } \frac{y}{b}=\frac{x}{a}; \\ \quad bz=cy, \quad \quad \quad \frac{z}{c}=\frac{y}{b}; \\ \text{and } cx=az, \quad \quad \quad \frac{x}{a}=\frac{z}{c}. \end{array} \right\}$$

$$\therefore x : y : z = a : b : c.$$

EXAMPLES 125.

(All the quantities below are supposed real.)

1. If $a^2+b^2+c^2=2c(a+b)$, prove that $a=b=c$.
2. If $x^2+y^2=2(x+y-1)$, prove that $x=y=1$.
3. If $a^2+b^2=4(a-b-2)$, prove that $a=-b=2$.
4. If $a^2+b^2+c^2+3d^2=2d(a+b+c)$, then $a=b=c=d$.
5. If $(1+x)^2+(1+y)^2+(1+z)^2=4(x+y+z)$, then $x=y=z=1$.
6. If $a^2+b^2+2=(1+a)(1+b)$, prove that $a=b=1$.
7. If $a^2+b^2+4c^2=ab+2bc+2ca$, then $a=b=2c$.
8. If $3x^2+2y^2+3z^2=2(xy+yz+zx)$, then $x=y=z$.
9. If $2x^2+y^2+2z^2=xv+yz+3zx$, then $x=y=z$.
10. If $1+xx'+yy'=\sqrt{(1+x^2+y'^2)}\sqrt{(1+x'^2+y^2)}$, then $x=x', y=y'$.

225. Theorem. If $A \times B=0$, either A must $=0$, or B must $=0$.

For if none of A and B be zero, then AB cannot $=0$.

\therefore either A must $=0$, or B must $=0$. Both A and B may be zero at the same time, but we are not justified in assuming it;

all that is *necessary* and *sufficient* in order that AB may be zero, is that *some one* of A and B must be zero.

Cor. 1. If $A \times B = 0$, and if A be *known* to be not zero, then must $B = 0$ *alone*; e. g., if $2x = 0$, x only $= 0$.

Cor. 2. If $AB = AC$, then must either $A = 0$, or $B = C$.

For, by transposition, $AB - AC = 0$, i. e., $A(B - C) = 0$.

\therefore either $A = 0$, or $B - C = 0$, i. e., $B = C$.

Cor. 3. If $AB = AC$, and if A be *known* to be not zero, then must $B = C$.

Ex 1. If $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}$, prove that

$$\frac{1}{a^{2n+1}} + \frac{1}{b^{2n+1}} + \frac{1}{c^{2n+1}} = \frac{1}{a^{2n+1} + b^{2n+1} + c^{2n+1}}.$$

We have $\left(\frac{1}{a} + \frac{1}{b}\right) + \left(\frac{1}{c} - \frac{1}{a+b+c}\right) = 0$;

i. e., $\frac{a+b}{ab} + \frac{a+b+c-c}{c(a+b+c)} = 0$; $\therefore (a+b) \left\{ \frac{1}{ab} + \frac{1}{c(a+b+c)} \right\} = 0$;

multiplying by $abc(a+b+c)$, $(a+b)(ca+cb+c^2+ab) = 0$.

$\therefore (a+b)(b+c)(c+a) = 0$.

Then some *one* of the following equations must hold good:

$$a+b=0, b+c=0, c+a=0.$$

If $a+b=0$, $a=-b$. $\therefore a^{2n+1} = (-b)^{2n+1} = -b^{2n+1}$,

and $a^{2n+1} + b^{2n+1} = 0$, by transposition.

$$\begin{aligned} \therefore \frac{1}{a^{2n+1}} + \frac{1}{b^{2n+1}} + \frac{1}{c^{2n+1}} &= -\frac{1}{b^{2n+1}} + \frac{1}{b^{2n+1}} + \frac{1}{c^{2n+1}} = \frac{1}{c^{2n+1}} \\ &= \frac{1}{a^{2n+1} + b^{2n+1} + c^{2n+1}}, \therefore a^{2n+1} + b^{2n+1} = 0 \end{aligned}$$

In like manner, the required result can be established if

$$b+c=0, \text{ or } c+a=0.$$

Ex. 2. Assuming $\frac{a+b-c}{a+b} = \frac{b+c-a}{b+c} = \frac{c+a-b}{c+a}$, and that $a+b+c$

is not zero, shew that $a=b=c$. C. U. 1873.

$$\text{Given } \frac{a+b-c}{a+b} = \frac{b+c-a}{b+c} = \frac{c+a-b}{c+a};$$

$$\text{subtracting each from 2, } 2 - \frac{a+b-c}{a+b} = 2 - \frac{b+c-a}{b+c} = 2 - \frac{c+a-b}{c+a};$$

$$\text{simplifying, } \frac{a+b+c}{a+b} = \frac{a+b+c}{b+c} = \frac{a+b+c}{c+a};$$

$$\therefore \text{ either } a+b+c=0, \text{ or } \frac{1}{a+b} = \frac{1}{b+c} = \frac{1}{c+a}.$$

But $a+b+c$ is not zero.....Given.

$$\therefore \frac{1}{a+b} = \frac{1}{b+c} = \frac{1}{c+a},$$

$$i. e., a+b=b+c=c+a; \text{ whence } a=b=c.$$

Ex. 3. If $x^2 - yz = y^2 - zx$, and if x and y be unequal, then will each of the expressions $\frac{1}{2}(x^2 + y^2 + z^2) = z^2 - xy$.

$$\text{Given } x^2 - yz = y^2 - zx.$$

$$\therefore \text{ by transposition, } x^2 - y^2 - yz + zx = 0.$$

$$\therefore (x-y)(x+y) + z(x-y) = 0, \text{ i.e., } (x-y)(x+y+z) = 0.$$

$$\therefore \text{ either } x-y=0, \text{ or } x+y+z=0$$

But by the question, since x and y are unequal, $x-y \neq 0$.

$$\therefore \text{ must } x+y+z=0 \dots\dots\dots(1)$$

Let each of $x^2 - yz$ and $y^2 - zx$ be now denoted by k .

$$\text{Then } k = x^2 - yz, \text{ also } k = y^2 - zx;$$

$$\therefore \text{ by addition, } 2k = x^2 + y^2 - z(x+y)$$

$$= x^2 + y^2 - z(-z), \because x+y = -z \text{ from (1),}$$

$$= x^2 + y^2 + z^2.$$

$$\therefore k = \frac{1}{2}(x^2 + y^2 + z^2) \dots\dots\dots(2)$$

$$\text{Also, } z^2 - xy = (x+y)^2 - xy, \because z = -(x+y) \text{ from (1),}$$

$$= x^2 + xy + y^2, \text{ expanding,}$$

$$= x^2 + y(x+y)$$

$$= x^2 + y(-z), \because x+y = -z,$$

$$= x^2 - yz = k \dots\dots\dots(3)$$

$$\therefore \text{ from (2) and (3); } x^2 - yz = y^2 - zx = \frac{1}{2}(x^2 + y^2 + z^2) = z^2 - xy.$$

EXAMPLES 126.

1. If $(1+a)(1+c)=2(1+ac)$, then either $a=1$, or $c=1$.
2. If $(a+b+c)(ab+bc+ca)=abc$, prove that either

$$b=-c, \text{ or } c=-a, \text{ or } a=-b.$$
3. If $\frac{x-y}{z} + \frac{y-z}{x} + \frac{z-x}{y} = 1$, and $x-y+z$ not 0, then $\frac{1}{x} = \frac{1}{y} + \frac{1}{z}$.
4. If $a+b+c+abc=abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc}\right)$, prove that one of the quantities a , b and c must be unity.
5. If $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$, shew that two of the quantities a , b and c must be equal.
6. If $ax+by+c=0$, $bx+cy+a=0$, and $x^2+y^2=2$, then either

$$a+b+c=0, \text{ or } a^3+c^3-2b^3=(a+c-2b)(bc+ca+ab)$$
7. If $x+y=az$, and $y+z=ax$, prove that $z+x=ay$, being given that z and x are unequal.
8. x , y and z being unequal, if $x^2+xy+y^2=y^2+yz+z^2$, then will each $=x^2+zx+x^2$.
9. If $\frac{y+z}{x} = \frac{z+x}{y}$, prove that each fraction $= \frac{x+y}{z} = -1$, x and y being unequal.
10. If $\frac{abc}{b+c} - a^2 = \frac{abc}{c+a} - b^2$, prove that each side $= \frac{abc}{a+b} - c^2$, being given that a and b are unequal.
11. If $a + \frac{bc-a^2}{a^2+b^2+c^2} = b + \frac{ca-b^2}{a^2+b^2+c^2}$, then each $= c + \frac{ab-c^2}{a^2+b^2+c^2}$, and each $= 0$, if $a+b+c=1$. (a, b, c are all unequal).
12. If $\frac{1-az}{1+az} \cdot \frac{1-bz}{1+bz} = \frac{1-z}{1+z}$, then $1-z^2 = \frac{(1-a)(1-b)}{ab}$, or $z=0$.
13. If $\left(\frac{a^2+c^2-b^2}{2ac}\right)^2 + \left(\frac{a^2+b^2-c^2}{2ab}\right)^2 = 1$, then either

$$a^2=b^2+c^2, \text{ or } a^2(b^2+c^2)=(b^2-c^2)^2.$$
14. If $l(1-m)=m(1-n)=n(1-l)$, then will $lmn=-1$, or $l=m=n$.
15. If $cx(b-y)=bz(a-x)=ay(c-z)$, then will $x, y, z = -abc$, or

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$
16. If $a^3+b^3+c^3=3abc$, prove that either $a+b+c=0$, or $a=b=c$.

17. $\frac{x}{a} + \frac{a}{x} = \frac{a}{b} + \frac{b}{a}$, then $\frac{x^n}{a^n} + \frac{a^n}{x^n} = \frac{a^n}{b^n} + \frac{b^n}{a^n}$

18. If $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$, prove that each of these fractions is equal to either $\frac{1}{2}$ or -1 .

226. Inequalities. a is said to be *greater* than b when $a-b$ is a *positive quantity*. Thus, since $5-3(=+2)$ is positive, 5 is greater than 3; $(-2)-(-6)=-2+6=+4$, which is positive, and $\therefore -2$ is *algebraically* greater than -6 .

227. Important Theorem. *The sum of the squares of two unequal quantities is always greater than twice their product.*

That is, $a^2 + b^2 > 2ab$.

For, a and b being supposed real, $(a-b)^2$ is always positive, whether $a-b$ be positive or negative, *i.e.*, whether $a > b$, or $b > a$.

$$\therefore a^2 + b^2 - 2ab \text{ is positive ;}$$

$$\therefore a^2 + b^2 > 2ab$$

N. B. If $a=b$, $a^2 + b^2 = 2a^2 = 2ab$. Hence, including this case also, we may say that $a^2 + b^2$ is *never less than* $2ab$.

Ex. 1. A man receives $\frac{x}{y}$ ths of Rs. 10 and afterwards $\frac{y}{x}$ ths of Rs. 10. He then gives away Rs. 20. Show that he cannot lose by the transaction. C. U. 188.

The total sum received $= \left(\frac{x}{y} + \frac{y}{x} \right)$ of 10 rupees,

and the sum given away = 20 rupees.

Evidently he does not lose,

$$\text{if } \left(\frac{x}{y} + \frac{y}{x} \right) \times 10 \text{ not } < 20 ;$$

$$\text{i.e., if } \frac{x}{y} + \frac{y}{x} \text{ not } < 2 ;$$

$$\text{i.e., if } x^2 + y^2 \text{ not } < 2xy ;$$

$$\text{i.e., if } x^2 - 2xy + y^2 \text{ not } < 0 ;$$

$$\text{i.e., if } (x-y)^2 \text{ not } < 0, \text{ i.e., not negative,}$$

which is true, because $(x-y)^2$ being a square quantity cannot be negative. Therefore the man cannot lose.

Ex. 2. If a , b and c be any unequal positive quantities, show that $(a+b)(b+c)(c+a) > 8abc$.

Multiplying out, we have

$$(a+b)(b+c)(c+a) = a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2) + 2abc. \quad (1)$$

Now, $b^2+c^2 > 2bc$; multiplying each side by a , and observing that a is given positive,

$$a(b^2+c^2) > 2abc.$$

Similarly,

$$b(c^2+a^2) > 2abc, \text{ and so on.}$$

\therefore the right side of (1) $> 2abc + 2abc + 2abc + 2abc$, i.e., $8abc$;

\therefore by (1), $(a+b)(b+c)(c+a) > 8abc$.

N. B. What would be the result, if only two of a , b and c be equal what again, if all the three were equal?

Ex. 3. Which is the greater, $\sqrt{13} + \sqrt{5}$ or $\sqrt{17} + \sqrt{3}$?

$$\sqrt{13} + \sqrt{5} > \text{ or } < \sqrt{17} + \sqrt{3},$$

according as $(\sqrt{13} + \sqrt{5})^2 > \text{ or } < (\sqrt{17} + \sqrt{3})^2$,

$$,, \quad 18 + 2\sqrt{65} > \text{ or } < 20 + 2\sqrt{51}, \text{ by expansion,}$$

$$,, \quad 2\sqrt{65} > \text{ or } < 2 + 2\sqrt{51}, \text{ by transposition,}$$

$$,, \quad \sqrt{65} > \text{ or } < 1 + \sqrt{51}, \text{ dividing by 2,}$$

$$,, \quad 65 > \text{ or } < 52 + 2\sqrt{51}, \text{ by squaring,}$$

$$,, \quad 13 > \text{ or } < 2\sqrt{51},$$

$$,, \quad 169 > \text{ or } < 4 \times 51, \text{ squaring.}$$

Now, $169 < 4 \times 51$, i.e., 204 ; $\therefore \sqrt{13} + \sqrt{5} < \sqrt{17} + \sqrt{3}$. *Ans.*

EXAMPLES 127.

(All the quantities below are real, positive and unequal.)

Prove that

$$1. \quad ab(a+b) + bc(b+c) + ca(c+a) > 6abc.$$

$$2. \quad (a+b+c)(ab+bc+ca) > 9abc.$$

$$3. \quad a^3+b^3 > a^2b+ab^2.$$

$$4. \quad a^3+b^3+c^3 > 3abc.$$

$$5. \quad (a+b)(b+c)(c+a) < 2(a^3+b^3+c^3+abc).$$

Which is the greater :

$$6. \quad 3 + \sqrt{7} \text{ or } 4 + \sqrt{3}?$$

$$7. \quad \sqrt{7} + \sqrt{5} \text{ or } \sqrt{10} + \sqrt{3}?$$

$$8. \quad 3(x^4+x^2y^2+y^4) \text{ or } (x^2+xy+y^2)^2? \quad 9. \quad (x^3+1)^2 \text{ or } x^3(x+1)^2?$$

228. Method of Detached Coefficients. When in an operation of multiplication both the multiplier and the multiplicand consist of powers of a single letter, or when they consist of homogeneous expressions of two letters, the method illustrated

below and known as the **method of detached coefficients** is a great simplification of the ordinary one. In this method *the coefficients are only written down*, and the powers of the symbol or symbols, are understood. Thus $4x^4 + 3x^3 + 2x^2 + x - 5$ is written

When any intermediate power is wanting, its place should be filled up *by writing a zero*.

This method is also applicable to Division.

Ex. 1. Multiply $2x^3 - x^2 + x - 3$ by $2x^2 + x - 3$.

Ordinary method :
$$\begin{array}{r} 2x^3 - x^2 + x - 3 \\ \underline{2x^2 + x - 3} \\ 4x^5 - 2x^4 + 2x^3 - 6x^2 \\ \underline{2x^4 - x^3 + x^2 - 3x} \\ 6x^3 + 3x^2 - 3x + 9 \\ \underline{4x^5 - 5x^3 - 2x^2 - 6x + 9} \end{array}$$

Detached coefficients : $\begin{array}{r} 2-1+1-3 \\ \underline{2+1-3} \\ 4-2+2-6 \\ \quad 2-1+1-3 \\ \quad \quad -6+3-3+9 \\ \quad \quad \quad - \\ 4+0-5-2-6+9 \end{array}$

The answer is now completed from $4 + 0 - 5 - 2 - 6 + 9$ thus :

Beginning from the end of the result, the last term of the required product is evidently 9; going backwards, we have successively the coefficients of x , x^2 , x^3 , &c. Thus we have got $4x^6 + 9x^5 - 5x^4 - 2x^3 - 6x + 9$, i.e., $4x^6 - 5x^4 - 2x^3 - 6x + 9$.

Ex. 2. Divide $x^6 - x^4y + 7x^2y^3 - 19xy^4 + 12y^5$ by $x^3 - 4xy^2 + 3y^3$.

The complete dividend $= x^5 - x^4y + 6x^3y^2 + 7x^2y^3 - 19xy^4 + 12y^5$.
and „ „ divisor $= x^3 + 0.x^2y - 4xy^2 + 3y^3$.

$$\begin{array}{r} \therefore \quad 1+0-4+3 \quad) \quad 1-1+0+7-19+12 \quad (\quad 1-1+4 \\ \underline{1+0-4+3} \\ -1+4+4-19 \\ -1-0+4-3 \\ \quad \quad \quad 4+0-16+12 \\ \quad \quad \quad 4+0-16+12 \end{array}$$

The required quotient $= 1.x^2 - 1.xy + 4.y^2 = x^2 - xy + 4y^2$. *Ans.*

N. B. The powers of x are written down, commencing from the end of the quotient as in the last example, while the powers of y are written down in the reverse order.

EXAMPLES 128.

Multiply by the method of Detached Coefficients :

1. $x^3 + 2x + 3$ by $x^3 - 2x + 3$. 2. $x^4 - 5x^2 + 4$ by $x^3 - 2x + 1$.
3. $x^3 - 3xy + 5y^2$ by $x^2 - 2xy + 3y^2$. 4. $x^4 + 2y^2$ by $x^2 - xy - 2y^2$.
5. $64a^5 - 16a^4 + 8a^3 - 2a + 1$ by $4a^2 + 2a + 1$.
6. $a^3 + b^3$ continuously by $a - b$ and $a^2 + ab + b^2$.

Divide

7. $a^6 - 2a^5 + 2a^4 - 4a^3 + 9a^2 + 16a - 34$ by $a^2 - 2a$.
8. $2a^7 - 5a^6 - 39a^4 - 15a^3 - 26a^2 - 12a$ by $a^4 - 4a^2 + a^2 - 3a$.
9. $x^8 + 8y^8$ by $x + 2y$, and $32x^6 + y^6$ by $2x + y$.
10. $a^7 - 6a^4b + 9a^2b^2 - 4b^3$ by $a^2 - b$, and $a^{10} - b^5$ by $a^2 - b$.

229 Symmetrical expressions. An expression is said to be **symmetrical** with respect to any two letters, when it is unaltered by interchanging these letters.

Thus $a + b$, $a^3 + b^3 + 3abd$, &c. are symmetrical with respect to a and b . It should be noticed that the second expression is not symmetrical with respect to a and d , or b and d .

An expression is said to be symmetrical with respect to any three letters, when it is unaltered by interchanging *any two whatever* of the three letters.

Thus $a + b + c$, $a^3 + b^3 + c^3 - 3abc$, &c. are symmetrical with respect to a , b and c .

The expression $k(a + b + c)$ is evidently symmetrical in a , b and c , k being constant. Conversely, if an expression of the first degree in a , b and c be symmetrical with respect to them, then it must be of the form $k(a + b + c)$. For, if possible, let $ka + lb + mc$ be symmetrical with respect to a , b and c . Then, interchanging any two letters, a and b , we get $kb + la + mc$, which is not the same as $ka + lb + mc$.

Similarly $k(a^2 + b^2 + c^2) + l(ba + ca + ab)$ is a *complete homogeneous expression of the second degree* in a , b and c , where k and l are any constants.

A similar expression of the third degree is

$$k(a^3 + b^3 + c^3) + l\{a^2(b + c) + b^2(c + a) + c^2(a + b)\} + n.abc.$$

Ex. 1. Shew that $(x + y)^6 - x^6 - y^6 = 5xy(x + y)(x^2 + xy + y^2)$.

The expression vanishes when (1) $x = 0$, (2) $y = 0$, (3) $x = -y$.

∴ x , y and $x + y$ are its factors. Art 119, Page 169.

∴ $k(x + y)^6 - x^6 - y^6 = xy(x + y).Q$,

where Q is the quotient of the left side by $xy(x + y)$.

Now, the left side is of the fifth degree in x and y , and $xy(x+y)$ is of the third degree; hence the quotient, Q , must be of the second degree

Again, since $(x+y)^5 - x^5 - y^5$ is homogeneous and symmetrical in x and y , Q must also be so.

$\therefore Q$ is of the form $k(x^2+y^2) + lxy$,

and $(x+y)^5 - x^5 - y^5 = xy(x+y)\{k(x^2+y^2) + lxy\} \dots\dots\dots (A).$

Since k and l are independent of x and y , they are unaltered by assuming any values for x and y

Put $x=1, y=1$ in (A); thus $2^5 - 1^5 - 1^5 = 2(2k + l)$,
i.e., $2k + l = 15 \dots\dots\dots (1)$

Again put $x=1, y=2$ in (A); thus $3^5 - 1^5 - 2^5 = 2.3(5k + 2l)$;
i.e., $5k + 2l = 35 \dots\dots\dots (2)$

Solving (1) and (2), $k = l = 5$;

\therefore from (A), $(x+y)^5 - x^5 - y^5 = xy(x+y)\{5(x^2+y^2) + 5xy\}$
 $= 5xy(x+y)(x^2+xy+y^2).$

Ex. 2. Factorise $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$.

The given expression vanishes when $b=c$, or $c=a$, or $a=b$

$\therefore a-b, b-c$ and $c-a$ are factors.

Again upon examination it will be found that the fraction $\frac{a(b-c)^3 + b(c-a)^3 + c(a-b)^3}{(b-c)(c-a)(a-b)}$ is symmetrical in a, b and c ; on reduction its value will evidently be of the first degree in a, b and c .

$\therefore a(b-c)^3 + b(c-a)^3 + c(a-b)^3 = (b-c)(c-a)(a-b).k(a+b+c), (A)$
where k is a constant.

Putting $a=1, b=-1, c=2$, we have $-3^3 - 1^3 + 2.2^3 = -3.1.2.k.2$;
 $\therefore k=1$;

\therefore by (A), the given expn $= (b-c)(c-a)(a-b)(a+b+c)$. *Ans.*

EXAMPLES 120.

1. A homogeneous expression of two dimensions is symmetrical in x, y and z ; its value is 83 when $x=1, y=2$, and $z=3$, and the same is 77, when $x=-1, y=3, z=4$. Find the expression.

Factorise

2. $a(b^3 - c^3) + b(c^3 - a^3) + c(a^3 - b^3)$. 3. $(a+b)^3 - a^3 - b^3$.
4. $a^2(b^3 - c^3) + b^2(c^3 - a^3) + c^2(a^3 - b^3)$. 5. $(b-c)^5 + (c-a)^5 + (a-b)^5$.
6. $(a+b)^2(a-b) + (b+c)^2(b-c) + (c+a)^2(c-a)$.
7. $a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc$. 8. $(a+b+c)^3 - a^3 - b^3 - c^3$.

Simplify

9. $(x+y+z)^3 - (y+z-x)^3 - (z+x-y)^3 - (x+y-z)^3$.
 10. $x(y+z-x)^2 + y(z+x-y)^2 + z(x+y-z)^2$
 $+ (y+z-x)(z+x-y)(x+y-z)$.
 11. $x(y+z)(y^2+z^2-x^2) + y(z+x)(z^2+x^2-y^2) + z(x+y)(x^2+y^2-z^2)$.
 12. Being given $s=a+b+c$, show that
 $(s-3a)^2 + (s-3b)^2 + (s-3c)^2 = 3\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$.

MISCELLANEOUS EXAMPLES III.

1. If $x=ka$, $y=(k-1)b$, $z=(k-3)c$, and $k = \frac{1+b^2+3c^2}{a^2+b^2+c^2}$, express $x^2+y^2+z^2$ in its simplest form in terms of a , b and c .
 2. Divide $\frac{7}{6}x^{\frac{3}{2}} - \frac{5}{4}x^{\frac{1}{2}} - \frac{1}{6}x - \frac{35}{12}x^{\frac{1}{2}} - \frac{1}{3}$ by $\frac{1}{3} + \frac{1}{4}x^{\frac{1}{2}} - \frac{1}{12}x^{\frac{1}{2}}$.
 3. Find a and b so that $x^3+ax^2+11x+\frac{1}{6}$ and $x^3+bx^2+14x+8$ may have a common factor of the form x^2+px+q .
 4. Factorize (1) $a(b-c)x^2+b(c-a)x+c(a-b)$;
 (2) $a^3(b+c)-b^3(c+a)-c^3(a+b)$;
 (3) $(a+1)^3+(a-1)^3-3a^2+4$.
 5. If $x+(p-1)y=a$, $x+(q-1)y=b$, $x+(r-1)y=c$, prove that
 $(q-r)a+(r-p)b+(p-q)c=0$.
 6. If $xy^{p-1}=a$, $xy^{q-1}=b$, $xy^{r-1}=c$, prove that $a^{q-r}b^{r-p}c^{p-q}=1$.
 7. If $a:b=c:d$, and $x:y=z:w$, prove that
 $ax+by:ax-by=cw+dw:cw-dw$.
 8. If $\frac{1}{x} + \frac{1}{z} + \frac{1}{x-y} + \frac{1}{z-y} = c$, prove that either $\frac{1}{z} + \frac{1}{x} = \frac{2}{y}$ or $z+x=y$.
 9. Solve the following equations :
 (1) $\frac{(x-1)(x-2)}{(x-5)(x-6)} = \frac{(x+3)(x-6)}{(x-1)(x-10)}$; (2) $\frac{x + \sqrt{(x^2-1)}}{\sqrt{(x+1)} + \sqrt{(x-1)}} = \frac{1}{\sqrt{6-2}}$;
 (3) $\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = 19$, $\frac{8}{x} + \frac{5}{y} - \frac{3}{z} = 12$, $\frac{5}{x} - \frac{6}{y} + \frac{3}{z} = 13$.
 10. A number of men is arranged into a hollow square a deep; had there been $4ab$ fewer men, they might have been re-arranged into a solid square having b men fewer in front. How many men are there?
 11. Find the square root of
 $(a^2+bc+ca+ab)(b^2+ca+ab+bc)(c^2+ab+bc+ca)$

12. Shew that $(x-xy-yz-zx)^2 + (x+y+z-xyz)^2$
 $= (1+x^2)(1+y^2)(1+z^2).$

13. Find the value of a for which the fraction $\frac{x^3-ax^2+19x-a-4}{x^3-(a+1)x^2+23x-a-7}$ admits of reduction. Reduce it to its lowest terms.

14. Resolve into factors as far as possible :

- (1) $a^3(a+2b) - b^3(2a+b)$; (2) $(a^3-b^3-c^3)(b^2c^3-c^2a^3-a^2b^3) - a^2b^2c^2$;
 (3) $(x+2)^3 + (x-2)^3 - 3x^2 + 13$

15. Simplify

$$\left\{ \frac{b^3-c^3}{bc(b-c)} + \frac{c^3-a^3}{ca(c-a)} + \frac{a^3-b^3}{ab(a-b)} \right\} \div \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \div \left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \right).$$

16. If $\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} = 0$, shew that $(x+y+z)^3 = 27xyz$.

17. If a, b, c, d and e be in continued proportion, prove that $(ab+bc+cd+de)^2 = (a^2+b^2+c^2+d^2)(b^2+c^2+d^2+e^2)$.

18. $\left(\frac{b}{b+c} \right)^2 + \left(\frac{c}{c+a} \right)^2 + \left(\frac{a}{a+b} \right)^2 = \left(\frac{c}{b+c} \right)^2 + \left(\frac{a}{c+a} \right)^2 + \left(\frac{b}{a+b} \right)^2,$

prove that two of the quantities a, b and c must be equal.

19. Solve the equations :

(1) $\sqrt{(x^2-3ax+4a^2)} + \sqrt{(x^2-5ax+8a^2)} = \sqrt{(x^2-4ax+7a^2)} + \sqrt{(x^2-6ax+11a^2)}.$

(2) $\frac{yz+zx}{5} = \frac{xy+yz}{6} = \frac{zx+xy}{7} = xyz.$ (3) $\frac{ax+by}{a^2+b^2} = \frac{bx+ay}{2ab} = c.$

20. A mixed fraction is twice misprinted by transposing the whole number to the numerator, consisting of a single digit, of the fractional part : in one case the two are in their original order, and in the other case in the reverse order ; the value of the whole fraction is diminished in the first case by $\frac{2}{3}$ of the integral part and in the second case by $\frac{1}{3}$ of the same. If the sum and difference of the numerator and denominator of the fractional part be to each other as 37 : 23, find the whole fraction.

21. Divide $(x-y)^4 + (x^2-y^2)^2 + (x+y)^4$
 by $(x-y)^2 + (x^2-y^2) + (x+y)^2.$

22. Find the continued product of $ax-c, bx-c$ and abx^2-c^2 , and simplify it when $x = \frac{2c}{a+b}.$

23. Shew that $(a+b^{-1})^2 - (b+a^{-1})^2$
 $= (1-a^{-1}b)(1+ab^{-1})(a+b^{-1})(b+a^{-1}).$

24. Factorize (i) $a^{2x} + a^{2-y} + a^{x+y} + 1$;

(ii) $a^{-9} - b^{-9}$; (iii) $8x^{\frac{1}{3}} - 2x^{\frac{1}{6}} - 1.$

25. Simplify $\frac{3a-b-c}{(a-b)(a-c)} + \frac{3b-c-a}{(b-c)(b-a)} + \frac{3c-a-b}{(c-a)(c-b)}.$

26. Extract the square root of

$$4ab - \frac{8a^{\frac{1}{2}}b^{\frac{3}{2}}}{3c^{\frac{1}{2}}} + \frac{4b^3}{9c} + \frac{16a^{\frac{1}{2}}b^3}{c^{\frac{3}{2}}} - \frac{16b^4}{3c^2} + \frac{16b^5}{c^3}.$$

27. If $a : b = b : c = c : d$, prove that

$$(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2.$$

28. $(a-bc)^{\frac{1}{2}} + (b-ca)^{\frac{1}{2}} = (a+b)^{\frac{1}{2}}(1-c)^{\frac{1}{2}}$, prove that c may be equal to any one of the quantities $\frac{a}{b}, \frac{b}{a}$ and 1.

29. Solve the following equations :

(1) $ax(x^4 + 10x^2 + 5)(a^4 + 10a^2 + 5)$
 $= (5x^4 + 10x^2 + 1)(5a^4 + 10a^2 + 1).$

(2) $(b-c)x + (c-a)y + (a-b)z = 0,$
 $x + y + z = a + b + c,$
 $(a-b)x + (b-c)y + (c-a)z = 2(bc + ca + ab - a^2 - b^2 - c^2).$

30. In a certain meadow there is a crop of 1044 stone of grass, which grows uniformly. It is given would consume all the grass in 48 days, and 9 oxen would require 58 days, what weight of grass does each ox eat in a day?

31. If $x = a - l$, $y = b - l$, and $z = c - l$, prove that

$$x^2 + y^2 + z^2 - yz - zx - xy = a^2 + b^2 + c^2 - bc - ca - ab.$$

32. Find the H. C. F

$$ax^6 + (a+b)x^5 + (a+b+c)x^4 + (a+b+c+d)x^3 + (b+c+d)x^2 + (c+d)x + d$$

and $ax^5 + (a+b)x^4 + (a+b+c)x^3 + (a+b+c)x^2 + (b+c)x + c.$

33. Factorize : (i) $a^{2x+y} + b^{2x+y} + (ab)^x + (ab)^y,$

(ii) $6(x^{\frac{1}{2}} + 1) - 4x^{\frac{1}{2}} - 9x^{\frac{1}{4}}.$

34. Shew that $a^{2n} + 2^{2n}(a^2 + 1)^n$ is exactly divisible by $(a^2 + 2)^2$, when n is any odd integer.

35. Shew that $\{4(1-x^2)^{\frac{3}{2}} - 3(1-x^2)^{\frac{1}{2}}\}^2 + (3x-4x^3)^2 = 1$.

36. If $a\sqrt{1-b^2} + b\sqrt{1-a^2} = \frac{1}{\sqrt{2}}$, prove that

$$\sqrt{(1-a^2)(1-b^2)} - ab = \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}}.$$

37. If $x : b+c = y : c+a = z : a+b$, prove that

$$a^2x(b-c) + b^2y(c-a) + c^2z(a-b) = 0.$$

38. If a and b be unequal, shew that $(1+a^2)(1+b^2) > (1+ab)^2$.

39. Solve the following equations :

$$(1) \frac{1+\sqrt{x}}{1+\sqrt{x}+\sqrt{1+x}} + \frac{1-\sqrt{x}}{1-\sqrt{x}+\sqrt{1+x}} = m;$$

$$(2) \left. \begin{aligned} \frac{3}{x} + \frac{2}{y} - \frac{3}{z} &= 0, \\ \frac{5}{x} - \frac{6}{y} + \frac{2}{z} &= 0, \end{aligned} \right\} \quad (3) \quad \begin{aligned} a^2 + b^2 + c^2 &= 3, \\ la^2 + mb^2 + nc^2 &= l+m+n, \\ (m-n)a^2 + (n-l)b^2 + (l-m)c^2 &= 0. \end{aligned}$$

$$2x + 3y + 4z = 3.$$

40. A train 132 yards in length, travelling at a uniform speed, overtook a man walking along the line at the rate of 6 miles an hour, and passed him in 12 seconds. Twenty minutes after overtaking the first man, the train overtook a second man and passed him in 11 seconds. How many hours after the train overtook the second man would the first man also overtake him, if both the men go in the same direction as the train?

41. Multiply $(1+a^{\frac{1}{2}}+a)l^2 + (1+a^{\frac{1}{2}})(1-b^{\frac{1}{2}})lm + (1+b^{\frac{1}{2}}+b)m^2$
by $(1-a^{\frac{1}{2}}+a)l^2 + (1-a^{\frac{1}{2}})(1-b^{\frac{1}{2}})lm + (1-b^{\frac{1}{2}}+b)m^2$.

42. If $x^2+y^2+z^2=2(x+y-1)$, prove that $x=y=1$, and $z=0$.

43. Express as the difference of two squares :

$$(i) (a-2b)^2(a^2+4b^2); \quad (ii) (x^2-4y^2)(9x^2-4z^2).$$

Shew that $(x+a)(x+b)(x+c)(x+d) + \frac{1}{4}(a+b-c-d)^2x^2$ is a perfect square, if $ab=cd$.

44. Factorize : (i) $e^{2x} - e^{2x} - e^x + 1$,

$$(ii) (a^2-1)^2x^2 - 2(a^2+1)x^2 + 1.$$

45. Find the quotient of $a^{2x} + b^{2x}$ by $a+b$, x being any positive integer.

46. Prove that

$$\frac{\sqrt{a+2} + \sqrt{a+1}}{\sqrt{a+2} - \sqrt{a+1}} - 3 \cdot \frac{\sqrt{a+3} - \sqrt{a}}{\sqrt{a+3} + \sqrt{a}} = \frac{4}{\sqrt{(a+1)(a+2)} - \sqrt{a(a+3)}}$$

47. If $a : b = c : d$, shew that $(a^2 - b^2 - c^2 + d^2) \sqrt{abcd}$ is a mean proportional between $(ac - bd)^2$ and $(ab - cd)^2$.

48. If a, b, c and d be any positive quantities, prove that

$$\frac{ac + bd}{a + b} \text{ is not less than } \frac{(a + b)cd}{ad + bc}.$$

49. Solve the following equations :

$$(1) \frac{a(a^2 + x^2)(a + x) + x^4}{a(a^2 + x^2)(a - x) + x^4} = \frac{1562(a + x)}{1563(a - x)};$$

$$(2) \left. \begin{aligned} 2^x + 3^y + 4^z &= 8, \\ 2^{x+1} + 3^{y+1} + 4^{z+1} &= 21, \\ 2^{x+2} + 3^{y+2} + 4^{z+2} &= 59. \end{aligned} \right\}$$

50. A mass of gold and silver weighing 9 lbs is worth £318; if the proportions of gold and silver in it were interchanged, it would be worth £618; it is known that 1 ounce of gold and two ounces of silver are worth £7; what are the prices of gold and silver per ounce?

51. Divide $10a(a^4 + 1) + 11(a^4 - 1) - 4(4a^2 + 1)$ by $5a^2 - 2a + 3$.

52. Factorize : (i) $(b - c)^6 + (c - a)^6 + (a - b)^6$;

$$(ii) (a - b)x^2 - (a^2 + ab + b^2)x + ab(2a + b)$$

53. If $x^2 = ab + c(a - b)$, shew that $\frac{(c - x)(x + a)}{x + b} = \frac{(c + x)(x - a)}{x - b}$

54. If $x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$ be a perfect cube, shew that $(a + b)(b + c)(c + a) = 8abc$.

55. Simplify $\{a \sqrt{(1 - b^2)} + b \sqrt{(1 - a^2)}\}^2 + \{\sqrt{(1 - a^2)} \cdot \sqrt{(1 - b^2)} - ab\}^2$

56. Find the square root of

$$a^{2x}c^{2y-2} - 8a^{x+1}b^{\frac{3}{2}}c^{y-1} + 16a^2b^3 - \frac{6a^2d}{c^2} + \frac{24ab^{\frac{3}{2}}d}{c^{y+1}} + \frac{9d^2}{c^{2y+2}}$$

57. If $\frac{3x + 2y}{3a - 2b} - \frac{3y + 2x}{3b - 2c} = \frac{3z + 2x}{3c - 2a}$, then will

$$5(x + y + z)(5c + 4b - 3a) = (9x + 8y + 13z)(a + b + c).$$

58. If a, b, c, x, y, z be real quantities, and

$$(a + b + c)^3 = 3(bc + ca + ab - x^3 - y^3 - z^3), \text{ prove that}$$

$$a = b = c, \text{ and } x = y = z = 0.$$

59. Solve : (1) $\frac{a^3 + x^3}{(a + x)^2} + \frac{a^3 - x^3}{(a - x)^2} = b$.

$$(2) (a + 1)x + (b + 1)y + (c + 1)z = 0,$$

$$(b + c)x + (c + a)y + (a + b)z = 0,$$

$$\frac{b^2 - c^2}{x} + \frac{c^2 - a^2}{y} + \frac{a^2 - b^2}{z} = 2(a + b + c).$$

60. If the increase in the number of male and female criminals be 1·8 per cent., while the decrease in the number of males alone is 4·6 per cent., and the increase in the number of females is 9·8 per cent., compare the numbers of male and female criminals respectively.

61. Divide $\left(\frac{a^2}{b^2} + \frac{b^2}{a^2} + 2\right)$ by $\frac{a}{b} + \frac{b}{a}$.

62. Find the square root of

$$(ax - by)^2(ay + bx)^2 - 4abxy(a^2 - b^2)(x^2 - y^2).$$

63. Simplify
$$\frac{a^2\left(\frac{1}{b} - \frac{1}{c}\right) + b^2\left(\frac{1}{c} - \frac{1}{a}\right) + c^2\left(\frac{1}{a} - \frac{1}{b}\right)}{a\left(\frac{1}{b} - \frac{1}{c}\right) + b\left(\frac{1}{c} - \frac{1}{a}\right) + c\left(\frac{1}{a} - \frac{1}{b}\right)}.$$

64. Resolve into elementary factors :

(i) $16x^3 - 81a^3$; (ii) $a^{2x} + (ab)^{x-y} + (ab)^{x+y} + b^{2x}$.

(iii) $aa'x^2 + bb'y^2 + (ab' + a'b)xy + (a'\epsilon + a\epsilon')x + (b'\epsilon + b\epsilon')y + \epsilon\epsilon'$.

65. If $s = \frac{1}{2}(a + b + c)$, shew that $(s - a)^3 + (s - b)^3 + 3(s - a)(s - b)c = c^3$.

66. Simplify $\sqrt[4]{729^3(9^{-1} \cdot 27^{-\frac{4}{3}})}$.

67. If $ad = bc$, shew that $(a + c)^n(b^n + d^n) = (b + d)^n(a^n + c^n)$.

68. If a , b and c denote the lengths of the sides of a triangle, then $a^2 + b^2 > \frac{1}{2}c^2$.

69. Solve the following equations :

(1) $[(x - 1) + \sqrt{\{(x + 4)(x - 6)\}}]^3 + [(x - 1) - \sqrt{\{(x + 4)(x - 6)\}}]^3 = 8(x - 1)^3$.

(2) $\left. \begin{aligned} x^2 - y^2 &= 21, \\ x - y &= 3; \end{aligned} \right\} \quad (3) \quad \left. \begin{aligned} \frac{x}{a} + \frac{y}{b} + \frac{z}{c} &= 3, \\ \frac{x}{2a - x} + \frac{y}{y - 2b} &= 0, \\ (ab)^{x+2y} \cdot (ab)^{x+2z} &= (ab)^{3(x+y+z)}. \end{aligned} \right\}$

70. In one chest of adulterated coffee the chicory is mixed by weight with pure coffee in the ratio of 3 : 5, and in another in the ratio of 1 : 7. What weights must be taken from these chests so as to form a mixture in which there will be 5 lbs. of chicory and 16 lbs. of coffee?

71. Find the quotient of $(a^3 - b^3)^3 + (b^3 - c^3)^3 + (c^3 - a^3)^3$ by $a(b^3 - c^3) + b(c^3 - a^3) + c(a^3 - b^3)$.

72. Find for what value of a the expression

$$(a+1)x^3 + (7a+6)xy + (a+6)y^3$$

may be a perfect square, and then find the square root.

73. If $8x - 5y : 11x - 7y = 1 : 2$, find the ratio of $3y^3 + 5x^3 : 3y^3 - 5x^3$.

74. Simplify $\frac{a^3 - a + 1}{a^2 + a + 1} + \frac{2a(a-1)^2}{a^2 + a^2 + 1} + \frac{2a^2(a^2 - 1)^2}{a^8 + a^4 + 1}$.

75. If $x = \left(\frac{a}{b}\right)^{\frac{a^2 - b^2}{2}}$, shew that $\frac{ab}{a^2 + b^2} \left(x^{\frac{a}{b}} + x^{\frac{b}{a}}\right) = \left(\frac{a}{b}\right)^{\frac{a^2 + b^2}{2}}$.

76. Shew that

$$x^3(3x - 4x^3) + (1 - x^2)^3 \{4(1 - x^2)^{\frac{3}{2}} - 3(1 - x^2)^{\frac{1}{2}}\} = (1 - 2x^2)^3.$$

77. If $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$, then will $b^2(a - c)^2 = 2\{c^2(a - b)^2 + a^2(b - c)^2\}$

78. If $x = cy + bz$, $y = az + cx$, $z = bx + ay$, shew that

$$\frac{x}{\sqrt{(1-a^2)}} = \frac{y}{\sqrt{(1-b^2)}} = \frac{z}{\sqrt{(1-c^2)}}, \text{ and find the relation between}$$

a, b and c .

79. Solve the following equations :

$$(1) \quad \frac{24x - 19}{4x - 3} + \frac{4(12x - 13)}{8x - 9} = \frac{40x - 37}{8x - 7} + \frac{2(14x - 17)}{4x - 5}.$$

$$(2) \quad \left. \begin{aligned} \sqrt{x+y+1} - \sqrt{x+y-2} &= 3 - \sqrt{6}, \\ \frac{\sqrt{x+1} - \sqrt{y-1}}{\sqrt{x-1} - \sqrt{y+1}} &= 2 \left(\frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{y-1}} \right), \\ xy + yz + zx &= xyz \end{aligned} \right\}$$

80. If 24 is taken from a certain integer, we get the square of a number ; if 5 is added to the integer, we get the square of the next greater number. Find the integer.

81. Find the product

$$(1 - x + x^2 - x^3 + \dots + x^{2n})(1 + x + x^2 + x^3 + \dots + x^{2n}).$$

82. Factorize :

$$(i) \quad (b+c)(c+a)(a-b) + (c+a)(a+b)(b-c) + (a+b)(b+c)(c-a) ;$$

$$(ii) \quad x^4 - 10x^3 + 35x^2 - 50x + 24 ;$$

$$(iii) \quad 2x^2 + 5xy + 3y^2 - 5yz - 5zx + 2x - 5y + 3y.$$

83. If $a_1 + a_2 + a_3 + \dots + a_n = \frac{1}{2}ns$, prove that

$$(s - a_1)^2 + (s - a_2)^2 + \dots + (s - a_n)^2 = a_1^2 + a_2^2 + \dots + a_n^2.$$

84. Simplify

$$\frac{1}{1-x} - \frac{1}{1+x} - \frac{2x}{1+x^2} - \frac{4x^3}{1+x^4} - \frac{8x^7}{1+x^8}$$

85. Divide $4x^3 + 6x - 1$ by $x^{\frac{3}{4}}/4 - \frac{3}{2} + 1$.

86. If $a : b = c : d$, prove that $a^2 - ab + b^2 : c^2 - cd + d^2$ is the duplicate ratio of $a : c$.

87. Find the square root of

$$9a^3 - 20a^{\frac{5}{2}}b^{\frac{3}{2}} + 7\frac{1}{2}a^2b - 13a^{\frac{3}{2}}b^{\frac{5}{2}} + \frac{1}{2}ab^3.$$

88. Simplify $\frac{1}{1+a^{x-y}+a^{y-z}} + \frac{1}{1+a^{z-y}+a^{x-y}} + \frac{1}{1+a^{x-z}+a^{y-z}}$ and express $(abc)^6$ in the form $(\sqrt{a}^{\frac{2}{3}}\sqrt[3]{b})^4(\sqrt[4]{b}^{\frac{4}{3}}\sqrt[3]{c})^4(\sqrt[4]{c}^{\frac{4}{3}}\sqrt[3]{a})^4$.

89. Solve the following equations :

$$(1) \sqrt{\frac{a}{x+c}} - \sqrt{\frac{b}{x+c}} - a = \frac{a+b}{\sqrt{x+c}};$$

$$(2) \frac{1}{x-y} + \frac{3}{x+y} = 1,$$

$$\frac{3}{x-y} - \frac{4}{x+y} = 5.$$

$$(3) (ab)^{2x+3y-2}(\frac{1}{2}c)^{3y+z-2x-8}(ca)^{2x-3y+8} \\ = (ab)^{x+3y-2}(bc)^{3y+z-2x-10}(ca)^{x-3y+8} \\ = (ab)^{x+3y-2}(bc)^{2y+z-2x-7}(ca)^{x-2y+8-1} \\ = (ab)^{2x+3y-6}(bc)^{3y-2x-4}(ca)^{2x-3y+4}$$

90. From a place A a messenger goes to a place B , distant 21 miles from A , and immediately turns back, going at the rate of 4 miles an hour; simultaneously with the messenger's departure from A , another messenger starts from B , at the rate of 3 miles an hour, goes to A , and immediately turns back; find the distance between the two points at which they cross each other.

91. Find the product of $a+b-c$, $a-b+c$ and $a^2+b^2+c^2-2bc$.

92. Shew that $(a^2+a+1)(b^2+b+1) \\ = (ab+a+1)^2 + (ab+a+1)(b-a) + (b-a)^2.$

93. Factorize : (i) $x^8 - 16x^{-8}$; (ii) $x^{2n} - 1 - 12x^{-2n}$; (iii) $2(a^2+b^2+a^2b^2) - a^4 - b^4 - 1$.

94. Find the L. C. M. of $x^{\frac{2}{3}} - 1$, $x^{\frac{2}{3}} + 1$, $x^{\frac{4}{3}} - 1$ and $x^2 - 1$.

95. If $\frac{l(x-b)(x-c)}{bc} = \frac{m(x-c)(x-a)}{ca} = \frac{n(x-a)(x-b)}{ab}$, then

$$l\left(\frac{1}{b} - \frac{1}{c}\right) + m\left(\frac{1}{c} - \frac{1}{a}\right) + n\left(\frac{1}{a} - \frac{1}{b}\right) = 0.$$

96. Shew that $\frac{a^3}{(a-b)(a-c)(a-d)} + \text{three similar terms} = 1$.

97. If $\frac{b}{a-c} = \frac{a+b}{c} = \frac{a}{b}$, find the ratios of $a : b : c$.

98. Prove that $(ab+xy)(ax+by)$ not $< 4abxy$. (a, b, x, y positive)

99. Solve :

$$(1) \frac{5(x+1)}{\sqrt{(7x+8)} - \sqrt{(2x+3)}} + \frac{5x-6}{\sqrt{(7x-3)} + \sqrt{(2x+3)}} = 11 ;$$

$$(2) \frac{\sqrt{(x+y)} + \sqrt{(x-y)}}{y} = \frac{1}{2} \{x - \sqrt{(x^2 - y^2)}\},$$

$$\sqrt[4]{(x+y)} + \sqrt[4]{(x-y)} = 2 + \sqrt{2}$$

100. There was a run on two bankers A and B ; after three days B stopped payment, when the daily demand on A was trebled, so that he failed after two days more. If A and B had joined their capitals, they might have stood the run as it was at first for 7 days, when B would have owed £4000 to A ; find the daily call on A at first.

101. If $x = (a-b)(c-l)$, $y = (b-c)(a-l)$, and $z = (c-a)(b-l)$, find the value of $x^3 + y^3 + z^3 - 3xyz$.

102. Express as the sum of two squares .

$$(i) (1+a^2)(1+b^2) ;$$

$$(ii) (a^2+b^2)(c^2+d^2).$$

103. Factorize : (i) $a^2 - 4(b^2 + a + 1)$; (ii) $a^2 - 4(b^2 - a - 1)$;

$$(iii) 10x^{2m} - a^{2m} - 21a^{2n}.$$

104. Shew that

$$\left\{ \frac{(xy+1)^2}{x^2y^2} - \frac{4}{xy} \right\} z^2 + 2 \left\{ \frac{(xy+1)^2}{xy} - 2xy - \frac{2}{xy} \right\} z + (xy+1)^2 - 4xy$$

is a perfect square, and find its square root.

105. Simplify $\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} + \frac{1}{c(c-a)(c-b)}$.

106. If $(b-c)x + (c-a)y + (a-b)z = 0$, prove that

$$\frac{b-c}{(a+b)y - (a+b)z} = \frac{c-a}{(b+c)x - (a+b)z} = \frac{a-b}{(c+a)x - (b+c)y}.$$

107. If $b^2x^4 + a^2y^4 = a^2b^2$, and $a^3 + b^3 = x^3 + y^3 = 1$, prove that

$$b^4x^6 + a^4y^6 = (b^2x^4 + a^2y^4)^2.$$

108. Prove that two of the quantities a, b, c must be equal to each other, if $\frac{b-c}{1+bc} + \frac{c-a}{1+ca} + \frac{a-b}{1+ab} = 0$.

109. Solve the following equations :

$$(1) \frac{(1+x)^2(1-x+x^2)+x^2}{(1-x)^2(1+x+x^2)+x^2} = \frac{121}{122} \frac{1+x}{1-x}.$$

$$(2) \frac{\frac{b}{y+b} + \frac{c}{z+c}}{a} = \frac{\frac{c}{z+c} + \frac{a}{x+a}}{b} = \frac{\frac{a}{x+a} + \frac{b}{y+b}}{c} = \frac{1}{a+b+c}.$$

110. In a certain community consisting of p persons, a per cent. can read and write ; of the males alone b per cent., and of the females alone c per cent. can read and write : find the number of males and of females in the community.

111. Shew that $(1-x)(1-2x)$

$$= (1-y)(1-y-x)(1+2y-2x) + y(y-x)(3-2y-2x).$$

112. Factorize : (i) $x^3 + x^2 - 17x + 15$;

$$(ii) x^4 - 4x^3 + 2x^2 + 4x - 3 ; \quad (iii) 4x^3 - 25y^3 + 10by - b^3.$$

113. Simplify $(5x+2y)^3 + 3(5x+2y)(x+y) - 10(x+y)^2$,

$$\text{and} \quad (5x+2y)^3 - 7(5x+2y)(x+y) + 10(x+y)^2.$$

114. Eliminate x and y from the equations :

$$x(x^4 + 10x^2y^2 + 5y^4) = a^5, \quad y(5x^4 + 10x^2y^2 + y^4) = b^5, \quad x^2 + y^2 = c^2.$$

115. Extract the cube root of

$$(x^{\frac{1}{2}} - 27y)(x^{\frac{1}{2}} + 27y) + 9x^{\frac{1}{2}}y^{\frac{1}{3}}(x^{\frac{2}{3}} + 81y^{\frac{4}{3}}) - 135x^{\frac{1}{2}}y.$$

116. If $2x = a + a^{-1}$, and $2y = b + b^{-1}$, find the value of

$$xy + \sqrt{\{(x^2 - 1)(y^2 - 1)\}}.$$

117. If $\frac{y+z}{pb+qc} = \frac{z+x}{pc+qa} = \frac{x+y}{pa+qb}$, shew that

$$\frac{2(x+y+z)}{a+b+c} = \frac{(b+c)x + (c+a)y + (a+b)z}{bc+ca+ab}.$$

118. If $x+y+z=a$, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{b}$, and $(y+z)(z+x)(x+y) = c^3$,

prove that $(a-b)xyz = bc^3$.

119. Solve

$$(1) \quad x \{ \sqrt{(a+x)} - \sqrt{(a-x)} \}^n = \frac{1}{2} b^{n+1} \{ \sqrt{(a+x)} + \sqrt{(a-x)} \} ;$$

$$(2) \quad \left. \begin{aligned} x+y+z &= 1+a+a^2, \\ a^2x+ay+z &= 3a^2, \\ x+2by+b^2z &= (1+ab)^2. \end{aligned} \right\}.$$

In the last equation, you are required to use the method of indeterminate multipliers.

120. A bill of £49. 10s is paid with 76 coins, including sovereigns, half-sovereigns, half-crowns and shillings. There are twice as many sovereigns as half-sovereigns, and three times as many gold as silver coins. Find the number of each.

121. For what value of x is $a^3 + b^3 + 6ab - x$ exactly divisible by $a + b - 2$?

122. If $x^3 + y^3 + z^3 = 3xyz$, either $x = y = z$, or $x + y + z = 0$.

123. Eliminate x and y from the equations

$$ax + by = c, \quad bx + ay = d, \quad \text{and} \quad x^2 + y^2 = l^2.$$

124. If $\tau = \frac{1}{2}(a + b + c)$, prove that

$$\begin{aligned} a(b^2 + c^2 - a^2) + b(c^2 + a^2 - b^2) + c(a^2 + b^2 - c^2) \\ = 2abc + 8(s - a)(s - b)(s - c). \end{aligned}$$

125 Simplify

$$\left\{ \frac{x^4 - y^4}{x - y} - 2y(2x^2 - xy + y^2) \right\}^{\frac{2}{3}} \div \left(\frac{x^3 - y^3}{x - y} - 3xy \right)^{-\frac{2}{3}}.$$

126. If $\sqrt{a-x} + \sqrt{b-x} + \sqrt{c-x} = 0$, shew that

$$(a + b + c + 3x)(a + b + c - x) = 4(bc + ca + ab).$$

127. If $\frac{x}{l(mb + nc - la)} = \frac{y}{m(nc + la - mb)} = \frac{z}{n(la + mb - nc)}$,

prove that $\frac{l}{x(by + cz - ax)} = \frac{m}{y(cz + ax - by)} = \frac{n}{z(ax + by - cz)}$.

128. Extract the square root of

$$x^4 - 4x^3 + 6x^2 - 8x + 9 - 4x^{-1} + 4x^{-2}.$$

129. Solve :

$$(1) \sqrt{x+6} \sqrt{x+9} + \sqrt{x-6} \sqrt{x+9}$$

$$= 2 \left\{ \sqrt{\left(\frac{\sqrt{x+2}}{\sqrt{x-2}} \right)} + \sqrt{\left(\frac{\sqrt{x-2}}{\sqrt{x+2}} \right)} \right\} ;$$

$$(2) x + y : 3y + 4z : 5(z + x) + 4 = 3 : 13 : 18.$$

130. Rs. 360 are divided among some men, women and children, numbering 60 in all. The total shares of the men, women and children are respectively as the numbers 5, 4 and 3, while the shares of each man, woman and child are respectively as the numbers 3, 2 and 1. Find the number of men, women and children separately.

131. Divide $8a^3 - 64b^3 + c^3 + 24abc$ by $4a^3 + 16b^3 + c^3 + 4bc - 2ca + 8ab$, and multiply the result by $a + 2b$.

132. Find the product

$$(a^{16} + a^{15}b + a^{14}b^2 + \dots + b^{16})(a^{16} - a^{15}b + a^{14}b^2 - \dots + b^{16}).$$

133. Shew that $(x^2 - 11x + 19)^2(x+2) + 3(x^2 + x - 11)^2(x-2) = 4(x-1)^5$.

134. Resolve into factors : (i) $(a+b+ab)^2 - (a-b)^2$;

(ii) $15a^{\frac{1}{2}} + 26a^{\frac{3}{4}} + 8$; (iii) $(a-b^2)^2 - (a^2-b)^2$.

135. Simplify

$$\frac{bca(b+c)^2}{(a-b)(a-c)} + \frac{ca(c+a)^2}{(b-c)(b-a)} + \frac{ab(a+b)^2}{(c-a)(c-b)}.$$

136. If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove that

$$\frac{x^3 + a^3}{x^2 + a^2} + \frac{y^3 + b^3}{y^2 + b^2} + \frac{z^3 + c^3}{z^2 + c^2} = \frac{(x+y+z)^3 + (a+b+c)^3}{(x+y+z)^2 + (a+b+c)^2}$$

137. If $a(b-c)x^2 + b(c-a)xy + c(a-b)y^2$ is a perfect square, then

$$\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$$

138. If $(a+b+c)^3 = a^3 + b^3 + c^3$, then will

$(a+b+c)^{2n+1} = a^{2n+1} + b^{2n+1} + c^{2n+1}$, n being any positive integer.

139. Solve the following equations :

$$(1) \quad \sqrt{x^2 + 2bx + c^2} + \sqrt{x^2 - 2bx + c^2} = \frac{2bx}{\sqrt{x^2 - c^2}} ;$$

$$\left. \begin{array}{l} (2) \quad xy = a(x+y), \\ \quad \quad \quad xz = b(x+z), \\ \quad \quad \quad yz = c(y+z); \end{array} \right\} \quad \begin{array}{l} (3) \quad (x+1)(y+1) = 70(x+y+2), \\ \quad \quad \quad (y+1)(z+1) = 140(y+z+2), \\ \quad \quad \quad (z+1)(x+1) = 84(z+x+2). \end{array}$$

140. A train passes a post in 8 seconds, and then passes in 10 seconds a man walking at the rate of 5 miles an hour. Find the length of the train.

141. Shew that $1 - x^n(1 - x^{n+1}) - x^{2n+1}$ is divisible by $1 - x^2$, n being any positive whole number.

142. Prove that $lx^3 + mx^2 + nx + p$ is divisible by $ax^2 + bx + c$,

$$\text{if } \frac{am - bl}{a} = \frac{an - cl}{b} = \frac{ap}{c}.$$

143. Factorise : (i) $(1+xy)^2 - (x+y)^2$;
 (ii) $x^4 - 8x^3 + 23x^2 - 28x + 12$;
 (iii) $12x^3 + 41xy + 35y^3 - 54x - 93y + 54$.

144. Simplify

$$\frac{(\sqrt{2} + \sqrt{3})(\sqrt{3} + \sqrt{5})(\sqrt{5} + \sqrt{2})(\sqrt{12} - \sqrt{8})}{(\sqrt{2} + \sqrt{3} + \sqrt{5})^2}.$$

145. Simplify $bc \frac{(a+b)(a+c)}{(a-b)(a-c)} + ca \frac{(b+a)(b+c)}{(b-a)(b-c)} + ab \frac{(c+a)(c+b)}{(c-a)(c-b)}$.

146. Eliminate x , y and z from the equations

$$x^2(y+z) = a^3, y^2(z+x) = b^3, z^2(x+y) = c^3, xyz = a^3.$$

147. If $a : b : c : d$, shew that $\sqrt{(abcd)(a^2 - b^2 + c^2 - d^2)}$ is a mean proportional between $(ac - bd)^2$ and $(ab + cd)^2$.

148. If $x + y + z = 0$, prove that

$$\frac{1}{y^2 + z^2 - x^2} + \text{two similar fractions} = 0.$$

149. Solve the following equations :

$$(1) \frac{x^2 - a^2}{x+a} + \frac{x^2 - b^2}{x+b} + \frac{x^2 - c^2}{x+c} = x + a + b + c.$$

$$(2) \frac{y+z-x}{b+c} = \frac{z+x-y}{c+a} = \frac{x+y-z}{a+b} = 1.$$

150. A starts to walk from P to Q at 11 A.M. ; B starts from Q at 11.55 A.M. They meet $5\frac{1}{2}$ miles from Q . B stays 20 min. at P , and A stays 2 hrs 17 $\frac{1}{2}$ min at Q and returning they meet at 6.30 P.M., when B has walked three quarters of the distance back. Find the distance from P to Q .

151. Given $x = \sqrt{1+a^2} - \sqrt{1+b^2}$, and $y = \frac{\sqrt{1+a^2} - 1}{\sqrt{1+b^2} - 1} \frac{b}{a}$, find

the value of $\frac{x}{y - y^{-1}}$ in its simplest form

152. If $x - \frac{1}{x} = a$, express $x^8 - \frac{1}{x^8}$ in terms of a .

153. Resolve into factors : (i) $(ab+1)^4 - 4ab(ab+1)^3 - (a^2 - b^2)^3$;
 (ii) $(a+b)^3 - (a+c)^3 - (b-c)^3$; (iii) $x^6 + 7x^3 - 8$.

154. If the two expressions $x^3 + ax^2 + bx + c$ and $x^3 + a'x^2 + b'x + c'$ have a common quadratic factor,

$$\text{then } \frac{c-c'}{a-a'} = \frac{a'c-ac'}{b-b'} = \frac{b'c-bc'}{c-c'}.$$

155. In the preceding Ex shew that the third factors are $x + \frac{a-a'}{c-c'} \cdot c$ and $x + \frac{a-a'}{c-c'} \cdot c'$ and that the quadratic factor is $x^2 + \frac{b-b'}{a-a'} x + \frac{c-c'}{a-a'}$.

156. Shew that $\frac{bcd}{(a-b)(c-d)(a-d)} + \text{three similar terms} = -1$.

157. If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ prove that each is equal to

$$\left\{ \frac{x^p(y-z) + y^p(z-x) + z^p(x-y)}{a^p(b-c) + b^p(c-a) + c^p(a-b)} \right\}^{\frac{1}{p+1}}$$

158. If $\frac{b-c}{y-z} + \frac{c-a}{z-x} + \frac{a-b}{x-y} = 0$, prove that $b-c : c-a : a-b = (y-z)(2x-y-z) : (z-x)(2y-z-x) : (x-y)(2z-x-y)$; and that $(b-c)^2(y-z)^2 + (c-a)^2(z-x)^2 + (a-b)^2(x-y)^2 = 0$.

159. Solve (1) $y + \sqrt{y^2 - b^2} = 9\sqrt{3} \left\{ \frac{\sqrt{(y+b)} - \sqrt{(y-b)}}{\sqrt{(y+b)} + \sqrt{(y-b)}} \right\}^{\frac{1}{2}}$,
 $\sqrt{(2+3y-a)} - \sqrt{(x+3y-2a)} = \sqrt{(9b+a)} - 3\sqrt{b}$,
 (2) $\left. \begin{aligned} x+y+2z &= 105, \\ x+2y+z &= 147, \\ 2x+y+z &= 176. \end{aligned} \right\}$

160. A, B, C, D working together can perform a piece of work in 8 days. A and B together take twice as long as A, B, C, D together to perform the same work. A works during the whole of the day, B during three-fourths, C during a half, and D during one-fourth of the day, and the work is finished in $13\frac{7}{8}$ days; had A, B and C worked whole time, the work would have been finished in $11\frac{1}{4}$ days. What is the equitable proportion of their respective shares of the total wages received?

161. Divide

$$(x+y+z)^4 + \{(x+y)^2 - z^2\}^2 + (x+y-z)^4 \text{ by } (x+y)^2 + 3z^2.$$

162. Resolve into factors:

- (i) $(a^2 - b^2)(a+b) + (b^2 - c^2)(b+c) + (c^2 - a^2)(c+a)$;
- (ii) $(x^6 - 8x^4 + 16x^2) + 6x(x-2)(x+2) + 9$.
- (iii) $ab(x^2 + y^2) - (a^2 + b^2)xy + (2a+b)x - (a+2b)y + 2$.

163. Prove that $\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2} = \left(\frac{1}{b-c} + \frac{1}{c-a} + \frac{1}{a-b} \right)^2$.

164. Extract the square root of

$$4 \left\{ \sqrt{\frac{x^2}{y^2}} - \sqrt{\frac{x}{y}} + \sqrt{\frac{y^2}{x^2}} - \sqrt{\frac{y}{x}} \right\} + 9$$

165. If $x^2 + y^2 = 1$, and $ax + by = c$, express $bx - ay$ independently of x and y .

166. If b is a mean proportional between a and c , prove that

$$\frac{a^8 - b^8 + c^8}{a^{-8} - b^{-8} + c^{-8}} = b^{16}.$$

167. If $a + b + c + d = 0$, then will

$$abc + bcd + cda + dab = \sqrt{\{ (bc - ad)(ca - bd)(ab - cd) \}}.$$

168. Shew that the sum of the squares of three consecutive odd numbers increased by 1 is divisible by 12, but not by 24.

169. Solve

$$\begin{aligned} (1) \quad & \left\{ \frac{x + \sqrt{(x^2 - a^2)}}{x - \sqrt{(x^2 - a^2)}} \right\}^3 = (\sqrt{2} + 1)^{10} \left\{ \frac{\sqrt{(x+a)} + \sqrt{(x-a)}}{\sqrt{(x+a)} - \sqrt{(x-a)}} \right\}; \\ (2) \quad & \left. \begin{aligned} a^x + b^y + c^z &= a + b + c, \\ la^x + mb^y + nc^z &= la + mb + nc, \\ a^{x+1} + b^{y+1} + c^{z+1} &= a^2 + b^2 + c^2. \end{aligned} \right\}. \end{aligned}$$

170. State the conventional laws that connect the letters and digits in the quantities ab and 23. Find the value of c , if '2c' when read according to the law by which we interpret '23' is three times the value of '2c', when read according to the law by which we interpret 'ab'.

171. What numerical value of l will make the expressions

$$\begin{aligned} & 2(l^3 + l^2)x^3 + (11l^2 - 2l)x^2 + (l^2 + 5l)x + 5l - 1 \text{ and} \\ & 2(l^2 + l)x^2 + (11l - 2)x + 4 \end{aligned}$$

have a common algebraical factor.

172. Find the square root of $\frac{a^4 x}{a - x}$ to four terms, and the sixth

$$\text{root of } \frac{a^6 b^6}{c^6} + \frac{64c^{12}}{a^6 b^6} - 160c^8 - 12 \cdot \frac{a^4 b^4}{c^3} + \frac{240c^6}{a^2 b^2} + 60a^2 b^2 - \frac{192c^9}{a^4 b^4}.$$

$$173. \text{ Reduce to its lowest terms } \frac{3x^3 - 23x^2 + 43x - 8}{x^4 - 5x^3 - 6x^2 + 35x - 7}.$$

174. Shew that $4x(x+a)(x+b)(x+c) + a^2 b^2$ is a perfect square, when $c = a + b$, and a perfect fourth power when also $c^2 = 2ab$.

$$175. \text{ Simplify } \frac{(a+3c+2b)(a+c)^3 - (2a+3c+b)(b+c)^3}{(a+2c+b)^3}.$$

176. Express as the difference of two squares :

$$(i) (a+2b+3c)(3a+2b-c); \quad (ii) (x+2a)(x-3a)(x+4a)(x+9a).$$

177. If $ax + c : bx + l = 2c : l - d$, prove that

$$(a+b)x - (c-d) : (a-b)x - (c+d) = (a+b)x + c + l : (a-b)x + c - l.$$

178. If x, y and z are unequal, and if $2a - 3y = \frac{(z-x)^2}{y}$,
and $2a - 3z = \frac{(x-y)^2}{z}$, then will $2a - 3x = \frac{(y-z)^2}{x}$, and $x + y + z = a$.

179. Solve (1) $\frac{1}{x^2+5x+6} + \frac{2(x+1)}{x^2+6x+8} + \frac{1}{x^2+7x+12} + \frac{6(x+12)}{x+4} = 16$;
(2) $\left. \begin{aligned} (x+y)(y+z) &= \frac{1}{4}(x+2y+z), \\ (y+z)(z+x) &= \frac{3}{8}(x+y+2z), \\ (z+x)(x+y) &= \frac{5}{8}(2x+y+z) \end{aligned} \right\}$

180. A man takes 1 hr. 30 min. to row from A to B and back, when there is no stream. When there is a stream, he takes 36 minutes to row from A to B ; how long will he take to row back?

181. Shew that $\frac{a-cx}{a+bx} = 1 - (b+c)\frac{x}{a} + (b+c)\frac{bx^2}{a^2} - (b+c)\frac{b^2x^3}{a^3} + \&c.$

182. If $ax^3 + by^3 = cz^3$, and $1/x + 1/y + 1/z = 1$, find the value of $\frac{2}{\sqrt{(ax^2 + by^2 + cz^2)}}$.

183. If $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$, shew that $x : y : z = a : b : c$.

184. Simplify $\frac{bc(a^2+1)}{(a+b)(a+c)} - \frac{ca(b^2+1)}{(b+a)(b-c)} - \frac{ab(c^2+1)}{(c+a)(c-b)}$.

185. If $x^2 + x + 1 = 0$, then $x^2 = 1$, and $x^{2^n} + x^{2^{n-1}} + 1 = 0$.

186. Shew that the following is a perfect square : $x^{-2}(x+4x^{-1})^2 + 3(3x^2 - 2 + x^{-2})(x^2 + 3x^{-2})$.

187. If $\frac{2y+2z-x}{a} = \frac{2x+2y-z}{b} = \frac{2x+2y-z}{c}$, shew that $\frac{x}{2b+2c-a} = \frac{y}{2c+2a-b} = \frac{z}{2a+2b-c}$.

188. Shew that the expression $x^2(1+y^2) + y^2(1+z^2) + z^2(1+x^2)$ is not less than $6xyz$.

189. Solve (1) $\cdot 45 + \frac{\cdot 27x - \cdot 45}{\cdot 09} = \frac{\cdot 256}{\cdot 4} - \frac{\cdot 27x - \cdot 54}{\cdot 9}$;

(2) $\frac{36x^2 + 63x + 101}{60x^2 + 77x + 150} = \frac{39x^2 + 65x + 96}{65x^2 + 78x + 144}$.

Find the value of a so that the following four equations may be consistent :

$x + 5y - 4z = 5$, $3x - 2y + 2z = 14$, $10x - 8y - z + 6 = 0$, and $(a-1)x + ay + (a+1)z = 32$.

190. Two trains running at rates of 25 and 20 miles an hour respectively on parallel rails in opposite directions are observed

to pass each other in 8 seconds, and when they are running in the same direction at the same rates as before, a person sitting in the faster train observes that he passes the other in $31\frac{1}{2}$ seconds; find the lengths of the trains.

191. Prove that $x^4 + ax^3 + bx^2 + cx + d$ is an exact square, if $a(4b - a^2) = 8c$, and $(4b - a^2)^2 = 64d$.

192. If $a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}} = 0$, shew that
 $(a^2 + b^2 + c^2 - 2bc - 2ca - 2ab)^{\frac{1}{2}} = 128abc(a + b + c)$.

193. Simplify

$$a \cdot \frac{(a+b)(a+c)}{(a-b)(a-c)} + b \cdot \frac{(b+a)(b+c)}{(b-a)(b-c)} + c \cdot \frac{(c+a)(c+b)}{(c-a)(c-b)}$$

194. Simplify

$$\frac{8\sqrt{12} - 5\sqrt{1200} - \sqrt{392} + 10\sqrt{243} + 2\sqrt{98}}{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}$$

195. Given $\frac{bz+cy}{b-c} = \frac{cx+az}{c-a} = \frac{ay+bx}{a-b} = \frac{ax+by+cz}{a+b+c}$, prove that
 $(b+c)x + (c+a)y + (a+b)z = 0$, $bzx + cay + abz = 0$,
 and either $a+b+c=0$, or $abc = (b-c)(c-a)(a-b)$.

196. If $a(bz+cy-ax) = b(cz+ax-by) = c(ax+by-cz)$, prove that

$$\frac{y+z-x}{a} = \frac{z+x-y}{b} = \frac{x+y-z}{c}$$

197. Eliminate x , y and z from the equations
 $(y-z)^2 = ayz$, $(z-x)^2 = bzx$, $(x-y)^2 = cxy$.

198. Show that a pair of the quantities a , b , c must be equal, if the following equations be consistent:

$$ax + by + c = 0, a^2x + b^2y + c^2 = 0, a^3x + b^3y + c^3 = 0.$$

199. Solve

$$(1) a^2(b-c)(x-a) + b^2(c-a)(x-b) + c^2(a-b)(x-c) = 0;$$

$$(2) \frac{\sqrt[3]{x+y} - \sqrt[3]{x-y}}{\sqrt[3]{x+y} + \sqrt[3]{x-y}} = \frac{1}{2}, x + 2y = 40.$$

$$(3) \frac{ax^3 - bx^2 + cx + d}{ax^3 - bx^2 - cx - d} = \frac{a'x^3 - bx^2 + c'x + d'}{a'x^3 - bx^2 - c'x - d'}$$

200. A man near the sea-shore sees the flash of a gun fired from a vessel steaming directly towards him and hears the report in 15 seconds. He then walks towards the ship at the rate of 3 miles an hour, and sees a second flash 5 minutes after the first, and immediately stops; the report follows in 10.5 seconds. Find the rate of the ship, the velocity of sound being 1200 feet per second.

ANSWERS.

Examples 1. Page 2.

1. 21. 2. 1. 3. 24. 4. 10 5. 27. 6. 5.
7. $9\frac{7}{8}$. 8. $1\frac{11}{16}$. 9. 27. 10. $109\frac{6}{11}$. 11. $2\frac{2}{7}$. 12. $25\frac{5}{16}$.

Examples 2. Page 3.

1. $\frac{2}{3}$. 2. $\frac{2}{3}$. 3. $30\frac{2}{7}$. 4. 42. 5. 54. 6. $1\frac{1}{2}$.
7. $66\frac{2}{3}$. 8. 150. 9. $\frac{1}{2}$. 10. $1\frac{1}{8}$. 11. 18. 12. $1\frac{1}{2}$.
13. 1. 14. $18\frac{3}{4}$. 15. 6. 16. $2\frac{2}{3}$. 17. $1\frac{1}{2}$. 18. $42\frac{3}{8}$.

Examples 3. Page 4.

1. 6. 2. 7. 3. $9\frac{1}{2}$. 4. $5\frac{1}{2}$. 5. $6\frac{1}{2}$. 6. 4. 7. 0.
8. 87. 9. 27. 10. 3. 11. 281. 12. 0. 13. $\frac{13}{8}$. 14. 42.

Examples 4. Page 4.

1. 0. 2. 0. 3. 0. 4. 30. 5. 0. 6. $98\frac{2}{3}$.

Examples 5. Page 6.

1. 243. 2. 1728. 3. 14641. 4. 216. 5. 24. 6. $1111\frac{1}{4}$. 7. 144.
8. 3. 9. $30\frac{8}{7}$. 10. $\frac{2}{3}$. 11. 59. 12. 98. 13. $\frac{4}{3}$. 14. 37.

Examples 6. Page 7.

1. 24. 2. 48. 3. 1. 4. $2\frac{1}{2}$. 5. $\frac{1}{8}$. 6. $\frac{1}{8}$. 7. $1\frac{1}{8}$. 8. $\frac{1}{4}$. 9. 64.

Examples 7. Page 8.

1. $155\frac{8}{11}$. 2. $8717\frac{1}{16}$. 3. £13 14s. $1\frac{1}{2}$ d.
4. 2 qrs. $5\frac{2}{3}$ lbs. 5. 4 yds. 2 ft. ; 35 yds.

Examples 8. Page 11.

1. -5 miles. 2. Gain = -10s. ; loss = 10s.
3. -2 yds longer ; 2 yds. shorter. 4. 2 ft 7 in. 5. 50 seers.
6. 45 min. past 9 ; 15 min. past 10. 7. -5°C. 8. (b-a)yrs ; (b+a)yrs.

Examples 9. Pages 12—13.

1. 0. 2. 8. 3. 48. 4. 19. 5. 721.
6. 9253. 7. 40207. 8. 70053. 9. 10000. 10. 684.
11. 554. 12. 31. 13. 98. 14. 66. 15. 11116.
16. 4. 17. 384 18. 256. 19. 512. 20. $316\frac{8}{15}$.
21. $2\frac{2}{3}$. 22. $361\frac{7}{17}$ 23. $16\frac{5}{16}$. 24. $7\frac{1}{8}$. 25. $9\frac{5}{8}$.
26. 1296. 27. $1\frac{5}{8}$ 28. 3. 29. $1\frac{5}{8}$. 30. $22\frac{7}{8}$.
31. $(x-y+z)$ or 31 sheep. 32. $3ab$, or 48600 lines.
33. y^2 sq. ft.; $\frac{y^2}{9}$ sq. yds. 34. $a-b^2+c^3-d^4$; 13 yds. 1 ft. 9 in.
35. $20a+10b+5c+2d+c+\frac{f}{12}$ shillings

Examples 10. Pages 15—16.

1. $12a$. 2. $19pqr$. 3. $23a^2b$. 4. $49a^2x$. 5. $-18a$.
6. $-24mp^2$. 7. $-46xyz$ 8. $-2\frac{2}{3}ab^2$. 9. 0. 10. 0.
11. $10a^2b$. 12. $-11a^2b$. 13. $23mnr$. 14. $-36a^2b$. 15. 0.
16. $-\frac{1}{6}x^2$. 17. $-\frac{2}{3}\sqrt{xy}$. 18. $-\frac{3}{4}p^3$. 19. $-\frac{6}{7}abc$.
20. $-\frac{4}{3}x^2y^3$ 21. 0 22. $\frac{1}{6}x^2$ 23. $-\frac{2}{3}\sqrt{(a^2b)}$

Examples 11. Pages 17—18.

1. $x+y$ 2. $2x+3y+z$. 3. $\frac{2}{3}x-7y+z$.
4. $14a^2-15b^2+12d^2-\frac{1}{2}lm$ 5. $2a^4+a^2-a$.
6. $x^2y^3+x^2y^2-xy-2$. 7. $-16y^7+y^6+\frac{1}{2}y^3+6y^2-\frac{1}{2}y+1$.
8. $\frac{2}{9}b^{10}+\frac{1}{9}b^7+11b^3-3b^2+b+5$. 9. $4b^3-b^2c+5bc^2-7c^3$.
10. $7l^4m^4+4l^3m^6-4l^2m^6-9lm^7$.
11. $-5x^5+27x^4y-4x^2y^2-5x^2y^3-3xy^4-7y^5$.
12. $2a+3b$. 13. $20a^2+\frac{1}{2}bc+9b^2$. 14. $3x^3-9y^3$.
15. $3x^2y^2-3xy^4-7y^5$. 16. $9a^2b^3-19ab^3+21b^5$.
17. $-\frac{1}{2}a^3c^2-\frac{1}{8}b^3-\frac{1}{4}xyz$. 18. $-\frac{3}{2}xyz$.
19. $-\frac{3}{8}a^2m$. 20. $2x^5+3x^4y-3x^2y^2-x^2y-27$.

Examples 12. Pages 18—20.

1. $2a$. 2. b . 3. 0 4. $x+y+z$. 5. $21x-2y+\frac{1}{2}z$.
6. $23a+13c-d$. 7. 0. 8. $4b^4+14c^4-4d^4$.

9. $12a^n - 7b^n + c^n + 33a^n$. 10. $-13a + 20$. 11. $14b^3 - 8b + 3$.
 12. $-5x^2 + 5xy + 8y^2$. 13. $8x^2$. 14. $-4x^3 - 14x^2 - x - 10$.
 15. $-x^4 + 39x^3 + 30x^2 - 39x + 6$. 16. $6x^4 + 6x^3 + 19x^2 - 6x$.
 17. $6y^3 - y^2 + 4y - 32$. 18. $6x^3 + 9x^2y - 14xy^2 - y^3$.
 19. $-5a^4 - 14a^3b + a^2b^2 - 6ab^3 + 9b^4$.
 20. $2x^5 - 2ax^4 + 3a^2x^3 - 11a^3x^2 + 7a^4x - a^5$.
 21. $2a^3 - b^2 - 3c^3 - ab + bc$. 22. $6a^3 - 19ab + 6b^2 - 5a + 12$.
 23. $x^5 + 3y^3 + 7z^3 + x^2y$. 24. $\frac{5}{4}x^4 + \frac{1}{8}x^3 - \frac{4}{3}x^2 + \frac{1}{14}x - \frac{1}{2}$.
 25. $\frac{5}{3}a^2 + \frac{2}{3}ab + \frac{2}{3}b^2$. 26. $\frac{7}{10}x - \frac{1}{8}by$. 27. $-\frac{1}{8}a + \frac{3}{8}b$.
 28. $\frac{1}{5}x^2 - \frac{1}{6}xy + \frac{1}{10}y^2$. 29. $\frac{1}{2}\sqrt{ab} - \frac{1}{8}\sqrt{bc} - \frac{1}{4}\sqrt{ca}$.
 32. $(x-y)$ degrees; -1^0 . 33. $(2a-3b)$ miles; -1 mile. 34. $3a^2 + c^2$.

Examples 13. Page 22.

1. $5a$. 2. $-\frac{1}{4}a$. 3. $\frac{5}{2}a$. 4. $\frac{23}{2}a$.
 5. $-\frac{5}{2}a$. 6. $-\frac{23}{2}a$. 7. $2a - 3b$. 8. $6x + 9y$.
 9. $20x - 9y$. 10. $b + c + d - a$. 11. $\frac{5}{2}c - 2a - 3b$.
 12. $a - 11b - 3c$. 13. $2b$. 14. $3b + 3c$.
 15. $\frac{1}{2}a - \frac{1}{3}b + c$. 16. $x + \frac{1}{10}y - \frac{1}{2}z$. 17. $2x - 2y$.
 18. $4x + y + 7z$. 19. $-5x + 3y - 11z$. 20. $3x + 3y - 11z$.
 21. $3x + 3y + 3z$. 22. $a + b - c - 2d$. 23. $x^2 - 5x$.
 24. $10x^3 + 7x^2 + 3x$. 25. $8x^4 + 2x^2y^2 + 4y^4$. 26. $9xz - 6xy$.

Examples 14. Page 23.

1. $-a^2 - a + 2$. 2. $2a^2 + 9a - 12$. 3. $-4x^2 + 15xy + y^2$.
 4. $2xy + 2yz$. 5. $\frac{7}{12}a^2 - \frac{1}{12}ab + \frac{1}{6}b^2$.
 6. $3x^4 + 2ax^3 + 2a^2x^2 - a^3x - a^4$. 7. $x^2 - bx$.
 8. $-ax^2 - ax + a^2 - b^2$. 9. $x^4 + ax^3 + a^2x^2 - a^4$.
 10. $2ab + 2bc - 2ca$. 11. $2x^4 + 4x^3y + x^2y^2 + xy^3 - 2y^4$.
 12. $-x^4 + 14x^3y + x^2y^2 + 6xy^3 + 2y^4$.
 13. $-9x^6 + 5x^4y - x^3y^2 + 10x^2y^3 - 3y^6$.
 14. $-2x^6 + 12x^5y + 9x^4y^2 + x^3y^3 - 4xy^5 + y^6$. 15. $a^3 - a^2 + a + 1$.
 16. $p^3 + 2ap^2 + ap - 2$. 17. $4x^5 - 3x^4 - 4x^3 - 7x^2 + 2x - 1$.
 18. $7x^6 + 5x^5 + 3x^4 + 7x^3 - 3x^2$. 19. $7a^3 - 2b^3 + \frac{1}{3}c^3 - \frac{1}{3}d^3 - 2f^3 + \frac{1}{2}g^3$.
 20. $2x^3 + 2y^3 - 2z^3 - xy$. 21. $a^4 + 2a^3 - 7a^2 + 3a - 1$.
 22. $3ax^3 - 2a^2x^2 - ax + 6$. 23. $3pq - 6qr$. 24. $3pq^2 + 2q^3$.

Examples 15. Page 27

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|---------------------------------|-----------------------------------|--------------------------------------|-------------------------------------|
| 1. $6ab.$ | 2. $15a^2.$ | 3. $14a^3b.$ | 4. $45a^3.$ |
| 5. $-21x^4.$ | 6. $-3a^2b^2.$ | 7. $-12x^2y^4.$ | 8. $2ax^2y^2z^4.$ |
| 9. $54a^{10}b^8c^3.$ | 10. $-2a^4b^2c^2.$ | 11. $-\frac{2}{3}ax^2yz.$ | 12. $\frac{1}{4}\frac{2}{3}a^5x^3.$ |
| 13. $\frac{1}{2}a^{2m}b^{n+r}.$ | 14. $a^2y^mz^{n+p}.$ | 15. $-\frac{1}{8}a^{r+1}b^{p+s}c^r.$ | 16. $a^2.$ |
| 17. $a^2b^2.$ | 18. $a^4x^2.$ | 19. $4x^4y^2z^2.$ | 20. $25a^2b^4c^4x^4y^{10}.$ |
| 21. $4a^4x^6y^8.$ | 22. $\frac{4}{9}x^8y^6z^4w^{20}.$ | 23. $\frac{1}{2}a^4b^6c^3.$ | |

Examples 16 Pages 28—29.

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|--|--|---------------------------------|----------------------------|
| 1. $18a^2b^2c^2.$ | 2. $12abc^2.$ | 3. $-144a^3bc^3d.$ | 4. $-24pqrst.$ |
| 5. $a^4b^3c^2.$ | 6. $24x^4y^2z^2.$ | 7. $8a^2b^2c^2.$ | 8. $4a^2bx^2y^2zfg.$ |
| 9. $l^5m^5n^5.$ | 10. $-p^{14}q^{14},^{14}$ | 11. $a^{10}b^{10}c^{10}.$ | 12. $-a^4b^4c^4x^4y^4z^4.$ |
| 13. $9a^{m+r+2}b^{m+5}c^4.$ | 14. $a^{2x}b^{2y}c^{2z}.$ | 15. $a^{2m}b^{2m}c^{2m}d^{2m}.$ | |
| 16. $-12abca^{m+n}b^{m+n}c^{2m+n}e^{n+1}.$ | 17. $a^{21}b^{2m}c^{3n}.$ | | |
| 18. $a^{2l+m+n}b^{m+n}c^{2+2m+n}e^{n+1}.$ | 19. $a^6.$ | 20. $-a^5b^3.$ | 21. $-a^{10}b^{10}c^{12}.$ |
| 22. $-8a^6x^9y^0.$ | 23. $81a^4b^4c^4.$ | 24. $\frac{1}{81}a^8b^4c^4.$ | |
| 25. $-\frac{2}{10}a^3b^{10}m^{15}n^{20}.$ | 26. $64x^6y^{12}z^{12}w^{15}.$ | 27. $-a^{10}b^{10}c^{10}.$ | |
| 28. $x^7y^7z^7.$ | 29. $-a^{10}b^{10}c^{10}.$ | 30. $-8944a^4x^{11}y^4z^5.$ | |
| 31. $-4l^4m^7n^6.$ | 32. $-32\frac{1}{4}a^{11}b^{12}c^{15}w^0.$ | | |

Examples 17. Page 30.

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|---|---|--------------------|-----------------------------|
| 1. $3x+12.$ | 2. $8x^2-12xy.$ | 3. $a^2bc-c^2c^2.$ | 4. $20a^4bc^2-30a^2b^3c^4.$ |
| 5. $2abx^3-3aby^3.$ | 6. $10x^5y^2-4x^3y^4+6pq$ | | |
| 7. $45a^5b^2c-63a^4b^2c+27a^3b^2c.$ | 8. $-4x^4y^3z^7-\frac{4}{3}x^2y^4z^8+\frac{1}{3}xy^8z^8.$ | | |
| 9. $-a^5b^2c^2d+\frac{4}{3}a^2b^4c^2d^3+\frac{2}{3}a^5b^2cd^2+\frac{4}{3}a^3b^2c^2d^2.$ | | | |
| 10. $-12a^4b^4c^2d^2+21a^3b^4c^2d^2-15a^2b^4c^2d^2.$ | 11. $21x^3-4x^2+3x.$ | | |
| 12. $8x^4-14x^3+x^2-9x+12.$ | 13. $3b^2x+bx-7b^2+3b.$ | 14. $0.$ | |
| 15. $a^2c-bc^2.$ | | | |

Examples 18. Pages 32—33.

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|---|---------------------------|-------------------|
| 1. $x^2+8x+7.$ | 2. $x^2+5ax+6a^2.$ | 3. $x^2-ax-6a^2.$ |
| 4. $x^2-5ax+6a^2.$ | 5. $18+3a-a^2.$ | 6. $x^2+2x-63.$ |
| 7. $a^2x^2-b^2y^2.$ | 8. $a^2-b^2.$ | 9. $a^2+2ab+b^2.$ |
| 10. $\frac{3}{2}a^2b^2+\frac{1}{2}ab^2d-6c^2d^2.$ | 11. $a^3+6a^2+13a+12.$ | |
| 12. $3a^3-a^2-a+15.$ | 13. $2a^3-21a^2+59a-35.$ | |
| 14. $x^3+6x^2y+12xy^2+8y^3.$ | 15. $x^6-4x^3+5x^2+4x-10$ | |

16. $x^6 + y^6$. 17. $-80x^3 + 228x^2y - 256xy^2 + 120y^3$.
 18. $ax^5 + a^2x^2 - bx^2 - 2abx + b^2$. 19. $a^7 - 4a^3b^2 - a^2b^3 + 2b^4$.
 20. $x^4 - 1$. 21. $a^2 + 2ab + b^2$. 22. $a^2 - 2ab + b^2$.
 23. $2a^2 + a - 91$. 24. $\frac{1}{2}x^2 - \frac{1}{3}xy + 2y^2$. 25. $3x^3 + 2x - 18$.
 26. $-3a^2 - 6a$. 27. -2 . 28. $16y^3$.

Examples 19. Pages 33-35.

1. $4x^4 + 1$; $6x^4 + 2x^3 - 5x^2 - 6x - 2$.
 2. $x^6 - 8x^4 + 16x^3 - 24x^2 + 71x + 120$;
 $12x^5 - 48x^4 + 59x^3 - 284x^2 - 5x + 24$.
 3. $4a^4 + a^2bx + (5a^3b - 3b^2 - 4a^4)x^2 + (3a^2b + 5ab^2)x^3 - 5a^3bx^4$;
 $4a^2 - (8a^3 + 3b)x + (4a^2 + 11ab)x^2 - (10a^2b + 3b)x^3 + 5abx^4$.
 4. $1x^4y + 2x^4 + x^3y - 6x^3y^2 - 10x^2y^3 - 3x^2y^2 + 12xy^4$;
 $-4x^4 - 8x^4y + 16x^3y^2 + 2x^2y^3 + x^2y^2 - 4xy^4$.
 5. $-2a^4 + 3a^3b - 4a^2b^2 + ab^3 + 2b^4$; $2a^4 - 6a^3b + 11a^2b^2 - 11ab^3 + 6b^4$.
 6. $a^2x^3 + (2ac - b^2)x + 2bc$; $apx^3 + (2aq - bp)x^2 + (2cp - 2bq)x + 4cq$.
 7. $3x^6y - 2x^5y^2 + x^4y^3 + 3x^3y^4 - 2x^2y^5 + x$;
 $6x^5y - 7x^4y^2 - 11x^3y^3 + 9x^2y^4 - 5xy^5$.
 8. $ax^4 - (a+b)x^2 + (a+b+c)x^2 - (b+c)x + c$;
 $a'x^4 + (am - b'l)x^3 + (an - bm + cl)x^2 + (cm - bn)x + cn$.
 9. $-20x + 17x^2 + 5x^3 + 14x^4 - 4x^5$;
 $10x - 21x^2 + 25x^3 - 14x^4 + 4x^5 - 16x^6$.
 10. $a^6 - 12a^4b^2 - 4a^2b^4 - b^6$. 11. $a^2 - b^4 + c^4 - 2a^2c^2 - 4b^2c^2 - 4b^3c$.
 12. $3x^2y^2 - x^4 - y^4$, $x^4 - 2\sqrt{5x^3y + 5x^2y^2 - y^4}$.
 13. $5m^5 - 4m^4 + 3m^3 - 5m^2 + 4m - 3$; $15m^4 - 32m^3 + 50m^2 - 32m + 15$.
 14. $a^3 - b^3 - c^3 - 3abc$. 15. $a^{2m} + 2a^mb^m + b^{2m}$; $x^{2n} - y^{2n}$.
 16. $x^6 + x^4 - x^2 - 1$. 17. $2a^3$. 18. 0. 19. $4ab + 4cd$.
 20. $x^8 - 4x^4y^4 + y^8$. 21. $x^5 + 2x^4 - 16x^3 - 5x^2 + 21x + 9$.
 22. $8x^6 - 18x^5 + 13x^4 - 10x^3 + 7x^2 - 5x + 2$.
 23. $a^2x^6 + abx^5 - (ab - ac)x^4 - b^2x^3 + (ac - bc)x^2 + b^2cx + c^2$.
 24. $4a^6 - 4a^5 + a^4 - 2a^3 + 6a^2 - 2a - 3$. 25. $a^3 - b^3 + c^3 + 3abc$.
 26. $a^3 - 8b^3 + 27c^3 + 18abc$. 27. $x^3 + y^3 + 6xy - 8$.
 28. $x^6 + 2x^4y + x^4y^2 - x^3y^4 + 2xy^5 - y^6$.
 29. $x^6 + 2x^5y - 8x^3y^3 + 4x^2y^2 - 16x^2y^4 + 8xy^3 + 16y^4$.
 30. $1 + 91x^6 + 180x^7$.

- 81 $a - (a+b)x + (a+b+c)x^2 - (a+b+c+d)x^3 + (b+c+d+e)x^4$
 $- (c+d+e)x^5 - (d+e)x^6 + ex^7$ 32 $a^3 - b^3$.
33. $x^8 - 116x^7 + 1789x^6 - 10460x^5 + 2502x^4 - 5382x^3 + 4033x^2 - 708x$
 $+ 1199$ 34 $x^2 - \frac{1}{2}a + \frac{3}{8}, \frac{1}{4}x^2 - \frac{1}{8}a + 1; x^2 - \frac{3}{4}a + 1$.
- 35 $\frac{1}{8}x^3 + \frac{1}{8}x^2 + \frac{2}{16}x + \frac{1}{8}$ 36 $\frac{1}{8}x^3 - \frac{9}{8}x^2 + \frac{2}{16}x - 27y^3, \frac{1}{8}x^4 - \frac{4}{8}x^3y + \frac{7}{8}x^2y^2 + \frac{3}{8}xy^3 - 6y^4$.
37. $\frac{1}{16}x^4 - \frac{3}{8}x^3y + \frac{3}{8}x^2y^2 + \frac{3}{8}x^2y^3$ 38 $\frac{1}{2}a^2x^5 - \frac{1}{2}ab^4xy^4$.
- 39 $2a^3x^4 - (3a^2y + 2aby)x^3 + (4a^2y^2 + 3aby^2 + 2b^2y^3)x^2 - (4aby^3 + 3b^2y^3)x$
 $+ 4b^2y^4$ 40 $a^5b^3 - (2b^2c^2 + b^4)a^4 + (3bc^3 + 3b^2c^2)a^3$
 $- (3b^2c^3 + 2b^2c^4 + 4c^4)a^2 + (4bc^4 + 3bc^5)a - 4c^6$
- 41 $a^6l^3 - 6a^5l^4 + 12a^4l^5 - 12a^3l^6 + 8a^2l^7 - 2al^8 - 2l^9$
- 42 $a^2x^4 + a^2y^3)x^3 + (4a^2y^2 + a^2y^4 - 2y^6)x^2$
 $+ (2ay^5 - a^2y^3 - 3ay)x - 2a^2y^4 + 3a^6$.
- 43 $14x^{10} - 29x^8 + 64x^6 - 34x^4 - 31x^2 - 23x^5 + 4x^4 + 10x^3 + x + x - 1$
44. $x^9 - 9x^8 - 34x^7 + 45x^6 + 76x^5 - 107x^4 - 65x^3 + 106x^2 + 12x - 32$
- 45 $- 16x^7 + 24x^6 - 45x^5 + 67x^4 + 30x^3 - 54x^2 + 11x + 15$
- 46 $18a^3 - 9a^7 - 2a^6 + 6a^5 - 21a^4 + 23a^3 - 89a^2 + 60$
- 47 $- 882x^5 + 357x^4 + 880x^3 - 379x^2 - 220xy^4 + 100y^6$
- 48 $616 - 252x + 14x^4 + 572x^3 - 152x^2 + 148x - 48$
- 49 $(b^3 + b^2c - bc^2 - c^3)a^3 + (3bc^3 + 2b^2c^3 - b^3c)a^2 - (b^3c^2 + 3b^2c^3)a + b^3c^3$
- 50 $\frac{1}{8}x^4 - \frac{1}{8}x^3 - \frac{1}{8}x^2 - x^2 + 4$ 51 $\frac{1}{4}x^2 - \frac{1}{4}x^4 + \frac{1}{80}x^3 - \frac{1}{80}x^2 - \frac{1}{80}x + \frac{1}{2}$
52. $\frac{3}{8}x^8 - \frac{3}{8}x^6 + x^5 - \frac{2}{8}x^4 + \frac{3}{8}$
- 53 $\frac{1}{8}a^8 - \frac{1}{8}a^6b^2 - \frac{1}{8}a^5b^3 - \frac{1}{8}a^4b^4 + 4a^4 + 10$
- 54 $l^5m^5 - 2l^3m^3 + lm$ 55 $\frac{1}{2}y^6 - 2x^5y^6 + 3x^4y^4 - 5x^3y^3 + 4x^2y^2 - xy$.
56. $2x^8 + 3x^7y - 8x^6y^3 + 3x^5y^5 - 8x^4y^5 + 10x^3y^6 - xy^7 - 1^8$
57. $28x^{10} - 35x^{15} + 7x^{14} - 16x^{12} + 20x^{11} - 4x^{10} + 12x^8 - 15x^7 + 3x^6$
 $- 4x^4 + 5x^3 + 11x^2 - 15x + 3$
- 58 $6x^{24} - 14x^{23} - 26x^{20} - 9x^{10} + 21x^{17} + 39x^{15} - 33x^{14} + 77x^{12}$
 $+ 143x^{10} + 27x^9 - 63x^7 - 117x^5 - 21x^4 + 49x^2 + 91$.

Examples 20. Pages 36—37

- $x^2 + 2xy + y^2; a^2 + 4ab + 4b^2$
- $4a^2 + 12ab + 9b^2, 16a^3 + 40ax + 25x^2$
- $49x^2 + 112xy + 64y^2; a^2x^2 + 2abxy + b^2y^2$.
- $x^3 - 4xy + 4y^3; 9x^2 - 24xy + 16y^2$

5. $100a^2 - 60a + 9$; $1 - 24x + 144x^2$.
 6. $36l^2 - 60lm + 25m^2$; $4l^2a^2 - 20lmab + 25m^2b^2$.
 7. $2x^3 + 2a^2$. 8. $p^3 - 4q^3$ 9. $16a^3 - 25b^3$ 10. $4x^2 - 9y^2$.
 11. $a^4 + a^2b^2 + b^4$ 12. $a^2 - b^2 - c^2 + 2bc$ 13. $a^2 + b^2 - c^2 - a^2 + 2ab - 2cd$.
 14. $a^2 + c^2 - b^2 - a^2 + 2ac - 2bd$ 15. $a^2 + c^2 - b^2 - a^2 - 2ac + 2bd$.
 16. $a^2 + b^2 - c^2 - a^2 - 2ab + 2cd$ 17. $x^2 + xy + y^2$ 18. $4x^2$.
 19. $25x^4 + 30x^3 - x^2 - 6x + 1$ 20. $25x^4 + 30x^3 + 9x^2 - 1$.
 21. $25x^4 - 9x^2 - 6x - 1$ 22. $25x^4 - 9x^2 + 6x - 1$.
 23. $x^6 - 3a^2x^4 + 3a^4x^2 - a^6$ 24. $4x^3$ 25. 0. 26. $4ac$ 27. $4y^2$.
 28. $(a^2 - b^2)x^2 + 4abxy - (a^2 - b^2)y^2$ 29. $x^4 + y^4$.
 30. $x^4 + 2\sqrt{2}x^2y + 2x^2y^2 - y^4$ 31. $a^4b^4 - 2a^2b^2pq + p^2q^2$.
 32. $\frac{l^4}{25} - \frac{l^2}{9} + \frac{l}{3} - \frac{1}{4}$; $\frac{l^4}{25} - \frac{1}{4}\frac{l^2}{9} + \frac{1}{4}$ 33. $\frac{a^4b^4}{16} - \frac{a^2b^2}{9} + \frac{2ab}{3} - 1$.
 34. $x^3 + y^3$ 35. $x^3 - y^3$ 36. $a^3b^3 + c^6$ 37. $a^3b^3 - c^6$.
 38. $a^6x^3y^3 + c^6$ 39. $a^6b^6 - m^3n^3$ 40. $a^6 + b^6$ 41. $a^6b^6p^3q^3 - 1$.
 42. $a^6 - 1$ 43. $a^6 - b^6$ 44. $-2ac$ 45. $2a^3$ 46. a^7
 47. 16. 48. $a^4x^4 + 2a^8$ 49. $2x^6y^2 - 2x^4y^4 - 2y^8$.
 50. 2116; 164025; 9409000000.
 51. 99960004; 63728289; 399480169; 809969400289.

Examples 21. Page 39.

1. $a^3x^3 - b^3y^3$ 2. $a^4 - 5a^2b^2 + 4b^4$ 3. $9x^4 - 52x^2y^2 + 64y^4$.
 4. $x^{16} - a^{16}$ 5. $1 - (a + b + c)x + (ab + ac + bc)x^2 - abcx^3$.
 6. $1 + x^4 + x^8$ 7. $a^3 + a^4x^4 + x^8$ 8. $a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abbc$.
 9. $x^4 - 4x^3 - 19x^2 + 46x + 120$
 10. $x^4 - (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 - (abcd + abcd + acd + bcd)x + abcd$ 11. $a^4 + 10a^3 + 35a^2 + 50a + 24$.
 12. $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.
 13. $a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc$
 14. $27x^3 + 54x^2y + 36xy^2 + 8y^3$; $a^2x^2 - 2abxy + b^2y^2$; $x^4 - 2x^2y^2 + y^4$.
 15. $x^2 + 3x + 2$ 16. $x^2 + 12ax + 35a^2$ 17. $x^2 + 7x - 8$.
 18. $x^3 - 3xy - 10y^3$ 19. $x^3 - 18x + 77$ 20. $x^2 - (a - b)x - ab$.
 21. $x^3 - (5a + 7b)x + 35ab$ 22. $6x^3 + 19x + 15$ 23. $35x^3 + 66x + 27$.
 24. $72x^2 - 151x + 77$ 25. $30x^2 + 91x + 55$ 26. $143x^2 - 19x - 12$.
 27. $165a^2 + 167a + 42$ 28. $108x^3 + 303x + 140$ 29. $40y^2 - 57y - 126$
 30. $168x^2 + 38xy - 45y^2$ 31. $66a^2 - 29ab - 27b^2$.

32. $42y^3 - 115yz + 75z^2$. 33. $104x^2 + 233xy + 104y^2$.
 34. $168 + 13x - x^3$. 35. $63 + 17a - 10a^2$ 36. $acx^2 + (ad + bc)x + bd$.
 37. $ac - (ad + bc)x + bdx^2$. 38. $8a^2x^2 - 2abxy - 15b^2y^2$.

Examples 22. Page 41.

1. -1 . 2. $-a$ 3. a^3 ; $-a$. 4. $\frac{3}{2}a^3$. 5. $\frac{5}{2}a^3$. 6. 4 ; $-a$.
 7. $-\frac{3}{2}x$. 8. $-4b$; -4 . 9. $\frac{5}{2}a^5b^3c$. 10. $\frac{6}{5}q$; $-\frac{3}{2}p$. 11. $-48a^2y^7z^6$.
 12. $\frac{1}{6}cx^{m-n}y^{m-n}z^{m-n}$. 13. -5 ; $\frac{5}{2}xy^4z^4$; $\frac{5}{2}yz^7$; $-\frac{2}{5}x^2y^2z^3$; $-\frac{5}{2}y^4$.
 14. $2a^2b^2c^2$; $-2lb^2c^2$; $-\frac{1}{2}a^2b^2c^2$; $\frac{l}{m}a^2b^2c^2$; $-\frac{2l}{m}a^2b^2c^2$.
 15. $\frac{p}{q}l^{2x-3y}m^{y-z}n^{z-x}$; $\frac{p}{r}l^{x-y}m^{y-z}n^{z-x}$; $\frac{p}{s}l^{x-y}m^{y-z}n^{z-x}$.
 16. $-4x^2y^2z^3$; $2a^2b^2c^3x$; al^2xy^2z ; $-4a^2b^2c^3z$.
 17. $\frac{3}{2}l^2m^2n^2x^4y^4z^4$; $\frac{7}{2}l^2m^2n^2x^4y^4z^4$; $-\frac{5}{2}l^2m^2n^2x^4y^4z^4$.
 18. $\frac{1}{4}a^2b^2c^2$; $-\frac{1}{4}a^2b^2c^2$.

Examples 23. Pages 44-45.

1. $x - y$ 2. $-x + 3y$, 3. $2a^3 - \frac{5}{2}a$. 4. $\frac{9}{2}a^2 - ab$
 5. $x^2 - \frac{3}{2}ax$. 6. $2ab^2 + 3b$. 7. $\frac{9}{2}ax - \frac{7}{2}ab$. 8. $l + 2m - 4n + 6p$.
 9. $4a^4b^3 - 3a^2b^5 + 12b^7$; $96a^4 - 72a^2b^2 + 288b^4$.
 10. $-2x^3 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + y^3$; $12x^3 - 2x^2y + \frac{3}{2}xy^2 - 6y^3$.
 11. $-4x^3y^2 + x^2y^2z + 2yz^3$; $3x^4y - \frac{3}{2}x^3yz - \frac{3}{2}xz^4$.
 12. $3l^3m^2n^3 - \frac{3}{2}l^2m^2n^2 + 2lmn^3$; $-\frac{1}{2}l^3mn + \frac{1}{2}l^2mn - \frac{1}{2}ln^3$.
 13. $6a^2x^3 - 2x^2ax^2 + \frac{1}{2}x$; $-\frac{1}{4}a^3x^2 + \frac{3}{4}a^2x - \frac{1}{4}a$; $-5a^2x^2 + \frac{4}{3}ax - \frac{3}{4}$.
 14. $-al^4m^5n^3 + bl^3m^4n^2 - clm^3n + dm$; $\frac{3}{2}al^5m^0n^4 - \frac{3}{2}bl^4m^1n^3$
 $+ \frac{3}{2}cl^3m^2n^2 - \frac{3}{2}dlm^2n$; $-5l^5m^5n^3 + 5\frac{b}{a}l^4m^4n^3 - 5\frac{c}{a}l^3m^3n + 5\frac{d}{a}lm$;
 $\frac{l^4m^4n^3}{bc} - \frac{l^3m^3n^2}{ac} + \frac{lm^2n}{ab} - \frac{d}{abc}$.
 15. $2a^{2x-2} + 3a^{x-2} + 4a$; $-3a^{2x-1} - \frac{9}{2}a^{x-1} - 6a^2$; $3a^{2x-2} + 3a^{2x-2} + 4a^{x+1}$;
 $-10a^{2x-3} - 15a^{2x-3} - 20a^x$.
 16. $x^3y^3z^3 - 4$; $-5x^3y^3z^3 + 2x^3y^3z^3$; $-\frac{1}{2}x^3y^3z^3 + x^3y^3z^3$.
 17. $ax - \frac{3}{2}bx - 3a + \frac{3}{2}b$. 18. $p^2 - 4q^2$.
 19. $-8x + 2a + 2$. 20. $a^3x - a^2x^2$.

Examples 24. Pages 44—46.

1. $x+4$. 2. $x+10$. 3. $x+6$. 4. $x+8y$. 5. $2x+3$.
6. $2x-1$. 7. $l+13$. 8. $8l-1$. 9. $1+6a$. 10. $1-8a$.
11. $-x-8$. 12. $3x+5y$. 13. $a-bx$. 14. $-5x-11y$. 15. $cx+d$.
16. $9x-11y$. 17. $ax-3by$. 18. $25l^2-15lm+9m^2$; $5l+3m$.
19. $25x^2+35ax+12a^2$; $25x^2+25ax+6a^2$. 20. $x+2y-3$.
21. $a^{11}+a^{10}b+a^9b^2+a^8b^3+a^7b^4+a^6b^5+a^5b^6+a^4b^7+a^3b^8+a^2b^9$
 $+ab^{10}+b^{11}$; $a^{10}+a^8b^2+a^6b^4+a^4b^6+a^2b^8+b^{10}$; $a^9+a^6b^3$
 $+a^3b^6+b^9$; $a^8+a^4b^4+b^8$.
22. k^2-k+2 ; $2k^2-3k-6$. 23. x^2+x-1 . 24. a^2+a-2 .
25. $1-r^2+r^3$. 26. $15x^3-3x^2+4x$. 27. $9x^2+15x+6$.
28. Quo. $=k^3+\frac{1}{3}k^2-\frac{1}{6}k+\frac{3}{2}$, rem. $=-\frac{7}{6}k-\frac{3}{2}$. 29. a^2+4y^2 .
30. $a^4-2a^3b+3a^2b^2-4ab^3+5b^4$. 31. k^2-2k+1 . 32. $a+6b$.
33. $a^2+2ab+2b^2$; $a^2-2ab+2b^2$. 34. $2x^2-xy+y^2$. 35. a^3+ax-x^2 .
36. $a^2+3a^2b+3ab^2+b^3$. 37. $2x^3-3x^2y+2xy^2$. 38. $x^2+x-a-a^2$.
39. $5a^4-a^3b+4a^2b^2-2ab^3+3b^4$. 40. $2x^3+3xy+y^2+3x-1$.
41. $x^4+a^2x^2+a^4$; $x^6-ax^5+a^3x^3-a^5x+a^6$. 42. $x^3+ax^2-a^2x+a^3$.
43. $-x^7-4x^2y-2xy^2-y^3$. 44. $8x^3-2x^2y-6xy^2-3y^3$.
45. $ax+by+c$. 46. $ax-by+z$. 47. $3a+2b+1$. 48. x^3+y^3 .
49. $a^3b^3+a^2b^2+ab+1$. 50. $4x^3+14xy+9y^2$. 51. $\frac{1}{4}a-\frac{2}{9}b$; $\frac{1}{3}a-\frac{1}{2}b$.
52. $\frac{5}{3}x-\frac{4}{3}y$; $12x-3y$. 53. $\frac{1}{2}x^3-\frac{2}{3}xy+\frac{5}{6}y^2$; $\frac{2}{5}x^2-\frac{2}{3}xy+\frac{2}{3}y^2$.
54. $12x-12$. 55. $\frac{1}{6}x^3+\frac{1}{12}xy^2-\frac{1}{3}a^2y$. 56. Quo. $=6x-\frac{4}{3}y+z$, rem. $\frac{1}{6}x^2z^3$.
57. $1-r^2+r^3-r^6$. 58. $2+x-4x^2-11x^3$.
59. $2+41+8x^2+16x^3+32x^4$. 60. $a^3+9a^2+12a+1-\frac{24a^2-22a-1}{a^3-2a^2+3a-2}$.

Examples 25. Pages 47—48.

1. $2a$. 2. $a-b$. 3. $3x-3y$. 4. 0 . 5. 1 . 6. $12a-b$.
7. $x^2-y^2-z^2$. 8. $2b$. 9. $y-2x-m-n$. 10. $-4x-5y$.
11. a . 12. 0 . 13. $3x+27$. 14. 0 . 15. $\frac{1}{3}x$.
16. $2y-25x$. 17. $7m-12n$. 18. 94 . 19. $1\frac{1}{3}$. 20. x^4-x^2+2x .

Examples 26. Pages 48—49.

1. $a+(b-c-d)$. 2. $a-(b+c+d)$. 3. $a-(b-c-d)$. 4. $x^2-(x+2)$.
5. $x^2+(x-2)$. 6. $x^4-(x^3-x^2+2x-3)$. 7. $x-y+[z-\{w+(a-b)\}]$.

8. $a - b - [c - \{d + (e + f)\}]$. 9. $2x - 3y - [4z - \{5p - (q + r + s)\}]$.
 10. $3x - y - [z + \{l + 2(m + n)\}]$. 11. $x^2 - y^2 - [z^4 + \{yz + (zx + xy + ax)\}]$.
 12. $2x - a - [2y + \{3b + (4c + d)\}]$.
 13. $ax - by - [cz - \{by - (cz + ax - 2cz)\}]$.
 14. $x^2 - lx + [y^2 - \{my - (z^2 - nz + a^2)\}]$.

Examples 27. Pages 49—50.

1. $a(a - b)$. 2. $a(a + 2b)$. 3. $x(2x - 3y)$. 4. $z(3y - 4x)$.
 5. $2b(3a + 2c)$. 6. $3b(3b - 2c)$. 7. $ab(c - 3d)$. 8. $2ab(c + 2b)$.
 9. $4xy(z^2 - 2xy)$. 10. $8xyz(x - y)$. 11. $3lmn(l + m)$.
 12. $11apq(r - 2ap)$. 13. $5a(a - b + 2)$. 14. $4b^3(b^2 - 5ab + 4a^2d)$.
 15. $2ab(ab - 2c^2 + 3bc + 6b^2d)$. 16. $5x^2y(xz - 2z^2 + 3yz - 4y)$.
 17. $3x^5y^6z^7(yz - 3xz^2 + 4x^2y)$. 18. $3a^2bh^2(1 - 2abh + 3a^2h^2 - 4a^4b^4h^3)$.
 19. $x^4y^5(1 - x^5y^6 + 2x^6y^3c)$. 20. $2x^2y^3z^2(az - 2by^3z^2 + cx^2y^3z^3)$.
 21. $4l^2m^2n^2(2mp - 3am^2n^2 + 4hlm^3n^4 - 5n^6)$.

Examples 28. Pages 50—51.

1. $(a + d)(b + c)$. 2. $(a + c)(a + b)$. 3. $(x^2 + a^2)(x + a)$.
 4. $(x - a)(x + a)^2$. 5. $(r^2 + a^2)(x - a)$. 6. $(x - a)^2(x + a)$.
 7. $(x^2 + 1)(a^2 + y^2)$. 8. $2(a - b)(x - 2y)$. 9. $(x + y)(y - z)$.
 10. $(l + a)(l - b)$. 11. $2(p - a)(p + x)$. 12. $(2x - q)(p + y)$.
 13. $(ax + by)(a + b)$. 14. $(ax - by)(a - b)$. 15. $(l + 3m)(2p + q)$.
 16. $x(x + 1)(ax - b)$. 17. $x(b + c)(2ax - b)$. 18. $3x(x - y)(y + a)$.
 19. $4ap(a + b)(p - q)$. 20. $cz(1 - x)(1 - y)$.
 21. $2(p - q)(a + b + c - d)$. 22. $a(a + b)(x - y - z + w)$.
 23. $(a - b)(m + p + q + r)$. 24. $(x + a)(x + y + z)$.

Examples 29 Pages 49—52.

1. $(1 + a)(1 - a)$. 2. $(2a + 3b)(2a - 3b)$. 3. $(4a + 5b)(4a - 5b)$.
 4. $(ab + xy)(ab - xy)$. 5. $a(x + y)(x - y)$. 6. $3(x + 3y)(x - 3y)$.
 7. $a(2b + 3c)(2b - 3c)$. 8. $y(bx + cy)(bx - cy)$. 9. $(al + bm)(al - bm)$.
 10. $ab(l + m)(l - m)$. 11. $3a(2x + 5y)(2x - 5y)$.
 12. $(25a + 13b)(25a - 13b)$. 13. $(x^2 + a^2)(x + a)(x - a)$.
 14. $(p^2q^2 + a^2b^2)(pq + ab)(pq - ab)$.
 15. $(p^3 + q^3)(p^4 + q^4)(p^2 + q^2)(p + q)(p - q)$.

16. $(9a^4 + 16b^4)(3a^3 + 4b^3)(3a^2 - 4b^2)$. 17. $(7x^2 + 1)(7x^2 - 1)$.
 18. $(1 + 4b^2)(1 + 2b)(1^2 - 2b)$. 19. $(x^3 + 2a^2)(x^3 - 2a^2)$.
 20. $(2p^2q^2 + 9)(2p^2q^2 - 9)$. 21. $(7a^2 + 10b)(7a^2 - 10b)$.
 22. $3(16a^4 + 1)(4a^2 + 1)(2a + 1)(2a - 1)$. 23. $(5a^6 + 2)(5a^6 - 2)$.
 24. $(4x^8 + 5y^3)(4x^8 - 5y^3)$. 25. $c(9 + c^2)(3 + c)(3 - c)$.
 26. $(ab^2c + x^3)(ab^2c - x^3)$. 27. $a^2(b^2 + a^2)(b + a)(b - a)$.
 28. $2y(x^2 + 9y^2)(x + 3y)(x - 3y)$. 29. $4ax$. 30. $15(a - b)(a + b)$.
 31. $(a + b)(a - b)(x + 1)(x - 1)$. 32. $(c + d)(c - d)(x + 1)(x - 1)$.
 33. $(a^2 + b^2 - 2)(a + b)(a - b)$. 34. $8ab(a^2 + b^2)$. 35. 720.
 36. 240. 37. 1002000. 38. 7600.
 39. 45000. 40. 99840000.

Examples 30. Pages 52-53.

1. $(a - 1)^2$. 2. $(r + 2)^2$. 3. $(a - 2b)^2$. 4. $(2a - 3b)^2$.
 5. $(4x - 5y)^2$. 6. $(\frac{2}{3}x + ty)^2$. 7. $(3x - 7y)^2$. 8. $(2x + 11)^2$.
 9. $(7ab - 9)^2$. 10. $(5r + 7y)^2$. 11. $2^2(3p - 2)^2$. 12. $(p - 13r)^2$.
 13. $(x^2 + y^2)^2$. 14. $(2x^3 + y^3)^2$. 15. $(3a^2 + 10b^4)^2$. 16. $(5a + 4b^6)^2$.
 17. $(9a^4 - 4b^8)^2$. 18. $(25abc + 3)^2$. 19. $(\frac{1}{2}a - \frac{1}{3}b)^2$. 20. $\frac{2}{3}x - \frac{3}{4}y^2$.
 21. $(\frac{1}{8} - \frac{3}{8}a)^2$. 22. $(1 - \frac{2}{3}a^2b^2c)^2$. 23. $(\frac{1}{2}y - 4x^2)^2$.
 24. $(\frac{2}{3}l - \frac{1}{3}m)^2$. 25. $2(a^3 - 2b)^2$. 26. $2a^2(x^2 + 5)^2$.
 27. $3(a - 11b)^2$. 28. $2(\frac{2}{3}l + 5m)^2$. 29. $5(3 - 2xy^2x^2)^2$.
 30. $m^2(2l - 7n)^2$. 31. $2^2a^3(5 + 5a)^2$. 32. $2ab(a^2 + 16)^2$.
 33. $7b^2c^2(2a - d)^2$. 34. $(x + a + b)^2$. 35. $(r + y - z)^2$.
 36. $(2a - 5b - 5c)^2$. 37. $(a + 1)^4$. 38. $(a - b)^2$. 39. $(a + 2)^4$.
 40. $2(a + d)^2(b + c)^2$. 41. 1. 42. 4. 43. 4. 44. 0009.
 45. 8100. 46. 25. 47. $49k^2$. 48. 1.

Examples 31. Page 56.

1. $(x + 1)(x + 2)$. 2. $(x + 1)(x + 3)$. 3. $(x + 7)(x + 1)$.
 4. $(x + 6)(x + 1)$. 5. $(x + 1)(x + 8)$. 6. $(x + 9)(x + 4)$.
 7. $(x + 7)(x + 6)$. 8. $(x + 2)(x + 8)$. 9. $(x + 6)(x + 4)$.
 10. $(x - 5)(x - 4)$. 11. $(x - 8)(x - 3)$. 12. $(x - 9)(x - 3)$.
 13. $(x - 7)(x - 6)$. 14. $(x - 7)(x - 8)$. 15. $(a - 8)(a - 9)$.
 16. $(x - 2)(x + 1)$. 17. $(x + 2)(x - 1)$. 18. $(c + 2)(c - 3)$.
 19. $(a + 7)(a - 4)$. 20. $(x - 3)(x + 12)$. 21. $(x - 12)(x + 4)$.

22. $(5+x)(8-x)$. 23. $(20+x)(1-x)$. 24. $(9+a)(7+a)$.
 25. $(10-x)(8+x)$. 26. $(12+a)(5-a)$. 27. $(12-y)(9-y)$.
 28. $(x+6y)(x+5y)$. 29. $(x-10y)(x-11y)$. 30. $(b-16a)(b+9a)$.
 31. $(c-8d)(c-15d)$. 32. $(ab-10)(ab-6)$. 33. $(y-15z)(y+12z)$.
 34. $(x+20y)(x+10y)$. 35. $(a+25b)(a-5b)$. 36. $(16+y)(5-y)$.
 37. $(x-a)(x-a-3)$. 38. $(x-a-1)(x-a-2)$.
 39. $(y-a-2)(y-a-3)$. 40. $(y+a+2)(y-a-3)$.
 41. $(x+a+2)(x-a+1)$. 42. $(x+a)(x-a+b)$.
 43. $(x+l+2m)(x-l-m)$. 44. $(y+a+b)(y-a-c)$.
 45. $(y+a+2b)(y-2a-b)$.

Examples 32. Page 59.

1. $(2x+1)(x+1)$. 2. $(x+2)(2x+1)$. 3. $(4x+1)(x+1)$.
 4. $(3a+1)(a+3)$. 5. $(2a+3)(a+2)$. 6. $(2b+1)(b+4)$.
 7. $(3x-1)(2x-1)$. 8. $(3x-2)(2x-3)$. 9. $(5x-4)(2x-3)$.
 10. $(3y-2)(4y-3)$. 11. $(9y-4)(y-3)$. 12. $(8y-9)(y-1)$.
 13. $(2a+1)(a-1)$. 14. $(3x-4)(x+5)$. 15. $(4x-1)(x+3)$.
 16. $(4x+3)(x-1)$. 17. $(11x-1)(10x+1)$. 18. $(7x+3)(x-18)$.
 19. $(2+x)(1-3x)$. 20. $(7-x)(4+x)$. 21. $(6+5x)(5-6x)$.
 22. $(15-2x)(1-5x)$. 23. $(6a-5)(a+10)$. 24. $(4-5b)(3+4b)$.
 25. $(8x-1)(x-8)$. 26. $(8x+1)(3x-4)$. 27. $(8x+7)(8x+9)$.
 28. $(3x+2y)(x+7y)$. 29. $(3x-20y)(x-10y)$. 30. $(9a-8b)(8a-9b)$.
 31. $(6-ab)(3-5ab)$. 32. $(3c-8d)(dc+9d)$.
 33. $(14x+5y)(3x-4y)$. 34. $2(3x-2)(x+7)$.
 35. $3(4x+3)(x+5)$. 36. $5(4x-5)(2x-7)$. 37. $a(7x-3)(x+18)$.
 38. $(7ax-3)(ax+18)$. 39. $ab(a+b)(a-2b)$. 40. $3x(2x+1)(x-3)$.
 41. $(2x^2+3)(x^2+2)$. 42. $(2x^2-7)(x^2-2)$.
 43. $x^2y^2(2x-5y)(7x+3y)$. 44. $3xy(6xy-1)(4xy-5)$.
 45. $(ax-b)(bx-a)$. 46. $c^2(4ab+3)(5ab-1)$.

Examples 33. Pages 62-64.

1. $2ax+2cy$. 2. $2mx^2+n^2x+2$. 3. $ap+2(b-c)q+(3c-a)r+2$.
 4. $(2b-2c-a)m-(a+2b-c)n+(4b-a)mn$.
 5. $(4m-r)ab-(n+r)bc+(5r-m)ca$.
 6. $(2x-y+z)a+(x+6y)b+(5y-x)c$.

7. $(x-4y+4z)a^m+(4y-x)a^mb^n+(2x+y+3z)b^n$.
 8. $(3l+2m+5n)a^{p+q}+(4l-4m+3n)a^{p+q}+(l-3m+3n)a^{r+p}$.
 9. $(2-x-y)pq+(z+1)qr+(x-2y+4z)rp$.
 10. $(l+m)x^mym-(2l+2n)x^m+(3m+3n-2l)y^n$.
 11. $(4a+3b-7c)x^m-(4c-a)x^n$.
 12. $(\frac{1}{2}a^2-c^2)l^2+\frac{1}{3}c^2m^2-(\frac{5}{3}l^2-ab)m^2+\frac{2}{3}$.
 13. $-2a^2x^3y^3-b^2y^3z^3-a^2z^3x^3$.
 14. $x^3-px^2+qx-a^3+a^2p-aq$.
 15. $x^3+(4ab-b^2)x-a^3+2a^2b-3ab^2+6b^3$.
 16. $x^3-(p-2)x^2+(q-2p+2)x^2-(2p-q-2)x^2-(p-2)x+1$.
 17. $(m+n)a^4+(m^2+2mn+n^2-1)a^3+(2m^2n+2mn^2-m-n)a^2$
 $+ (m+n-2mn)a-1$.
 18. $(a-b)x^3-(a^2-ab)x^2-(ab^2-b^3)x+a^2b^2-ab^3$.
 19. $x^3(z-y)+x(y^3-y^2z+y^2z-z^3)+2y^2z-2yz^3$.
 20. $a^3(b-c)+b^3(c-a)+c^3(a-b)$.
 21. $2y^2z^2+2z^2x^2+2x^2y^2-x^4-y^4-z^4$.
 22. 3. 23. -12. 24. -19. 25. -7. 26. $ac-b^2$. 27. 0.
 28. $x^2-bx-cx+bc$; $x^2+ax-bx-ab$.
 29. x^2+bx-a^2+ab ; $x^2+ax-bx-ab$. 30. $a+b$. 31. $b-c$.
 32. $ab-bc-ac+1^2$; $a^2-ab-ac+b^2$.
 33. $x^3-xz-yz+z^2$; $x^3-xy-xz+yz$.
 34. $x^2y-x^2z+xy^2-xyz-y^2z+z^3$; $x^2y-x^2z-xy^2+xz^2+y^2z-yz^2$.
 35. $x^2+y^2+z^2+xy+yz+zx$ 36. $-x^2+xy+xz-yz$
 37. $x-y$. 38. $x^3+5(a-1)x-b$

Examples 34. Pages 66-67.

- | | | | |
|------------------------|------------------------|------------------------|-------------------------|
| 1. $x=6$. | 2. $x=4$. | 3. $x=3$ | 4. $x=5$. |
| 5. $x=5$. | 6. $x=-1$ | 7. $x=5$. | 8. $x=3$. |
| 9. $x=3\frac{1}{2}$ | 10. $x=6\frac{2}{3}$ | 11. $x=-\frac{5}{2}$ | 12. $x=-1\frac{1}{2}$. |
| 13. $x=\frac{1}{2}$. | 14. $x=2\frac{1}{4}$. | 15. $x=\frac{1}{18}$. | 16. $x=3$. |
| 17. $x=\frac{3}{2}$. | 18. $y=4\frac{1}{2}$. | 19. $y=1\frac{1}{2}$. | 20. $y=\frac{1}{12}$. |
| 21. $x=1$. | 22. $x=-\frac{1}{2}$. | 23. $x=1$. | 24. $x=-\frac{1}{4}$. |
| 25. $x=1$. | 26. $x=\frac{1}{4}$. | 27. $x=-13$. | 28. $x=-\frac{3}{2}$. |
| 29. $x=2\frac{1}{2}$. | 30. $x=7$. | 31. $x=13$. | 32. $x=\frac{7}{10}$. |
| 33. $x=-4$. | 34. $x=2$. | 35. $x=\frac{2}{3}$. | 36. $x=0$. |

37. $x=0$. 38. $x=7$. 39. $x=36$. 40. $x=2$.
 41. $x=3$. 42. $x=-2$. 43. $x=1$. 44. $x=-1$.
 45. $x=1\frac{5}{8}$. 46. $x=9\frac{3}{4}$. 47. $x=-1$. 48. $x=-17$.
 49. $x=4\frac{1}{8}$. 50. 3. 51. $\frac{3}{8}$. 52. $m=-\frac{2}{3}$.

Examples 35. *Pages 67—69.*

1. $a-x$. 2. $8-a$. 3. $\frac{3}{2x}$. 4. $\frac{x}{y}$. 5. $16x+11$.
 6. $\frac{a-b}{x}$. 7. $a-8$. 8. $5y$. 9. $\frac{b}{16x}$. 10. $\frac{x}{5}$ pence.
 11. $\frac{20(a-5)}{x}$. 12. $\frac{x}{10} + \frac{a}{80} = 11b$. 13. $\frac{x}{12}$. 14. ap miles
 15. $\frac{x}{24m}$ miles. 16. $2n-1, 2n, 2n+1$.
 17. $8x^3+12x^2-2x-3; 4x^2+4x$. 18. $(x+a)$ and $(x-a)$ years.
 19. $(2x+15)$ years. 20. $2(x-7)$ years. 21. $\frac{w}{x} \cdot \frac{w}{x} + \frac{w}{y}$.
 22. $\frac{mb}{n}$ hrs. 23. $A, \mathcal{L}(x+\frac{1}{2}y-\frac{3}{40}); B, \mathcal{L}(\frac{1}{2}y+\frac{3}{40})$.
 24. $240(x-y)-12x$.

Examples 36. *Pages 70—72.*

1. 11. 2. 8. 3. 4. 4. 180. 5. 25. 6. 80. 7. 240.
 8. 21. 9. 10. 10. 77. 11. 12. 12. Rs. 8. 13. Rs. 180.
 14. Rs. 480 15. $\mathcal{L}1440$ 16. 4 yds. 17. 100 mds.
 18. $\mathcal{L}3\frac{1}{2}$. 19. Rs 90. 20. $\mathcal{L}10$. 21. 8 srs.
 22. 4 lbs. 23. 20. 24. Rs. 30. 25. $\mathcal{L}1080$.
 26. 60. 27. 20. 28. 800.

Examples 37. *Pages 73—75.*

1. 48; 12. 2. 50; 38. 3. 35; 15. 4. 45; 46.
 5. 25; 45. 6. 28; 70. 7. 20; 12. 8. 20; 25.
 9. 60; 100 10. 51; 42. 11. Rs. 85, A 's; Rs. 115, B 's.
 12. $\mathcal{L}14-A; \mathcal{L}12-B; \mathcal{L}4-C$. 13. $A-\mathcal{L}26, B-\mathcal{L}20, C-\mathcal{L}12$.
 14. $A-\text{Rs. } 64, B-\text{Rs. } 54, C-\text{Rs. } 60$.
 15. $\mathcal{L}375-A, \mathcal{L}250-B, \mathcal{L}125-C$.

16. $A = \text{£}31$, $B = \text{£}51$, $C = \text{£}87$. 17. Each child $\text{£}10$, each man $\text{£}20$.
 18. Each child $\text{£}5$, each man $\text{£}14$. 10s.
 19. Each girl $\text{£}3$ 15s., each boy $\text{£}15$.
 20. Each girl Rs. 2. 12 as., each woman Rs. 6. 12 as., each man Rs. 13. 8 as.
 21. Each man Rs. 16, each woman Rs. 8, each girl Rs. 2.
 22. $\text{£}55 - A$, $\text{£}85 - B$, $\text{£}35 - C$.
 23. $\text{£}80$ —eldest, $\text{£}120$ —middle, 160 —youngest.
 24. 16 men, 8 women. 25 Rs. 35— A , Rs. 45— B , Rs. 55— C .
 26. Rs. 60— A , Rs. 40— B . 27 Son—8 yrs., father—24 yrs.
 28. 30 years—father, 10 yrs—son 29 6 yrs—son, 36 yrs.—father.
 30. 15 yrs.—son, 31 yrs—father. 31. 8 pence and 16 oranges.
 32. 320 lbs at 1s. 3d., 200 lbs. at 1s 6d. 33. 42 hens and 23 geese.
 34. 20 lbs of the first kind, and 25 lbs of the 2nd kind.
 35. 14 yards of the first set and 11 of the 2nd.
 36. A for 5 days and B for 2 days. 37. 4 seers of water.
 38. 24 days ; 16 days. 39. $\text{£}52$ in one and $\text{£}2$. 12s. in the other.
 40. 58—Rs., 42—two-anna pieces.

Miscellaneous Examples 1. Pages 76—81.

1. 0. 2. $6x^4 - 4x^3 - 7x^2 - 26x + 21$. 3. $a - 2b + 6c - 24d + 24e$.
 4. $-7x^6 + 11x^5 - 16x^4 + 7x^3 - 7x^2 + 5x + 21$.
 5. $2x^6 - 9x^5 + 20x^4 - 37x^3 + 44x^2 - 34x + 24$.
 6. $8a^6 + 4a^5 + 12a^4 - 8a^3 + 24a - 32$. 7. $2a^3 - a^2 + 3a - 4$. 8. 11.
 9. 0. 10. 11. -1. 12. $mx^4 + nx^3 + 2mx^2 + 2nx$. 13. $8x + 2y$.
 14. $x^2(b^2 + c^2) + y^2(c^2 - a^2)$ 15. $1 - a + 3a^2 - 2a^3 - 4a^4 + a^5 - 12a^6$.
 16. $x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc$; $x^3 - 6x^2 + 12x - 8$.
 17. $x^6 + 2ax^5 + 2a^2x^4 - 4a^4x^3 - 8a^5x - 8a^6$.
 18. $(p + 8q)(p - 9q)$; $(3a + 4b)(4a - 5b)$. 19. $-\frac{1}{3}$. 20. 7 srs.
 21. 8 22. $(n^2 + 1)a^2 + (n^2 + 2)ab + (n^2 + 1)b^2$.
 23. $(a - 2b + c)xy + (2a - b - c)y^2$. 24. $-29x + 28y + 12g$.
 25. $12(a^2 + 1)$. 26. $x^3 - 2x^2y + 2xy^2 - y^3$. 27. $-2ab^3$.
 28. $(3a + 3b + 4)(3a + 3b - 4)$; $(x - 16)(x - 18)$; $a^2(a + 1)(a - 1)(a^2 + 1)$;
 $(5y - 3)(y - 7)$ 29. $x = 5\frac{2}{3}$; $x = -\frac{2}{3}$. 30. 15 years before.
 31. $-\frac{7}{8}$. 32. $\frac{7}{2}x^3 - \frac{3}{2}x^2y + 2xy^2 - \frac{1}{2}y^3$. 33. $(a + b)^2$.

34. $x^3 + y^3 + 1 - 3xy$. 35. $a^4 - ab^3 + b^4$. 36. $-4a(b+c)$.
 37. $a = 18$. 38. (1). $-(a^2 + b^2)(a+b)(b-a)$. (2). $ab^3(5-9ab)(15+ab)$
 40. 25 sovereigns and 100 crowns. 41. $\sqrt{(21)}$.
 42. $-3cx^3 + ax^3 + cx - b - c$. 43. $15a^4 - 38a^3b + 50a^2b^2 - 51ab^3 + 8b^4$.
 45. $30x^3y^3 - 497bxy + 20a^2b^2$.
 46. $a^8 - 2a^7 + a^6 + a^5 - 4a^4 + 2a^3 + a^2 - 2a + 1$.
 47. (1) $(10y+7)(5y-4)$; (2) $(25a+6)(a-2)$;
 (3) $(5x^2-7)(x+1)(x-1)$.
 48. (1) $x=3$; (2) $x=10$. 49. 93. 50. 120 mds. 51. 35.
 52. $-46a + 57b - 9c$. 53. $-2x^3 + (1-m)x^2 + (1-n)x - 1$.
 54. 0. 55. $x^3 + y^3 - 2x^2 - xy - 2y^2 + 2x + 2y - 1$.
 56. $x^3 + bx - 2ax + b^3 + 3a^3$. 57. $x^6 - 4x^4 + 2x^3 - 2x^2 - x + 8$.
 59. (1) $(3x-5)(7x+6)$; (2) $3x(x+3)(x-3)$; (3) $4x^3(x-8)(x-7)$
 60. $\frac{xz}{192y}$ shillings. 61. $\frac{7}{7+4a}$. 62. 13.
 63. $(a+b+c)x^4 - (a+b+c)x^3 + (a+b+c)x^2 + dx + c$;
 $(a+b+c-d)x + a+b+c-e$
 64. $8a^3 - 24ax - x^3 + 6x^2 + 12x - 8$. 65. $2x^2 + \frac{4}{3}ax - \frac{1}{3}a^2$. 66. -3 .
 67. $2ab$. 68. $27-9a+9b-3x^2+3b^2$ a^3+b^3 . 70. $2b-a$
 71. 9. 72. $x-1$. 73. $-a^2 - 7a^2 - 7a^3 + 4b - 4(a+7ab)$.
 74. $\frac{a^3}{256} - \frac{b^3}{6561}$. 75. $x^7 + 2x^4 - 8x^2 - 16x$. 77. $1-2m+1$.
 78. (1) $x(3+5x)(7-4x)$; (2) $(5a+b)(5a-b)(a+3b)(a-3b)$;
 (3) $(l+m)(l-m)(n+r)(n-r)$.
 79. $x=0$. 80. 10 lbs. 81. 9.
 82. $x^3 + xy^2 + 3y^3$; $-3y^3$. 83. $2a^3b^3$. 84. $m^3 + m^4n^4 + n^6$
 85. $y+z+2$. 86. x^2y . 88. (1) $(x-18)(x+17)$;
 (2) $(5x-4)(5x-17)$; (3) $(1+10x)(3-2x)$; (4) $5m(l+m)$.
 89. $\frac{1}{2}(ax^6 + bxy^4 - cx^3y^2 - dx^2y^3 + exy^4 + fy^6)$. 90. $ar+c=b(x-x-y)$
 91. $5\frac{1}{2}; -1$. 92. $-a^4d^2 + 6a^3bcd - 2a^2c^3 - 3a^2b^2c^2 - 4a^2b^3d$
 $+ 6ab^4c - 2d^4$
 93. $alx^6 + (am+bl)x^5 + (an+bm+cl)x^4 + (ra+nb+mc+ld)x^3 + (r$
 $+n+md)x^2 + (rc+nd)x + rd$; $9x^6 + 69x^5 + 100x^4 - 7x^3$
 $- 85x^2 + 76x - 1$.
 94. $15a - \frac{7}{2}b + \frac{1}{3}c$ 95. $2x^3 - xy - 3y^3$.

96. $1 - 2x^3 + x^{10}$; 7. 97. $4(ap + bq + cr)$.
 98. $(7a - 11b)^2$; $(x + \frac{1}{2})^2$; $(x + \frac{1}{2})(a + \frac{2}{3})$; $-a(a - b)^2$.
 99. $2a - 5b$. 100. 1770 101. $(x - \frac{3}{2}y)(x^2 + y^2)$.
 102. $3x(x - 7)(x + 9)$. 103. $1 - 2x + 3x^2 - 4x^3$.
 104. $x^4 + (a - 1)x^3 - (2a + 1)x^2 + (a^2 + 4a - 5)x + 3a + 6$. 107. $a = 4$.
 108. $x = \frac{1}{2}$. 109. Diminution of area $= (a + 2)$ sq. ft. 110. 24.

Examples 38. Pages 83—84.

- $15x^2 + 40xy + 25y^2$; $49x^2 + 112xy + 64y^2$.
- $9x^2 - 42xy + 49y^2$; $36a^2 - 96ab + 64b^2$.
- $x^2 + y^2 + z^2 - 2xy - 2yz + 2zx$; $x^2 + y^2 + z^2 - 2xy + 2yz - 2zx$.
- $a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2ad - 2bc + 2bd - 2cd$;
 $a^2 + b^2 + c^2 + d^2 - 2ab - 2ac - 2ad + 2bc + 2bd + 2cd$.
- $a^2 + b^2 + c^2 + d^2 + e^2 + 2ab - 2ac - 2ad + 2ae - 2bc - 2bd$
 $+ 2be + 2cd - 2ce - 2de$.
- $a^2 + b^2 + c^2 + d^2 + e^2 + 2ab - 2ac + 2ad - 2ae + 2bc - 2bd + 2be - 2cd$
 $+ 2ce - 2de$.
- $4x^2 + 9y^2 + 25z^2 + w^2 - 12xy + 20xz + 4xw - 30yz - 6yw + 10zw$.
- $4x^2 + 11y^2 + 25z^2 + w^2 - 16xy - 20xz - 4xw + 40yz + 8yw + 10zw$.
- $a^6 + 2a^5b + 3a^4b^2 + 4a^3b^3 + 3a^2b^4 + 2ab^5 + b^6$.
- $a^6 + 2a^5b - a^4b^2 - 4a^3b^3 - a^2b^4 + 2ab^5 + b^6$.
- $x^4 + 2x^2y^2 + y^4 + 4ax(x^2 + y^2) + 4by(x^2 + y^2) + 4a^2x^2 + 8abxy + 4b^2y^2$.
- $x^4 + 2x^2y^2 + y^4 - 4ax(x^2 + y^2) - 4by(x^2 + y^2) + 4a^2x^2 + 8abxy + 4b^2y^2$.
- $x^4 + 8ax^3 + 2(11a^2 + 2y^2)x^2 + 8a(3a^2 + 2y^2)x + 9a^4 + 12a^2y^2 + 4y^4$.
- $x^4 - 8ax^3 + 2(5a^2 - y^2)x^2 + 8a(3a^2 + 2y^2)x + 9a^4 + 12a^2y^2 + 4y^4$.
- $\frac{4}{3}x^3 + \frac{1}{2}y^3 + \frac{1}{4}z^3 - \frac{4}{3}xy - \frac{1}{2}yz + \frac{3}{4}zx$.
- $\frac{1}{4}a^4x^4 - 3a^3bx^3y + \frac{9}{4}a^2b^2x^2y^2 - 4ab^3xy^3 + \frac{1}{4}b^4y^4$.
- $a^4 + b^4 + c^4 - 2a^3(b + c) - 2b^3(c + a) - 2c^3(a + b) + 3(a^2b^3 + b^2c^3 + c^2a^3)$.
- $a^4 + b^4 + c^4 - 2a^3(b + c) - 2b^3(c + a) - 2c^3(a + b) - a^2b^3 - b^2c^3 + 3c^2a^3$
 $+ 4a^2bc - 4abc^2$.
- $x^5 + 4x^3y^2 + 6x^4y^4 + 4x^2y^6 + y^5 - 4a^2(x^6 + 3x^4y^2 + 3x^2y^4 + y^6)$
 $+ 6a^4(x^4 + 2x^2y^2 + y^4) - 4a^6(x^2 + y^2) + a^8$.
- $4(a^2 + b^3 + c^2 + d^2)$. 21. $2(x^2 + y^2 + z^2)$. 22. c
- $-(a^4b^4 + b^4c^4 + c^4a^4)$.

Examples 39. *Pages 85—86.*

1. $(a-b)(a+b)(a^2+b^2)$; $(a-b)(a+b)(a^2+b^2)(a^4+b^4)$.
2. $(ax-y)(ax+y)(a^2x^2+y^2)(a^4x^4+y^4)$;
 $(ab-cd)(ab+cd)(a^2b^2+c^2d^2)(a^4b^4+c^4d^4)$.
3. $(a-2)(a+2)(a^2+4)$; $(x-3)(x+3)(x^2+9)(x^4+81)$.
4. $(4-a)(4+a)(16+a^2)$; $(1-a)(1+a)(1+a^2)(1+a^4)(1+a^8)$.
5. $(2x-3a)(2x+3a)(4x^2+9a^2)$; $2(a-2b)(a+2b)(a^2+4b^2)$.
6. $a(ab-5c)(ab+5c)(a^2b^2+25c^2)$; $2b^2(5a-2c)(5a+2c)$.
7. $(a+b+c)(a+b-c)$. 8. $(a-b+1)(a-b-1)$.
9. $(a-c)(a+2b+c)$. 10. $(a+b-c)(a-b+c)\{a^2+(b-c)^2\}$.
11. $(6+5p+6q)(6-5p-6q)$ 12. $(6+x^2)(2+x)(2-x)$.
13. $(2a+3b-12c)(2a-3b+12c)$ 14. $(5x^2+9y^2+4z^2)(5x^2+y^2-4z^2)$.
15. $(1-x)^2(1+x)^2$. 16. $(a-b)^2(a+b)^2$.
17. $(a-2b+1)(a-2b-1)$. 18. $(a-b)(a+b+c)$.
19. $9(a-b)(a+b)$. 20. $11(a-b)(a+b)(a^2+b^2)$.
21. $5(ax-by)(ax+by)$. 22. $(a-b)(a+b)(x-y)(x+y)$.
23. $(b-p)(b+p)(a-q)(a+q)$. 24. $(1-y+x)(1-y-x)$.
25. $(2l-m+n)(2l-m-n)$. 26. $(ab+b+1)(ab-b+1)$.
27. $(6+y-z)(6-y+z)$. 28. $(ac+ab-1)(ac-ab+1)$.
29. $(2a-3b+2)(2a-3b-2)$ 30. $(p-3q+4r)(p-3q-4r)$.
31. $(5p+6q+5r)(5p-6q-5r)$. 32. $(a+b+c+d)(a+b-c-d)$.
33. $(a+b-2x+3y)(a-b-2x-3y)$ 34. $(x-y+z-1)(x-y-z-1)$.
35. $(a+b)(a-b-2)$. 36. $(a+c)(a-2b-c)$.
37. $(1-a+b+c)(1+a-b+c)$.
38. $(a-1)(a+1)(a^2+1)(x-1)(x+1)(x^2+1)$.
39. $(a^2-2ab+2b^2)(a^2+2ab+2b^2)$; $(x^2-4x+8)(x^2+4x+8)$.
40. $(x^2-6x+18)(x^2+6x+18)$; $(1-a+a^2)(1+a+a^2)$;
 $3(2a^2-1)(2a^2+1)(2a^2-2a+1)(2a^2+2a+1)$.
41. 28899. 42. 249879. 43. 2755375. 44. 3097519.

Examples 40. *Page 88.*

1. x^2-5x+6 . 2. $x^2+7x-44$. 3. $x^2+13x+42$.
4. $x^2+x-132$. 5. $a^2-6ab-40b^2$. 6. $a^2-6ab-91b^2$.
7. $4x^2+28x+45$. 8. $4y^2+4y-99$. 9. $4z^2-10z-50$.

10. $9x^3 - 3xy - 20y^3$. 11. $121x^2 - 44xy - 117y^2$.
 12. $49a^3 + 217ab + 240b^2$. 13. $x^3 + 6x^2 + 14x + 6$.
 14. $x^3 + 7x^2 - 14x - 120$. 15. $y^3 + 2y^2 - 85y + 154$.
 16. $x^3 - 3x^2 - 6a^2x + 8a^3$. 17. $x^3 - 3b^2x + 2b^3$.
 18. $8x^3 - 122x - 180$. 19. $16x^3 - 57x + 14$.
 20. $9x^3 - 9x + 10$. 21. $12 - 4a - 3b + 3ab$. 22. $3b^3(a^2 + 3b^3)$. 23. $4c^3$.

• Examples 41. Page 90.

1. $8x^3 + 36x^2y + 54xy^2 + 27y^3$. 2. $61a^3 - 48a^2b + 12ab^2 - b^3$.
 3. $27a^3 + 54a^2b + 36ab^2 + 8b^3$. 4. $216a^3 - 540a^2b + 450ab^2 - 125b^3$.
 5. $125a^3b^3 - 150a^2b^2 + 60ab - 8$. 6. $8a^3b^3 + 12a^2b^2c + 6abc^2 + c^3$.
 7. $27a^3b^3 + 54a^2b^2 + 36ab + 8$.
 8. $543a^3b^3 - 588a^2b^2cd + 356abc^2d^2 - 64c^3d^3$.
 9. $a^3 + 3a^2b + 6a^2c + 3ab^2 + 12abc + 12ac^2 + b^3 + 6b^2c + 12bc^2 + 8c^3$.
 10. $9a^3 - 12a^2b - 12a^2c + 6ab^2 + 12abc + 6ac^2 - b^3 - 3b^2c - 3bc^2 - c^3$.
 11. $27x^3 - 27x^2y + 27x^2z + 9xy^2 - 18xyz + 9xz^2 - y^3 + 3y^2z - 3yz^2 + z^3$.
 12. $a^3x^3 - 3a^2bx^2y + 3a^2cx^2z + 3ab^2xy^2 - 6abctxyz + 3ac^2xz^2 - b^3y^3 + 3b^2cy^2z - 3bc^2yz^2 + c^3z^3$. 13. $27a^3$. 14. $8a^3$. 15. $(a+b)^3$.
 16. $z\{a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(c+a) + 3c^2(a+b) - 6abc\}$.
 17. $Quo = 3(y+z)(z+x)$. 18. $Quo = 5(2x+y)(2x-z)$. 20. 91.

Examples 42. Page 92.

1. $(4a+5)(16a^2 - 20a + 25)$. 2. $(2a-7b)(4a^2 + 14ab + 49b^2)$.
 3. $(ax+6by)(a^2x^2 - 6abxy + 36b^2y^2)$.
 4. $(x^2 + 2z^2)(x^2y^4 - 2xy^2z^2 + 4z^6)$.
 5. $(4x-3y-2z)(16x^2 - 24xy + 8xz + 9y^2 - 6yz + 4z^2)$.
 6. $(4-7x+4y)\{16 + 4(7x-4y) + (7x-4y)^2\}$.
 7. $(x-1)(x-2)(x^4 + 3x^3 + 13x^2 + 11x + 4)$.
 8. $(x+1)(x+3)(x^4 - 4x^3 + 22x^2 - 12x + 9)$.
 9. $(x+2)(x+3)(x^4 + 10x^3 + 19x^2 - 30x + 36)$.
 10. $(2x-1)(x-3)(4x^4 + 14x^3 + 61x^2 + 21x + 9)$.
 11. $(3x+2)(2x-3)(36x^4 + 30x^3 - 47x^2 - 30x + 36)$.
 12. $(x+a-b)(x^3 + 2ax + bx + a^3 + ab + b^3)$.
 13. $(x-a)(x^3 + 4ax + 7a^2)$. 14. $(x-2)(x^2 + 5x + 13)$.
 15. $(x-4)(x^3 - 2x + 4)$. 16. $2(2x+1)(2x^2 - x + 1)$.

17. $(x-y)^2(x^4+2x^2y+6x^2y^2+2xy^3+y^4)$.
 18. $(a+2b)(a^3-2ab+4b^2)(a^6-8a^3b^3+64b^6)$
 19. $(a-1)(a-2)(a^2+a+1)(a^2+2a+4)$.
 20. $(a-1)(a-2)(a+2)(a^2+a+1)(a^2+2a+4)(a^2-2a+4)$.
 21. $a^3+5ab+13b^2$ 22. $9a^4-3a^2b+b^2$
 23. $x^3-4xy+7y^2$ 24. $(a^4b^4+4a^2b^2c^2+16c^4)(ab+2c)$.
 25. $(a+b)(a^6-a^2b^3+b^6)$ 26. $(a^2-ab+b^2)^2$
 27. x^2+xy+y^2 28. $4a^2+2ab+b^2$
 29. $27(7x^2+13xy+7y^2)$, 30. $2(31x^2-56xy+56xz+64y^2-48yz+36z^2)$.

Examples 43. Pages 94-95

1. $(a+b-c)(a^2+b^2+c^2-ab-bc-ca)$.
 2. $(a-b-c)(a^2+b^2+c^2+ab-bc+ca)$
 3. $(x+y+1)(x^2+xy+y^2-x-y+1)$
 4. $(x-y+1)(x^2+xy+y^2-x+y+1)$
 5. $(x-y-1)(x^2+xy+y^2+x-y+1)$
 6. $(x+2y+3z)(x^2+4y^2+9z^2-2xy-6yz-3zx)$.
 7. $(2x-3y-4z)(4x^2+9y^2+16z^2+6xz-12yz+8zx)$.
 8. $(1-a+2b)(1+a-2b+a^2+2ab+4b^2)$.
 9. $(a^2+b^2-c^2)(a^4+b^4+c^4-a^2b^2+b^2c^2+c^2a^2)$.
 10. $(ab+a+1)(a^2b^2-a^2b+a^2-ab-a+1)$
 11. $(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2-a^2b-ab^2-b^2c-ca^2-ab^2)$.
 12. $(ab-bc+1)(a^2b^2+ab^2c-ac^2+b^2c^2+bc+1)$
 13. $(a-3b-2)(a^2+5ab+9b^2+2a-6b+4)$.
 14. $(a^2+a-1)(a^4-a^3+2a^2+a+1)$.
 15. $(x^2+2x-4)(x^4-2x^3+8x^2+8x+16)$.
 16. $a^3(1-2b+2bc)(1+2b+4b^2-2bc+4b^2c+4b^3c^2)$.
 21. $a^4-2a^3b+3a^2b^2+a^2c^2-2ab^3-4ab^2c+b^4+b^3c^2+c^4$.
 23. 1. 24. 0

Examples 44. Page 95.

5. $(a+b)(b-c)(a-c)$. 6. $(a-b)(b+c)(c-a)$.
 7. $(a+b)(a-b)(c+a)(c-a)(b^2+c^2)$. 8. $(x+2y)(2y+z)(z+x)$.
 9. $(x+y)(y+3z)(3z+x)$. 10. $(x+2y)(2y+3z)(3z+x)$.

Examples 46. Pages 97-98.

1. $(b-c)(c-a)(a-b)$. 2. $(a-b)(b-c)(a-c)(a+b)(b+c)(c+a)$.
3. $(a-b)(b-c)(c-a)(a+b)(b+c)(c+a)$.
4. $(a-b)(b-c)(a-c)(a+b)(b+c)(c+a)$.
5. $(a-b)(b-c)(a-c)(a+b+c)$. 6. $(a-b)(b-c)(c-a)(a+b+c)$.
7. $(a-b)(b-c)(a-c)$. 8. $(a-b)(b-c)(c-a)$. 9. $(a-b)(b-c)(a-c)$.
10. $n(a-b)(b-c)(c-a)$ 11. $(a-b)(b-c)(a-c)$.
12. $(a-b)(b-c)(a-c)$ 13. $(a-b)(b-c)(a-c)(a+b+c)$.

Examples 47. Page 99.

1. $(a^2+a+1)(a^2-a+1)$. 2. $(a^2+2ab+4b^2)(a^2-2ab+4b^2)$.
3. $(4x^2+2xy+y^2)(4x^2-2xy+y^2)$. 4. $(a^2+3a+9)(a^2-3a+9)$.
5. $(a^2+3ab+9b^2)(a^2-3ab+9b^2)$. 6. $(4x^2+6xy+9y^2)(4x^2-6xy+9y^2)$.
7. $(x^2+5xy+5y^2)(x^2-5xy+5y^2)$.
8. $(4x^2+10xy+25y^2)(4x^2-10xy+25y^2)$.
9. $\{(x+y)^2+(x+y)+1\}\{(x+y)^2-(x+y)+1\}$.
10. $\{16(a-b)^2+4(a-b)+1\}\{16(a-b)^2-4(a-b)+1\}$.
11. $(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$.
12. $(a^2+a+1)(a^2-a+1)(a^4-a^2+1)$.
13. $(4a^2+2a+1)(4a^2-2a+1)(6a^4-4a^2+1)$.
14. $(x^2+3xy+9y^2)(x^2-3xy+9y^2)(x^4-9x^2y^2+81y^4)$.
15. $a^2+b^2+c^2-2ab+ac-bc$. 16. a^2-a+1 .
17. $x^2-3xy+3y^2$. 18. $x^6+2x^3y^3+4y^6$. 19. 4

Examples 48. Page 100.

1. $(x-1)(x-2)(x+3)$. 2. $(x+1)(x+2)(x-3)$. 3. $(x-2)^2(x+4)$.
4. $(x-1)(x-3)(x+4)$. 5. $(x-2)(x-3)(x+5)$. 6. $(x-a)(x^2+ax-b^2)$.
7. $(x-1)^2(2x+1)$. 8. $(x-1)(x-2)(3x+2)$. 9. $(x+1)(x-2)^2$.
10. $(x+1)(x^2+4)$ 11. $(x-2)(x^2+1)$. 12. $(x+1)(2x^2-x+3)$.
13. $(y-2)(y+1)(y+4)$. 14. $(x-1)(x+1)(x^2+x+1)$.
15. $(x+2)^2(x-2)(x-3)$. 16. $(x+a)(x-a)(x^2-3)$.
17. $(x-2)^2(x+2)(x^2+5)$. 18. $(x+1)(x-1)(x-2)(x^2+2x+4)$.
19. $(x-1)(x^2-3)(x^2+3)$. 20. $a(2a-b)(a^2-ab+b^2)$.
21. $(1+x+y)(1+x-y)(1-x+y)(1-x-y)$.
22. $(x+y)^2(x-y)^2$. 23. $(a+1)^2(a-1)^2$.

24. $(1-a)(1-b)(1+a)(1+b)(1+a+b-ab)(1-a-b-ab)$.
 25. $(a+d)(b+c)(a-d)(b-c)(ab-cd+ac+bd)(ab-cd-ac-bd)$.
 26. $(a+2)(a+3)(a+5)$; $(x-1)(x-2)(x^2-x+2)$.
 27. $(x+a)^2(x^2+5a^2)$. 28. $(x-1)^2(y+1)^2$.

Examples 49. Pages 109—112.

8. $(x+1)^2(x-3)^2(x+2)^2$. 9. $(x-1)^2(x+1)^2(2x-1)^4(2x+1)^4$.
 10. $(x+1)(x+2)(x+5)(x-2)$. 11. $(x^2+2x+3)(x+1)^2$.
 12. $(x-1)(x+2)(x-2)(x+3)$. 13. $(x+y-7)(x+1)$
 14. $(x+2y+2)(x-2y+3)$. 15. $(2x+3y-2)(y+2)$.
 16. $(x+y+2)(x+y+4)$ 17. $(x-1)(x+2y+c)$.
 18. $(a-2b+3c)(a+b+c)$. 19. $(2a-b+c)(a+b)$.
 20. $(x+y-b)(x-ay+b)$. 21. $(x+ay+b)x+cy+d$.
 22. $(2x+3)(3x+2)(4x+1)$; $2^5 \times 2^3 \times 4^1$.
 23. (1) $(a-b)(b+c)(c+a)$; (2) $(a+b)(b-c)(c+a)$;
 (3) $(b-c)(b+c)(c^2+a^2)(a^2+b^2)$;
 (4) $(b-a)(b+c)(c+a)(a^2+ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)$
 24. (1) $(a-b)(b+c)(c-a)$;
 (2) $(a-b)(c-b)(c+a)(a^2+ab+b^2)(b^2+bc+c^2)(c^2-ca+a^2)$;
 (3) $(a-b)(a+b)(c-b)(c+b)(c^2+a^2)$
 27. $3(a-b)(b-c)(c-a)(a+1)(b+1)(c+1)$.
 28. $3abc(bz-cy)(cx-az)(ay-bx)$.
 29. $(x^2+2x+4)(x^2-2x+4)$; $(a^2+4a+7)(a^2+3)$.
 30. $2(y-1)(3x^2+y^2+4y+4)$ 31. 0. 32. 0.
 33. $3(2x+2y-z)(2x-y-z)(2x-y+2z)$.
 36. $3(a+c)(b+2c+a)(c+2a-b)$.
 37. $2(x^4+y^4+z^4)-4x^2(y+z)-4y^2(z+x)-4z^2(x+y)$
 $+6(x^2y^2+y^2z^2+z^2x^2)$ 39. $-4(b-c)(c-a)(a-b)$.
 40. $a^3(b-c)+b^3(c-a)+c^3(a-b)$. 49. $(x^2+5x+5)^2-1^2$.
 50. $(x^2+7ax+16a^2)^2-(7ax+17a^2)^2$ 51. $(x^2+8x+2)^2-(3x+8)^2$.
 52. $(x^2+ax-a^2)^2-(5a^2)^2$
 58. (i) $(ax+by)^2+(bx-ay)^2$; (ii) $(a^2+ab+b^2)^2+(a^2-ab+b^2)^2$.
 60. $(x-z)^2+(z-1)^2+(x-1)^2$.

Examples 50. Pages 113—114.

1. $x^2; xy^2; x^2y^2$.
2. $a^3b^2c^3; a^4b^4c^3; a^2b^3$.
3. $2a^2x^3y^2; a^3c^2; 2l^2m^3$.
4. $4a^2x^2; 3x^2; 5x^4y^4$.
5. $12a^5b^3c^4d^4; 12a^4x^4y^3$.
6. $ab; a^3b^3c^2$.
7. $xyz; l^3m^2$.
8. $2a^2x^3; a^4b^3$.
9. $x^3y^4c^5; 3x^3y^3z^3$.
10. $12a^2b^3p^3q^2$.
11. $8a^3b^2x^3$.
12. $x^2y^2z^2$.
13. $2a^2b^3l^2m^2$.
14. $7a^2bc^2d$.
15. $2lm$.

Examples 51. Pages 114—116.

1. $a-b; b-a$.
2. $a-b; a+b$.
3. $x+1; a+x$.
4. $x^2-y^2; 3x-2y$.
5. $ab; ab$.
6. $a-2b; 1-2x$.
7. $2a+x; 3x+2y$.
8. $2a-b; 3a+b$.
9. $a-1; 3a+2b$.
10. $x-2; ax+1$.
11. $x(x+5); x(x-5)$.
12. $x+y; y-3$.
13. $x+2y; 3a-2b$.
14. $4(ax+by); -x^2$.
15. $3(x^2+y^2); y(y-1)$.
16. $ab(a-b); 3a(3a+2b)$.
17. $a+b; a^3(a+1)$.
18. $8(a-b)^2; 8(a^3+ab+b^3)$.
19. $(x-2y)^2; x^2(x-z)^2$.
20. $y^4(x-y); x^2(x+7)$.
21. $x^2(x-5); x^2(x-4)$.
22. $xy(x-2y)$.
23. $x+1$.
24. $ab(a-b)$.
25. $a(2a-3b)$.
26. $7ax(2a-x)$.
27. $2a^2x^2(2a-3x)$.
28. $(y-1)(y+2); y^2-y+1$.
29. $a+b$.
30. $1+x^2; 1+x+x^2$.
31. $x-1; 1+l+l^2$.
32. $(2x-1)(3x-1); 1-x^2$.
33. $a-b; 1+a$.
34. $a-b; l^2+m^2$.
35. $m-1$.
36. $a+2b$.
37. $x-1$.
38. $(a+1)(b+1)$.
39. $a(a-2)$.
40. $3m^3(l-m)$.
41. $4a^2-2a+1$.
42. $x-a$.
43. $(y+z)(y-z)^2$.
44. $a+b-c$.
45. a^2+b .
46. $a+b-1$.

Examples 52. Pages 119—120.

1. $x-1; 2x-1$.
2. $3x+2; 4x+5$.
3. $x+1; x+7$.
4. $x-7$.
5. $2(x+2y)$.
6. $4x^2-1$.
7. $x-2a$.
8. $x(2a-3x)$.
9. x^2-2x+5 .
10. $x(x-3)$.
11. $x-1$.
12. $x(x-4)$.
13. $2(x-2y)$.
14. x^3-3 .
15. $2x-5$.
16. $x-2y$.
17. x^2+x+1 .
18. $2x^2+5x-3$.
19. x^2+2x-1 .
20. x^2+2x-1 .
21. x^2-3x .
22. $3x-7$.
23. $2x^2-5x+1$.
24. $2a^2b^3(2a-3b)$.

25. $a-1$; $a-1$. 26. $x-2$. 27. $2x^2-2x+1$
 28. $2x^6-10x^5+16x^4-8x^3$ 29. $2x-3y$.
 30. $x-3$. 31. $2x^2-3x+5$. 32. $6x^3+2x-5$.
 33. $x^2-2xy+4y^2$ 34. $x^3-2x^2y+3xy^2-4y^3$.
 35. $x+1$ 36. $ab-2$ 37. $x-a$.
 38. $mx-1$. 39. $r-b$. 40. $(x-1)^2$.
 41. $ax+a+1$. 42. $(a+1)x+a-1$.

Examples 53. Page 121

1. $r-2$. 2. $3x^2+2x+1$ 3. $x+2$.
 4. $2x^2-3x+1$. 5. $x-3$. 6. $x-1$
 7. $x-1$. 8. r^2-r-12 . 9. $a-b$ 10. $x^2-3xy+2y^2$.

Examples 54. Page 126

1. $x-4$ 2. $r-1$ 3. 3 ; $3x+1$. 4. 3 ; $x-1$.
 5. 3 , $x+3$ 6. 4 ; $x-4$ 7. 10 ; $10x+8$
 8. $(p^2+q^2)(p-q)=(p+q)^2$.

Examples 55. Pages 128—129

1. x^4y^4 2. $x^4y^6z^3$ 3. $18a^3b^4$
 4. $21a^3b^4c^4$. 5. $12a^2b^3c^3d^2$ 6. $20a^3b^2x^3y^2$.
 7. $150a^4b^4c^4x^8$ 8. $240p^3q^3r^4$. 9. $30x^3y^4z^4$.
 10. $72x^5y^6z^7$ 11. $x^3j^3z^3$ 12. $15a^3b^3c$.
 13. $210a^4b^3c^5y^3$. 14. $120a^3b^2x^5y^5$. 15. $156x^3x^3y^3$.
 16. $7200a^3b^3c^3d^2$. 17. $288a^2b^2c^3l^6m^6n^6$. 18. $10a^3b^3r^3$
 19. $5x^4y^4z^4$. 20. $10a^6b^6c^6d^2$. 21. $b(a^2-b^2)$
 22. $(a-b)(a+b)^2$. 23. $ab(a-b)$ 24. $a^2x^3(a-x)$
 25. $x^{11}-x^9$. 26. $xy(x^2-y^3)$. 27. a^6-b^6 .
 28. $4a^4(9a^2-1)$. 29. $72a^3b^3(a+b)(a-b)^2$.
 30. $6a^2(1+x)(1-x^3)$ 31. $84a^4x^4(a^2-x^2)^2$.
 32. $12(a^2-b^2)^2(p^2-q^2)^4(q+r)$. 33. $96x^4y^3z^4(a^3-b^3)$
 34. $40ab(a^6-b^6)(a^2+b^2)$. 35. $168x^2y^4(a-x)^3$ 36. $6x^2(x^6-729)$.
 37. $(a-1)(a-2)(a-3)$. 38. $(a-2)^2(a+2)$. 39. $ab(a+b)(a^3-b^3)$.
 40. $(8a^2-1)(a+1)$. 41. $(4x-1)(4x^2-1)$.
 42. $(64x^2-529y^2)(x-2y)$ 43. $(5x+4)(5x+3)(5x-2)$.

44. $(a^2 - b^2)^2(a^2 + b^2)$. 45. $6ab(a^6x^6 - b^6y^6)$.
 46. $(x-1)(x-2)(x-3)(x-4)$. 47. $(a+7b)(a+2b)(a-3b)(a-5b)$.
 48. $(4a+1)(3a-1)(3a-2)$. 49. $48a^6x^9(2a-x)^2(a-x)$.
 50. $ab(4a-b)^2(a-2b)(a-4b)$. 51. $72a^2b^2(a^6 - b^6)$.
 52. $abc(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.
 53. $4(9x^2 - 4y^2)(x^2 - 9y^2)$. 54. $1800ab^7(a-b)(a-2b)(a-3b)$.
 55. $500x^3y^2z^2(a^2 - x^2)^2(a+2x)$ 56. $x(x+1)(x^2-1)^2(x^2-4)$.

Examples 56. Page 131

1. $2x^5 - x^4 - 34x^3 + 61x^2 + 8x - 48$ 2. $x^3 + 6x^2 + 18x^2 + 27x + 14$.
 3. $x^4 - 5x^2y^2 + 4y^4$ 4. $x^5 - 6x^4 - 5x^3 + 90x^2 - 176x + 96$.
 5. $x^5 + x^4 - 13x^3 - 13x^2 + 36x + 36$
 6. $12x^5 + 26x^4y + 36x^3y^2 + 58x^2y^3 + 37xy^4 + 6y^6$
 7. $(3x^4 + 8a^3 + 3a^2 - 5)(3a^3 + a^2 + a - 1)$.
 8. $(1-1)(x-2)(x-4)$ 9. $(2x+1)(x^2-4)(x^2-4x+5)$
 10. $(a^2-1)(a^2-4)(a^2-9)(a-4)$.

Examples 57 Pages 137-138.

1. $\frac{ay}{bx}$ 2. $\frac{ay}{cx}$ 3. $\frac{3a^2c}{4x^4}$ 4. $\frac{3ab^3}{4x^2y}$ 5. $\frac{2yz^3}{3p}$.
 6. $\frac{6acx}{5b^2y^2z^2}$ 7. $\frac{a-b}{a+b}$ 8. $\frac{1}{1-x}$ 9. $\frac{a-5}{ba}$ 10. $\frac{ab}{a^2+b^2}$.
 11. $\frac{b+m}{c+b}$ 12. $\frac{b}{x-y}$ 13. $\frac{z^2(x^2+y^2)}{x^4}$ 14. $\frac{a^4+b^2c^2}{a^2(a^2-c^2)}$.
 15. $\frac{7x}{5z}$ 16. $\frac{3a+2x}{3a}$ 17. $\frac{10(a^2+2ab+b^2)}{(a-2b)^2}$ 18. $\frac{(x+y)^2(x^2-y^2)}{(x^2+xy+y^2)^2}$.
 19. $\frac{x+5}{x+3}$ 20. $\frac{2x+1}{2x-1}$ 21. $\frac{x-1}{x+4}$ 22. $\frac{2+3x}{1+5x}$ 23. $\frac{a-7b}{a-15b}$.
 24. $\frac{x-3a}{x+11a}$ 25. $\frac{x+8}{x+12}$ 26. $\frac{x+a}{x-a}$ 27. $\frac{x(x+9)}{3(x-3)}$ 28. $\frac{a+b-c}{a-b-c}$.
 29. $\frac{a}{a-b}$ 30. $\frac{c}{a+b-c}$ 31. $\frac{x-b}{x-a}$ 32. $\frac{x+c}{x-b}$ 33. $\frac{c+d}{c-d}$.
 34. $\frac{3(4ax-1)}{4(ax-1)}$ 35. $\frac{nx-m}{nx+m}$ 36. $\frac{a^2+ab+b^2}{a^2-b^2}$ 37. $\frac{2ab-b^3}{4a^2-2ab+b^2}$.
 38. $\frac{a^2-a-1}{a^2+a+1}$ 39. $\frac{ax+by+cz}{ax-by+cz}$ 40. $\frac{bx+c^2}{bx-c^2}$.

Examples 58. Page 139.

1. $\frac{x^2-3x-12}{x^2+x-5}$
2. $\frac{2x^2-x+1}{3x^2-2x+1}$
3. $\frac{(x+2)^2}{x^2+2x+4}$
4. $\frac{x-3}{x^2-3x-3}$
5. $\frac{(x-4)(x+1)}{(x-1)^2}$
6. $\frac{2a^2-3b^2}{3a-2b}$
7. $\frac{4a^2-ab+b^2}{3a^2+2ab-b^2}$
8. $\frac{2a^2+a+1}{5a^2+3}$
9. $\frac{2x-3}{4x-3}$
10. $\frac{3a^2-2a-4}{4a^2+8a-3}$
11. $\frac{4a^2+4a+9}{6a^2+6a-1}$
12. $\frac{(2a-b)(3a+b)}{(a-b)(4a-b)}$
13. $\frac{2a^2+6ab-b^2}{3a^2-ab-3b^2}$
14. $\frac{y'(x-y)^2}{x(x-3y)^2}$
15. $\frac{m(l-4m)}{2(l-m)}$
16. $\frac{9a^2+8ab+8b^2}{3a^2-4ab-14b^2}$
17. $\frac{a^2-ma+n}{a^2-na+m}$
18. $\frac{b(a-4)}{c(a+3)}$
19. $\frac{a+2b}{a-b}$
20. $\frac{p^2+pq+3q^2}{p^2+6pq+4q^2}$

Examples 59. Page 141.

1. $\frac{a}{b} + \frac{b}{a}$
2. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$
3. $\frac{x^2+y-3z}{y^2+x-\frac{3z}{y}}$
4. $\frac{a^2}{cd} + \frac{b^2}{cd}$
5. $\frac{a}{bc} + \frac{b}{ac} - \frac{1}{c}$
6. $\frac{l^2}{m^2n^2} + \frac{m^2}{l^2n^2} + \frac{n^2}{l^2m^2}$
7. $\frac{x^2}{2y} - 2x + \frac{y}{2} - \frac{y^2}{x}$
8. $\frac{4a}{9b} + 1 - \frac{b}{3a}$
9. $\frac{x^2}{3z} - \frac{xy}{4z} + \frac{y^2}{2z}$
10. $\frac{2p}{3qr} - \frac{q}{2pr} + \frac{1}{3} - \frac{2}{p^2q^2r^2}$

Examples 60. Pages 142-143.

1. $\frac{2x}{12}, \frac{8a}{12}, \frac{5b}{12}$
2. $\frac{48x}{60}, \frac{27x}{60}, \frac{2z}{60}$
3. $\frac{x^2}{xyz}, \frac{y^2}{xyz}, \frac{z^2}{xyz}$
4. $\frac{a^2}{abc}, \frac{b^2}{abc}, \frac{c^2}{abc}, \frac{d^2}{abc}$
5. $\frac{(a-x)^2}{a^2-x^2}, \frac{(a+x)^2}{a^2-x^2}$
6. $\frac{5(x-y)}{20}, \frac{2(y-z)}{20}, \frac{z-x}{20}$
7. $\frac{18bc(1+a)}{90abc}, \frac{15ca(2-a)}{90abc}, \frac{15ab}{90abc}$
8. $\frac{a(a-b)}{a^2-b^2}, \frac{b(a+b)}{a^2-b^2}, \frac{ab}{a^2-b^2}$
9. $\frac{2(x^2-4a^2)}{(x-a)(x^2-4a^2)}, \frac{-3(x-a)(x+2a)}{(x-a)(x^2-4a^2)}, \frac{4(x-a)}{(x-a)(x^2-4a^2)}$
10. $\frac{x(x+y)}{2y(x+y)}, \frac{2y^2}{2y(x+y)}, \frac{2z}{2y(x+y)}$
11. $\frac{a(a+2b)}{(a+b)(a^2-4b^2)}, \frac{b(a+b)}{(a+b)(a^2-4b^2)}$

12. $\frac{b^2(x-y)^2}{a^2b^2(x^2-y^2)}, \frac{a^2(x+y)^2}{a^2b^2(x^2-y^2)}, \frac{x^2+y^2}{a^2b^2(x^2-y^2)}$
13. $\frac{c(a-b)}{abc}, \frac{a(b-c)}{abc}, \frac{b(2c-a)}{abc}$
14. $\frac{2x(x^2-a^2)}{(x+a)(x-a)^2}, \frac{a(x-a)^2}{(x+a)(x-a)^2}, \frac{a(x+a)^2}{(x+a)(x-a)^2}$
15. $\frac{-a^2(b-c)}{(a-b)(b-c)(c-a)}, \frac{-b^2(c-a)}{(a-b)(b-c)(c-a)}, \frac{-c^2(a-b)}{(a-b)(b-c)(c-a)}$
16. $\frac{5x^2(x+2y)}{(x-3y)(x^2-4y^2)}, \frac{4y^2(x-2y)}{(x-3y)(x^2-4y^2)}, \frac{3(x-2y)(x-3y)}{(x-3y)(x^2-4y^2)}$
- $\frac{x^2(x-3y)}{(x-3y)(x^2-4y^2)}$

Examples 61. Pages 145-146.

1. $\frac{3a}{8}$. 2. $\frac{3x}{10}$. 3. $\frac{x}{6}$. 4. $\frac{5a}{6}$. 5. $\frac{a}{2}$.
6. $\frac{11a}{15}$. 7. $\frac{2b+3c}{48}$. 8. $\frac{8a-15b}{36}$. 9. $\frac{ax-by}{xyz}$.
10. $\frac{3ab+4ac}{4bc}$. 11. $\frac{24ab}{35c}$. 12. $\frac{5a^2-3b^2}{abc}$. 13. $\frac{x^3+y^3}{xyz}$.
14. $\frac{a^3d^2+b^2c^2}{b^2d^2}$. 15. $\frac{a^2b-bc^2}{acd}$. 16. $\frac{a^2}{12b^2}$. 17. $\frac{bc}{6a^2}$.
18. $\frac{x^2-y^2}{5xy}$. 19. $\frac{5x}{24}$. 20. $\frac{41x}{12y}$. 21. $\frac{2ab}{315}$.
22. $\frac{a^2-b^2+c^2}{abc}$. 23. $\frac{60a^2c-9abc-8a^2}{12bc}$. 24. $\frac{a^2-2ab-b^2}{b(a-b)}$.
25. $-\frac{2a^2+4a+2}{(3a+4)(5a+6)}$. 26. $\frac{4ab}{a^2-b^2}$. 27. $\frac{5a+4}{(3a+2)(6a+2)}$.
28. $\frac{2a^2}{b(a^2-b^2)}$. 29. $\frac{1}{(x-2)(x-3)}$. 30. $\frac{1}{2}$.
31. $\frac{10x-24}{(x+3)(x-3)^2}$. 32. $\frac{4x^2}{(x^2-a^2)^2}$. 33. $\frac{2b^3}{a^4+a^2b^2+b^4}$.
34. $\frac{2(b+y)}{b-y}$. 35. $\frac{2}{1-2a}$. 36. $\frac{n(m+3n)}{m(m+2n)}$. 37. $\frac{2x+4y}{x+2y}$.
38. $\frac{1+2x}{1-x^2}$. 39. $\frac{y^3}{x^2}$. 40. $\frac{y^3}{(x+y)^2}$.

41. $\frac{x^2}{x^2 - y^2}$. 42. $\frac{a^2 - 11a + 30}{11(a^2 + 11a + 30)}$ 43. $\frac{3}{(x-1)(x-2)}$
 44. $\frac{1}{x^2 - 1}$ 45. $\frac{a^2 + 2a + 4}{a(a^2 + 1)}$ 46. -2 .
 47. $\frac{1}{y^2 + 2y + 2}$ 48. $\frac{x^2 + 24x + 12}{4(x^2 - 4)}$ 49. $\frac{2x}{(x-a)(x-b)}$
 50. $-\frac{5}{a}$ 51. $\frac{1}{x+y}$ 52. $\frac{x^2 + 17x - 9}{x^2 - 9}$
 53. $\frac{3 + 4b + 4b^2}{(1-2b)(1-4b)(1-6b)}$ 54. $\frac{4a}{a+x}$ 55. $\frac{2\{(a+b)^2 + (ab+1)^2\}}{(a+b)^2 - (ab+1)^2}$
 56. $\frac{4x^2}{1-x^4}$ 57. 1. 58. 0. 59. $\frac{b^2}{(a-b)(a-2b)(a-3b)}$
 60. $\frac{1}{a+b}$ 61. $\frac{18}{(x-1)(x+2)(x+5)}$ 62. $\frac{a+b+c}{(a+b-c)(a-b+c)(b+c-a)}$

Examples 62 Page 148.

1. $\frac{b^2}{c^2}$ 2. $\frac{b}{1}$ 3. $\frac{xy}{x^2}$ 4. $\frac{1}{6}$ 5. $\frac{2x}{3z}$ 6. b^2c .
 7. $\frac{2bd^2}{ac^2}$ 8. $\frac{x}{2a^2}$ 9. $\frac{2bc}{3a^2}$ 10. $\frac{a}{8bc^2}$ 11. $\frac{x^2}{6y^2}$ 12. 1.
 13. $\frac{(x+y)(a-b)}{(x-y)(a+b)}$ 14. $\frac{(2a+1)^2}{12}$ 15. $\frac{(a+1)(a^2+a+1)}{(a-1)(a^2-a+1)}$ 16. -5
 17. $\frac{y}{2(2-xy)}$ 18. $\frac{x^2 - (b+c)x + bc}{x^2 + (a+c)x + a}$ 19. $\frac{1}{6}$ 20. $a+b$

Examples 63 Page 149.

1. $\frac{a^2n^2}{b^2m^2}$ 2. $\frac{a^2a^2}{b^4}$ 3. $\frac{10b^2}{a}$ 4. $\frac{2xy}{3ab}$ 5. $\frac{qx^2}{py^2}$
 6. $\frac{20x^2}{3ab}$ 7. $\frac{a}{b}$ 8. $\frac{a+x}{a}$ 9. $\frac{(1-a)(1-b)}{ab}$ 10. $\frac{y}{x}$
 11. $-\frac{1}{ax}$ 12. $-\frac{a}{b}$ 13. $\frac{bc}{6a^2}$ 14. $\frac{16}{75}$ 15. $\frac{16xy}{75a^2}$
 16. $\frac{18a^{15}n}{5b^3}$ 17. $\frac{a^4}{2bc^2d}$ 18. $\frac{5b}{7z}$ 19. 1 20. $\frac{x^2 - 7x + 6}{x^2 - 9x + 20}$

Examples 64. Pages 152-153.

1. $\frac{d}{c}$ 2. $\frac{x^2}{a^2}$ 3. $\frac{a}{b}$ 4. $\frac{b^3}{a^3}$ 5. $a+b$ 6. $-\frac{1}{ab}$ 7. $\frac{c}{b}$
 8. $\frac{x^2(x^2-y^2)}{x^4-x^2y^2+y^4}$ 9. $\frac{3(x+3)}{5(x-3)}$ 10. $\frac{ab}{a^2-b^2}$ 11. $\frac{3x-4y}{4x-3y}$
 12. $\frac{2ax}{1+x^2}$ 13. y 14. $\frac{x-6}{x-10}$ 15. xy 16. $\frac{ax(x+a)}{x-a}$
 17. $\frac{x-7}{x-1}$ 18. $\frac{x-7}{x-5}$ 19. $\frac{x-y}{xy}$ 20. $\frac{x^3-y^3}{x^2y^2}$ 21. $\frac{a^3+b^3}{ab}$
 22. 1. 23. $\frac{q(p-q)}{p}$ 24. $\frac{a(3a^4+4)}{a^6+6a^4+8}$ 25. $\frac{(x+4)(3x-44)}{3x^2-8x+48}$
 26. $\frac{x-4}{x+6}$ 27. $\frac{(a+1)^2}{a^3-a^2+1}$ 28. 1. 29. $\frac{5x+12}{3x+9}$ 30. $\frac{x-1}{3x+2}$
 31. $-\frac{5}{2x+2}$ 32. x 33. $\frac{3x-8}{x-3}$

Solutions

Examples 65. Pages 161-168.

1. $\frac{c^2+d^2}{(c+d)^2}$ 2. 1. 3. $\frac{a+b}{a-b}$ 4. $\frac{a^2-b^2}{c^2-d^2}$ 5. $\frac{xz}{(x+2y+z)(x+y+2z)}$
 6. a 7. 1 8. $\frac{l+m-n}{l-m+n}$ 9. $\frac{(y+z-x)(z+x-y)(x+y-z)}{(x+y+z)^3}$
 10. 1. 11. 1. 12. 1. 13. $\frac{(x-y)^4(x^2+y^2)}{(x+y)^3}$ 14. 1.
 15. $(b-c)(c-a)(a-b)$ 16. $(x-y)^2$ 17. 3. 18. $\frac{x^2-y^2}{xy}$
 19. $\frac{1}{abc}$ 20. $\frac{a-b}{abc}$ 21. $\frac{(b-c)(c-a)(a-b)}{2}$ 22. $\frac{(b-c)(c-a)(a-b)}{abc}$
 23. $(b-c)(c-a)(a-b)$ 24. $a+b+c$ 25. 1. 26. $\frac{x^3-49}{x^3+x-20}$ 27. 1.
 28. $\frac{x(1-x)}{2+x}$ 29. $\frac{c}{a}$ 30. 0 31. 1. 32. $\frac{x}{(a+c-x)(c+bx)}$
 33. $\frac{2a^3}{(a^2+b^2)^2}$ 34. $\frac{(a-b)^2}{a}$ 35. $\frac{3(p+3q)}{4q^2}$ 36. m^2+n^2
 37. 4. 38. 1. 39. 3. 40. $a+b+c$ 41. $2(a+b+c)$
 42. $-\frac{1}{2}$ 43. $2abc$ 44. 3. 45. 1. 46. 1. 47. 1.

48. $\frac{1}{ab^2c}$. 49. 1. 50. 0. 51. $\frac{ab+bc+ca}{a^2b^2c^2}$. 52. 0. 53. 0. 54. -1.
55. 0. 56. 0. 57. $\frac{a+b+c}{abc}$. 58. $a+b+c$. 59. -3. 60. 4.
61. $(x-\frac{1}{a})(x+\frac{1}{b})(x-\frac{1}{c})$. 62. $(1+a)(1+b)(1+c)$. 63. l^2 .
64. $a+b+c+3$. 65. $-(a+b+c)$. 66. -1.
67. $\frac{1}{a^2b^2c^2}$. 68. $\frac{(x-m)^2}{(x+a)(x+b)(x+c)}$. 69. -4. 70. $bc+ca+ab$.
71. $\frac{m(a+b+c)}{(a+b)(b+c)(c+a)}$. 72. $\frac{1}{(a-d)(b-d)(c-d)}$.
73. $1+\frac{d^3}{(a-d)(b-d)(c-d)}$. 74. $\frac{abc}{(a-d)(b-d)(c-d)}-1$.
75. $\frac{x^4}{b^4y^2}+\frac{2ax^3}{b^3y}+\frac{3a^2x^2}{ab^2y}+\frac{x^2}{a^2}+\frac{6x^2}{b^2}+\frac{a^2x^2}{b^4}+\frac{3axy}{b^3}+\frac{2xy}{ab}+\frac{y^2}{b^2}$.
76. $\frac{a^4}{b^4}+\frac{b^4}{a^4}-\frac{a^2}{b^2}-\frac{b^2}{a^2}$. 77. $\frac{y}{b}-\frac{2x}{a}-a$. 78. $\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+1$. 79. 0.
80. 0. 81. 0. 82. $-\frac{a+b+c}{abc}$. 83. 1. 84. 2.

Examples 66 Page 170

11. 132. 12. -1714. 13. 2. 14. 2.

Examples 67 Pages 177-178.

2. $x^7+x^6y+x^5y^2+x^4y^3+x^3y^4+x^2y^5+xy^6+y^7$.
3. $1+y+y^2+y^3+y^4$. 4. $1+y+y^2+y^3+y^4+y^5$.
5. $1+y^2+y^4+y^6+y^8$. 6. $a^6+a^4b+a^2b^2+b^3$.
7. $x^4-x^2y+x^2y^2-xy^3+y^4$.
8. $x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-x^2y^5+y^6$.
9. $x^9-x^8y+x^7y^2-x^6y^3+x^5y^4-x^4y^5+x^3y^6-x^2y^7+xy^8-y^9$.
10. $x^6-x^3y+y^2$. 11. $1+2x+4x^2+8x^3+16x^4$.
12. $1-2x+4x^2-8x^3+16x^4-32x^5$. 13. $16x-24x^2+36x^3-54x^4+81x^5$.
14. $1+x+x^2+\dots+x^{n-1}$. 15. $(a^2-ab+b^2)(c^2-cd+d^2)$.
16. $x^5+2x^4+3x^3+4x^2+5x+6$. 17. When $a^2m-a^2n+ar-s=0$.
18. (1) 51; (2) -4; (3) $4a(a^2+b^2)$.
31. $1+x+x^2+x^3+\dots+x^{20}+x^{21}$.

Examples 68. Pages 187-190.

1. \sqrt{a} . 2. $\sqrt[3]{a}$. 3. $\sqrt[4]{(x^2)}$. 4. $\sqrt[5]{(y^4)}$. 5. $\sqrt[3]{(a^2)}$, $\sqrt[5]{(b^3)}$.
6. $\frac{1}{\sqrt{a}}$. 7. $\frac{1}{\sqrt[3]{(x^2)}}$. 8. $\frac{1}{y^4}$. 9. $\sqrt[5]{(x^3)}$. 10. $\frac{1}{\sqrt[6]{(x^2y^3)}}$.
11. 9. 12. $\frac{1}{6}$. 13. $\frac{3}{2}$. 14. 5. 15. $\frac{1}{125}$. 16. $1\frac{1}{2}$. 17. $2\frac{6}{7}$.
18. $3\frac{3}{8}$. 19. $2\frac{1}{2}$. 20. 2. 21. 216. 22. 3^2 . 23. $\frac{1}{2\sqrt{2}}$. 24. 1.
25. $\sqrt[6]{(x^{10})}$. 26. $\sqrt{(x^5)}$. 27. $\sqrt[4]{(a^{11})}$. 28. $\sqrt{(b^6)}$. 29. a^3 .
30. x^3 . 31. $b^{\frac{1}{3}}c^{\frac{1}{3}}$. 32. $2a^{\frac{1}{2}}b^0c$. 33. $4x^{\frac{1}{2}}y^{-\frac{1}{2}}z^{\frac{1}{2}}$.
34. $a^{-\frac{1}{5}}b^{\frac{1}{2}}$. 35. $a^3b^{-\frac{1}{2}}$. 36. $x^{\frac{1}{3}}y^{\frac{1}{2}}z^{\frac{5}{6}}$. 37. $a^{-\frac{5}{6}}b^{-\frac{1}{3}}c^{-\frac{1}{6}}x^{\frac{5}{6}}y^{-\frac{5}{6}}$.
38. ab^{-2} . 39. $2a^{-2}xy^{-2}z$. 40. $\frac{3}{2}l^{-2}mn$. 41. a^0 . 42. b^{-m} .
43. a^2 . 44. $2(2x^2 - y^2)$. 45. $\sqrt[7]{(a^{-686})}$, b^{-12} .
46. $p^{n(n+m)} + p^{m(n-m)}$. 47. $x^{\frac{1}{3}}y^{\frac{2}{3}}$. 48. abc . 49. $x^{n(n-1)}$.
50. a^{l-m} , b^{m-n} , c^{n-2} , $x^{l-3m+2n}$, y^{m-2n+l} , z^{m-2l+n} .
51. $\left(\frac{a}{b}\right)^{m+n}$. 52. $(-1)^n x^{m+n} y^n$. 53. $a^{\frac{2}{m}} - a^{-\frac{2}{m}}$; $a^{\frac{2}{m}} - a^{-\frac{3}{m}}$.
54. $a^6 - a^{\frac{9}{2}} - a^3 - a^2 + a^{\frac{3}{2}} - a^{\frac{1}{2}}$; $a^7 - 2a^6 - 2a^4 + a^3 + 2a^2 + a - 1$.
55. $6(x^{\frac{2}{3}} - x^{-\frac{2}{3}}) + 4y^{\frac{1}{2}} - 9y^{-\frac{1}{2}}$; $6(x^{\frac{2}{3}}y^{-\frac{1}{2}} - x^{-\frac{2}{3}}y^{\frac{1}{2}}) - 5$.
56. $2x^{\frac{1}{2}}y^{\frac{1}{2}} - 7xy^{\frac{2}{3}} + 11x^{\frac{1}{2}}y^{\frac{1}{2}} - 7x^{-\frac{1}{2}}y^{-\frac{1}{2}} - 3x^{-\frac{2}{3}}y^{-\frac{1}{2}}$.
57. $a^{x+1}b^y - 4a^{x+y-1}b^{2y} - 27a^{x+y-2}b^{3y} + 42a^{x+y-3}b^{4y}$.
58. $a^{\frac{1}{3}} + 3 + 6a^{-\frac{1}{3}}$. 59. $\frac{2a}{b^3} + \frac{\sqrt{2}\sqrt[3]{a}}{b} + \sqrt[3]{a^2}$. 60. $a^m - 3b^n + 2c^p$.
61. $x^{2n-1} - y^{2n-1}$; $x^2 s^{2n-1} + x^{2n-1} s^2 + y^2 s^{2n-1}$.
62. $x^3 + x^{-3} + 3(x + x^{-1})$. 63. $a^{\frac{1}{3}}b^{\frac{1}{2}} - a^{\frac{2}{3}}b^{\frac{1}{3}}c$.
64. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y)$. 65. $(a^{\frac{2}{3}} - b^{\frac{2}{3}})(a^{\frac{2}{3}} + b^{\frac{2}{3}})$.
66. $(a^{-2} + 2b^{-3})(a^{-2} - 2b^{-3})$. 67. $(a^{-1} - 1)(a^{-2} + a^{-1} + 1)(a^{-6} + a^{-3} + 1)$.
68. $(a^{\frac{1}{3}} + 1)(a^{\frac{1}{3}} + 1)$. 69. $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2$. 70. $e^x + e^{\frac{5}{2}}x^{\frac{1}{2}} - x$.
71. $a^{x-11}b^y - b^{y-11}x$; $a^{y+11}z - a^{x+y}b^{y-11}x + a^{x-11}b^y z^2 + b^{y+11}y$.
72. $e^x + 1$. 73. $a^{\frac{1}{2}}(a^{\frac{1}{2}} - \frac{1}{2}a^{\frac{1}{2}})$. 74. $(x^{\frac{2}{3}} - y^{\frac{2}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{1}{2}} + y^{\frac{1}{2}})^{\frac{1}{2}}$.
75. $\frac{a^2 + b^2}{a(a+b)}$. 76. 2. 77. $\left(\frac{a-b}{a+b}\right)^n + 2$. 78. $a^x b^{-y}$.

80. $\frac{x^{-6}}{64} - \frac{x^{-4}y^{-3}}{80} + \frac{x^{-2}y^{-6}}{100} - \frac{y^{-9}}{125};$
 $-x^{\frac{1}{2}}y^{\frac{1}{2}}\left(x^{\frac{5}{2}}+x^2y^{\frac{1}{2}}+x^{\frac{3}{2}}y+x^{\frac{1}{2}}y^2+y^{\frac{5}{2}}\right);$
 $x^{-\frac{5}{2}}+x^{-2}y^{-\frac{1}{2}}+x^{-\frac{3}{2}}y^{-1}+x^{-1}y^{-\frac{3}{2}}+x^{-\frac{1}{2}}y^{-2}+y^{-\frac{5}{2}}.$
81. m = a positive multiple of 6.
84. $1+x^2+x^4+\dots\dots\dots+x^{4n}.$

Examples 69. Pages 192—194.

1. $\sqrt[3]{9}.$ 2. $\sqrt[3]{49}.$ 3. $\sqrt{(x^4y^3)}.$ 4. $\sqrt{\{4a^4(b+c)^3\}}.$
 5. $\sqrt[3]{(64)}.$ 6. $\sqrt[3]{(64a^6b^6)}.$ 7. $\sqrt[3]{(a^3)}.$ 8. $\sqrt[3]{(x^3y^3)}.$
 9. $\sqrt{(12)}.$ 10. $\sqrt[3]{(32)}.$ 11. $\sqrt{(a^2b)}.$ 12. $\sqrt[5]{(243x^6y^6z)}.$
 13. $\sqrt[5]{(\frac{1}{128})}.$ 14. $\sqrt[3]{(\frac{8}{27})}.$ 15. $\sqrt{(a^6)}.$ 16. $\sqrt[3]{(\frac{a^6}{b^6})}.$
 17. $\sqrt[5]{(16a^4b^5)}.$ 18. $\sqrt[3]{(3^n x^{n-1}y)}.$ 19. $\sqrt[n]{(a^{mn+1}b^{mn-1})}.$
 20. $\sqrt[3]{(x^{mn-1}y^{1-mn})}.$ 21. $6\sqrt{2}.$ 22. $2\sqrt[3]{9}.$ 23. $12\sqrt{2}.$
 24. $5\sqrt[3]{4}.$ 25. $2\sqrt[5]{3}.$ 26. $2\sqrt[4]{(20)}.$ 27. $a\sqrt{b}.$ 28. $a^2bca^2\{\sqrt[3]{(c^2d)}\}.$
 29. $(a+b)\sqrt{c}.$ 30. $(x+y)\sqrt{\{(x+y)z\}}.$ 31. $2x(y-z)\sqrt[3]{\{x(y-z)^2\}}.$
 32. $(x-1)(x+2)(x+3)\sqrt{(x+1)}.$ 33. $\frac{\sqrt[3]{6}}{3}; \frac{\sqrt[3]{(18)}}{3}; \frac{\sqrt[5]{(16)}}{2}.$
 34. $\sqrt[5]{(27)}.$ 35. $\sqrt[12]{(256)}.$ 36. $\sqrt[12]{(a^3b^3)}.$ 37. 1st. 38. 1st. 39. 2nd. 40. $3\sqrt{2}-5\sqrt{3}.$ 41. 0
 42. 0. 43. 8. 44. $2\sqrt{2}$ 45. 16. 46. 3. 47. $11\sqrt{5}.$
 48. $2+\sqrt{2}+14\sqrt{3}.$ 49. $3\sqrt{2}; 5\sqrt{2}.$ 50. $2\sqrt{6}; 18.$
 51. 1. 52. 1. 53. a^2+x-x^3 54. $b.$ 55. $14+3\sqrt{30}.$
 56. $16+5\sqrt{15}.$ 57. 2. 58. $\sqrt{2}.$ 59. $\sqrt{(\frac{8}{3})}.$ 60. $\sqrt[3]{(\frac{1}{2})}.$ 61. 140.
 62. $4(1+\sqrt{2}+\sqrt{6})-\sqrt{3}.$ 63. $\frac{2\sqrt{(a^2-b^2)}}{b}.$ 64. 48.

Examples 70. Pages 195—196.

1. $\sqrt{x}-\sqrt{y}.$ 2. $2-\sqrt{3}.$ 3. $\sqrt[3]{x^2}+\sqrt[3]{(xy)}+\sqrt[3]{y^2}.$ 4. $\frac{\sqrt{6}+\sqrt{2}}{2}.$
 5. $\frac{2+\sqrt{6}}{2}.$ 6. $\frac{\sqrt{(x^2-a^2)}}{x+a}.$ 7. $\frac{\sqrt{(a^2-b^2)}}{a-b}.$ 8. $\frac{(\sqrt{x}+\sqrt{a})^2}{x-a}.$
 9. $\frac{2x^2-y^2-2x\sqrt{(x^2-y^2)}}{y^2}.$ 10. $\frac{a+\sqrt{(a^2-b^2)}}{b}.$ 11. $\frac{2+\sqrt{3}}{1}.$

12. $\frac{5+2\sqrt{5}}{1}$. 13. $\frac{119-12\sqrt{42}}{61}$. 14. $\frac{\sqrt{5}}{5}$. 15. $\frac{2+\sqrt{2}-\sqrt{6}}{4}$.
 16. $\frac{2\sqrt{2}+\sqrt{6}-\sqrt{3}-1}{2}$. 17. $(1-x)^{\frac{1}{2}}$. 18. $\frac{1}{\sqrt{x}}$. 19. 1.
 20. $26+15\sqrt{3}$. 21. $3\cdot7321$. 22. $8\cdot2426$. 23. $17\cdot9443$. 24. $1\cdot7321$.

Examples 71. Pages 199-200.

1. $4a^3+b^3+9c^3+4ab-6bc-12ca$; $a^{13}+2a^6b^6+b^{13}$.
2. $4x^4+25y^4+36z^4-20x^2y^2+6cy^2z^2-24z^2x^2$;
 $a^{-20}-2a^{-16}b^{-5}+a^{-10}b^{-10}$.
3. $a^{2m}+b^{2m}+c^{2m}+2b^{1m}c^m+2c^ma^m+2a^mb^m$.
4. $4^{4m}+2a^{2m}b^m+3a^{2m}b^{2m}+2a^mb^{3m}+b^{4m}$.
5. $\sqrt[3]{x^2}+\sqrt[3]{9}bx-6\sqrt[3]{b^6}x^{15}-6\sqrt[3]{b^5}x\sqrt{x}+2\sqrt[3]{(bx)^2}+\sqrt[3]{b^2}$.
6. $a^{\frac{5}{2}}-4a^{\frac{5}{4}}b^{\frac{1}{4}}-2ab^{\frac{1}{2}}+14a^{\frac{3}{4}}b^{\frac{3}{4}}+5a^{\frac{1}{2}}b-6a^{\frac{1}{4}}b^{\frac{5}{4}}+b^{\frac{3}{2}}$.
7. x^4-2+x^{-4} ; $x^3-4x^2+2x+4+x^{-1}$.
8. $\left(x^4+\frac{1}{x^4}\right)-4\left(x^3+\frac{1}{x^3}\right)+10\left(x^2+\frac{1}{x^2}\right)-16\left(x+\frac{1}{x}\right)+19$.
9. $8x^3+y^3-27z^3+12x^2y-3yz+3y^2(2x-3z)-27z^2(2x+y)-36xyz$;
 $a^{15}-3a^{10}b^5+3a^5b^{10}-b^{15}$.
10. $8x^{13}+12x^{12}y^6+6x^6y^{12}+y^{18}$; $8a^{30}-36a^{20}b^{10}+54a^{10}b^{20}-27b^{30}$.
11. $a^{-12}-3a^{-8}b^{-4}+3a^{-4}b^{-8}-b^{-12}$; $x+3\sqrt[3]{(x^2y)}+3\sqrt[3]{(xy^2)}+y$.
12. $a^{-2}+3\sqrt[3]{(a^{-4}b^{-2})}+3\sqrt[3]{(a^{-2}b^{-4})}+b^{-2}$; $x^{\frac{7}{2}}+3x^{\frac{3}{2}}y^{-\frac{5}{4}}+3x^{\frac{3}{4}}y^{-\frac{3}{2}}+y^{-\frac{11}{4}}$.
13. $x^6-3x^4+12x^2-19+\frac{36}{x^2}-\frac{27}{x^4}+\frac{27}{x^6}$; $x^{3m}-3x^{2m}y^m+3x^my^{2m}-y^{3m}$.
14. $x^{\frac{6}{m}}-3x^{\frac{3}{m}}+3-x^{-\frac{3}{m}}$; $a^{\frac{3m}{m}}+6a^{\frac{2m}{m}}b^{\frac{m}{m}}+12a^{\frac{m}{m}}b^{\frac{2m}{m}}+8b^{\frac{3m}{m}}$.
15. $a^8-4a^6b^2+6a^4b^4-4a^2b^6+b^8$.
16. $a^{13}-8a^9b^4+24a^6b^7-32a^3b^{10}+16b^{13}$.
17. $a^2b^2+4a^{\frac{3}{2}}b^{\frac{3}{2}}c^{\frac{1}{2}}d^{\frac{1}{2}}+6abcd+4a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{3}{2}}d^{\frac{3}{2}}+c^2d^2$.
18. $ab^{-1}-4a^{\frac{1}{2}}b^{-\frac{1}{2}}+6-4a^{-\frac{1}{2}}b^{\frac{1}{2}}+a^{-1}b$.
19. $a^8-8a^7+28a^6-56a^5+70a^4-56a^3+28a^2-8a+1$.
20. $a^{\frac{4m}{n}}-4a^{\frac{3m}{n}}b^{-\frac{m}{n}}+6a^{\frac{2m}{n}}b^{-\frac{2m}{n}}-4a^{\frac{m}{n}}b^{-\frac{3m}{n}}+b^{-\frac{4m}{n}}$.

21. $x^6 + 5x^4y + 10x^2y^2 + 10x^2y^3 + 5xy^4 + y^5$ and $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$; $x^6 - 5x^4y + 10x^2y^3 - 10x^2y^3 + 5xy^4 - y^5$ and $x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$.
22. $32x^5 - 240x^4y + 720x^2y^3 - 1080x^2y^3 + 810xy^4 - 243y^6$ and $64x^6 - 576x^5y + 2160x^4y^2 - 4320x^3y^3 + 4860x^2y^4 - 2916xy^5 + 729y^6$.
23. $a^{15} - 5a^{13}b^{-2} + 10a^{11}b^{-4} - 10a^9b^{-6} + 5a^7b^{-8} - b^{-10}$ and $a^{18} - 6a^{15}b^{-3} + 15a^{12}b^{-6} - 20a^9b^{-9} + 15a^6b^{-12} - 6a^3b^{-15} + b^{-18}$.
24. $x^7 - 7ax^6 + 21a^2x^5 - 35a^3x^4 + 35a^4x^3 - 21a^5x^2 + 7a^6x - a^7$; $128a^7 - 448a^6b + 672a^5b^2 - 560a^4b^3 + 280a^3b^4 - 84a^2b^5 + 14ab^6 - b^7$.
25. $x^{14} + 7a^2x^{12} + 21a^4x^{10} + 35a^6x^8 + 35a^8x^6 + 21a^{10}x^4 + 7a^{12}x^2 + a^{14}$; $x^{-21} - 7a^{-2}x^{-18} + 21a^{-6}x^{-15} - 35a^{-10}x^{-12} + 35a^{-14}x^{-9} - 21a^{-18}x^{-6} + 7a^{-22}x^{-3} - a^{-21}$.
26. $1 - \frac{7a}{b} + \frac{21a^2}{b^2} - \frac{35a^3}{b^3} + \frac{35a^4}{b^4} - \frac{21a^5}{b^5} + \frac{7a^6}{b^6} - \frac{a^7}{b^7}$; $\left(\frac{x^7}{a^7} - \frac{a^7}{x^7}\right) - 7\left(\frac{x^5}{a^5} - \frac{a^5}{x^5}\right) + 21\left(\frac{x^3}{a^3} - \frac{a^3}{x^3}\right) - 35\left(\frac{x}{a} - \frac{a}{x}\right)$.
27. $16x^4y^2z^3$; $-a^{10}b^{11}$. 28. $-x^{17}yz^{-16}$. 29. $2(a^{2^{n+1}} + b^{2^{n+1}})$.
30. $2(a^{2^n} + 3a^{2^{n-1}}b^{2^{n-1}})$ 33. $-xyz$ 34. $-\frac{8}{ab}$ 35. $\frac{c+a-b}{a+b-c}$.
36. $\frac{3-x}{1+x}$ 37. $3a^2$; -12 38. 1 39. -11 40. -3
41. 3. 42. 1. 43. -240 . 44. $-35a^4b^3$. 45. 1.

Examples 72. Pages 201—202.

1. $3xy$. 2. $6a^4b^3c^2$. 3. $7a^{-1}b^2$. 4. $11x^{-2}y^2z^4$. 5. $\frac{x^2}{2yz}$.
6. $\frac{4a^2b^4}{5ca^6}$. 7. $\frac{2xy^2}{z^3}$. 8. $\frac{4ac^3}{3b^2d}$. 9. $3ab^2$. 10. $5a^2b^{-1}$.
11. $-4abd^2$. 12. $-7x^{-2}y^2z^3$. 13. $\frac{2ax}{3}$. 14. $\frac{5ab^2}{6xy^2}$.
15. $-\frac{8x^4}{3yz^3}$. 16. $-\frac{9ab^3}{5c^2}$. 17. $-abc^2$. 18. $2a^2bx^2$. 19. $2abc^2$.
20. $-\frac{2a}{bc}$ 21. $\frac{3ab}{c}$. 22. $-\frac{2}{3}ab\sqrt{b}$. 23. $-\frac{1}{ab^2}$.
24. $\frac{2}{3}\sqrt[3]{(15)} - 2\sqrt[3]{(243)} + 9\sqrt[3]{9}$ 25. $\sqrt[3]{12}$ 26. $-x^6$

Examples 73. Page 204.

1. $2a+5$.
2. $3a-4x$.
3. $2x+\frac{5}{2}y$.
4. $\frac{7}{4}ab-\frac{8}{3}c$.
5. $10x^2y+3z$.
6. $ab+1$.
7. $7x+3y$.
8. $9x-5y$.
9. $x-\frac{1}{x}+2$.
10. $\frac{a}{b}+\frac{b}{a}-3$.
11. x^2-ax+1 .
12. $x^2-5ax-3$.
13. $\frac{x}{y}-\frac{y}{x}-4$.
14. $x-\frac{1}{2x}-7$.
15. $(x-2)(2y-1)$.
16. $(a-1)(a+1)(a+2)$.
17. $(2a+3x)(3a+2x)(3a-2x)$.
18. $2x^2+3x+1$.
19. $x^2-4ax-8a^2$.
20. $x^2+5xy-2y^2$.
21. $a+b+1$.
22. $x^2+2x-1-3$.
23. $\frac{4a}{x}+\frac{x}{8a}$.
24. $x-\frac{y}{3}+\frac{z}{4}$.
25. a^5-b^6 .
26. a^2-a^3 .
27. $3i^3-1$.
28. $3+x^{\frac{1}{2}}-2x$.
29. $3-x^n+x^{-n}$.
30. $5x^{m+1}-2x^{-m}$.
31. $p-\frac{8}{p}+4$.
32. $\frac{a^2+b^2}{a^2-b^2}$.
33. $(p+q)/-(p-q)m$.
34. x^2+y^2 .
35. $\frac{a}{x}-\frac{2b}{y}+\frac{c}{2z}$.
36. $1+x+x^2$.
37. $4x^2-16x+11$.
38. $3a^2+b^2$.
39. $\frac{a+1}{a-1}$.
40. $1-\frac{2p}{p^2-1}$.

Examples 74. Pages 210-211.

1. $3x^2+2x-2$.
2. $a^2+5ab-11b^2$.
3. $\frac{1}{2}+2a-6a^2$.
4. $5a^2+2a^3-4a+5$.
5. $6x^2+\frac{5}{2}x-\frac{1}{2}$.
6. $\frac{3}{2}a^2-2a-\frac{2}{3}$.
7. $\frac{3}{2}x^2-\frac{1}{2}ax-\frac{3}{2}b$.
8. $x^2-\frac{1}{2}xy-\frac{3}{2}$.
9. $1-\frac{3}{2}x^2+x^3$.
10. $\frac{3}{2}b^2-\frac{1}{2}ab-\frac{1}{5a}$.
11. $\frac{2}{3}+\frac{4x}{y}-\frac{5x^2}{2y^2}$.
12. $2p-3q-4r$.
13. $a^2x-2ay-3z$.
14. $\frac{a}{b}+\frac{a}{3}-\frac{b}{2}$.
15. $8a^3+4a^2b+\frac{3}{2}ab^2+\frac{1}{2}b^3$.
16. $\frac{a}{b}+\frac{2b}{c}-3$.
17. $\frac{x}{2y}-\frac{2y}{3z}-\frac{3z}{4x}$.
18. $a^2+a(2b-3c)+4c^2$.
19. $5x^4-4x^3+3x^2-2x+1$.
20. $\frac{x}{y}-\frac{y}{x}+3$.
21. $\frac{5}{6}\frac{ab}{c}-\frac{1}{2}+\frac{3}{5}\frac{c}{ab}$.
22. $\frac{x^3}{y^2}-\frac{y}{x}+2$.
23. $\frac{x^3}{y^2}-\frac{y^3}{x^2}+3\left(\frac{x}{y}-\frac{y}{x}\right)$.
24. $\frac{2x^3}{y}-\frac{1}{2xy}-\frac{1}{3x^2}$.

25. $a^{-3} - 2b^{-4} + 2$ 26. $a^{\frac{2}{3}}b^{\frac{1}{2}} + a^{\frac{1}{3}}b^{\frac{1}{2}} - 3$ 27. $7\sqrt[5]{\frac{a^2}{b^4}} + 3\sqrt[5]{\frac{b^2}{a^2}} - \sqrt[5]{b}$
 28. $\left(1 + \frac{a}{2b}\right)x^{\frac{2}{3}} - bx^{-\frac{2}{3}}$ 29. $2a^m + 3a^m - 2$ 30. $\frac{a^mb^{-n}}{2^2} + \frac{a^mb^{-m}}{3^2} - 2^3 \cdot 3^2$
 31. $x^2 - x + 2$ 32. $x - 1$ 33. $a = 0; x + 3$
 34. $l = 4; 5x - 2$ 35. $a = 0; x + y$ 36. $c = a^2b^2; x^2 - (a + b)x + ab$

Examples 75 Page 215.

1. $x + 3$ 2. $2x + 1$ 3. $x - 4$ 4. $2x - 5$ 5. $4x - 3a$
 6. $3ab - 2c^2$ 7. $\frac{x}{y} + \frac{2y}{x}$ 8. $\frac{a}{3b} - \frac{2b}{a}$ 9. $4a^{-1} - b^{-1}$
 10. $x^{\frac{1}{2}} + 2y^{-\frac{1}{2}}$ 11. $a^2 - b^2$ 12. $-x(x + 2)$ 13. $x + a - b$
 14. $x^2 - a + 2a^2$ 15. $xy - 5yz$ 16. $b^2 - ab + 2a^2$ 17. $\frac{1}{2} + 5a^2 - 3a^4$
 18. $a - 2b + 1$ 19. $\frac{3}{2}(x + y) - \frac{5}{3}$ 20. $12a^2 + 4b^3$ 21. $\frac{x^2}{4} - \frac{ax}{6} - \frac{a^2}{9}$
 22. $\frac{x^2}{y^2} + \frac{y^2}{x^2} - 2$ 23. $\frac{x}{3y} - x + \frac{3y}{x}$ 24. $3x + \frac{y}{2}$ 25. $1 - \frac{y}{2}$
 26. $2 + \frac{1}{12}x - \frac{1}{288}x^2; 3 - \frac{1}{21}x - \frac{1}{187}x^2; 2'1527; 2'9241$

Miscellaneous Examples II Pages 216—221.

1. $3a^4 - 5a^3 + 9a^2 - 16a - 2$
 3. $(x + 1)(x - 3)(x - 1)^2(x + 3)^3; 1 + 2x^{-1} + x^{-2}$
 4. (1) $(x - 2)(x^2 + 2x + 4)(x^3 + 3)$, (2) $(x + 2)(x^2 - 2x + 4)(x^3 - 3)$;
 (3) $(x + 2y - 3)(x - 2y - 3)$, (4) $(ay - bx + az)(by + ax - bz)$
 5. $\frac{4(2x + 1)(x - 3)(x + 1)}{3(1 - 1)(x + 3)(x - 2)}$ 6. $x^{2^n} + a^{2^{n-1}}x^{2^{n-1}} + a^{2^n}$ 7. $2x^3 \div 5x + 4$
 8. $-\frac{2(x^2 + 2)}{x^2}$ 9. (1) $x = 3\frac{1}{2}$; (2) $x = 1$ 11. $2a^4 - 5a^3 - 4a^2 + a + 2$
 12. $ad' + a'd + bc' + b'c$ 13. $2a + b$
 14. (1) $(2x + y)(2x - y + 6a)$; (2) $4(3x - 2)(5x - 1)$;
 (3) $(a + b + c)(a + 2b - 3c)$
 15. 2. 16. $6'464, \dots$ 17. $2(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) - 1$ 18. 6.
 19. $5\frac{1}{2}$ 20. 16; 26. 21. $a^2 - ab + b^2; 3a^2 + 3ab + 7b^2$
 22. $x^2 - (a - b)x + b^2$ 23. $3x(10x + 7y)$
 24. (1) $(6 + x)(2 - 5x)$, (2) $(2a + 1)(a^2 - 2a + 4)$ (3) $(x - 6y + 1)(2x + y - 3)$

25. 0. 26. $x^{\frac{4}{3}}y^{-\frac{4}{3}} + x^{\frac{2}{3}}y^{-\frac{2}{3}} + x^{-\frac{2}{3}}y^{\frac{2}{3}} + x^{-\frac{4}{3}}y^{\frac{4}{3}}$. 28. x . 29. $\sqrt{3}$.
 30. Rs. 5. 5as. 4p.; Rs. 4. 31. 84. 32. $4(b-a)x^2$.
 33. $(a-c)(bc-ad) = (b-d)^2$. 34. (1) $(a+2b+2)(a-2b+2)$.
 (2) $(a+7b)(a-63b)$; (3) $(x-1)(x-3)(x+4)$.
 36. $x^{\frac{1}{2}}y^{-\frac{1}{2}} + x^{-\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{1}{4}}y^{-\frac{1}{4}} - x^{-\frac{1}{4}}y^{\frac{1}{4}}$. 37. $a^3 - 4ab - 13b^2$.
 40. $b - \frac{ac}{2a-2c}$ miles 41. $7(x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}})$.
 42. $x^3 - x^{\frac{5}{2}} - 4x^2 + 6x^{\frac{3}{2}} - 2x$. 43. $-9ab$. 44. $(3a-b)(a^2-b^2)^2$.
 45. $\frac{a(a+1)}{a^2+4a+1}$. 47. $\frac{x^2}{y^3} - \frac{14x}{y} + 45 + \frac{28y}{x} + \frac{4y^2}{x^2}$;
 $\frac{16x^4}{y^4} - \frac{96x^2}{y^2} + 216 - \frac{216y^3}{x^2} + \frac{81y^4}{x^4}$. 48. $\left(\frac{a-b}{a-c}\right)^m$; x^a .
 49. $-(u+c)$. 50. 200. 51. $1+4a+10a^2+20a^3+25a^4+24a^5+16a^6$.
 54. (1) $(x-\frac{7}{3}y)(x+2y)$; (2) $(3-4x)(6+7x)$;
 (3) $(x^2+4ax+8a^2)(x^2-4ax+8a^2)$. 55. (1) $\frac{a}{b}$; (2) 1. 56. 1.
 57. $\frac{1}{2}x^{\frac{3}{2}} + x + \frac{3}{2}x^{\frac{1}{2}} + 2$. 58. $x^{2n} + x^{-3n} + 2$.
 59. $(x^2 + \frac{1}{2}ax + \frac{3}{2}a^2)^2 - (\frac{3}{2}ax + \frac{3}{2}a^2)^2$; $(x^2 + \frac{1}{2}ax + \frac{2}{3}a^2)^2 - (\frac{2}{3}ax + \frac{2}{3}a^2)^2$;
 $(x^2 + \frac{1}{2}ax + \frac{1}{2}a^2)^2 - (\frac{1}{2}ax + \frac{1}{2}a^2)^2$. 60. 350 and 450.
 62. $1-3x^6+3x^{12}-x^{18}$; $(1+x^{\frac{1}{3}})(1+x)(2x+3x^2)$. 63. ax^2+bx-c .
 64. (1) $2xyz(x+y+z)$; (2) $(3a+\frac{8}{3})(a-\frac{4}{3})$; (3) $(x+ay)(x+\frac{y}{a})$.
 65. b^2 . 66. m^2 .
 67. (1). $a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6$;
 (2) $a^6 + a^5x^2 + a^4x^4 + a^3x^6 + x^8$; (3) $16-24x+36x^2-54x^3+81x^4$.
 68. $a^3+b^3+c^3$. 70. £1000. 71. $4x^2-16xy+16y^2$.
 73. $2a^2(a+b)(a-2b)$.
 74. (1) $(x-y+z)^2(x^2+y^2+z^2+2xy+2yz-2zx)$;
 (2) $(x-a)(x+2a)(x^2+ax+2a^2)$;
 (3) $(x-y)(x+y)(x^2+y^2)(x^2-xy+y^2)(x^2+xy+y^2)(x^4-x^2y^2+y^4)$;
 (4) $\frac{3}{2}(1+a)(16-4a+a^2)$. 75. (1) -6 ; (2) $\frac{c+2b}{1+2bc}$.
 76. $\frac{x^2}{(x-a)(x-b)(x-c)}$. 77. 1.

78. $5\{(a+b)x^4 + 2(a^2-b^2)x^3 + 2(a^3+b^3)x^2 + (a^4-b^4)x - a^4b - 2a^3b^2 - 2a^2b^3 - ab^4\}.$
79. $a+b+c.$ 80. 13. 81. $(a^2-ax+x^2)(a^4-a^2x^2+x^4)$
83. $(x-1)^3(x+1)(3x+1)$
84. (1) $-(a-b)(b-c)(c-a)$; (2) $4(7-3x)(9-8x)$;
 (3) $(a-b)^2(a^4+2a^3b+6a^2b^2+2ab^3+b^4)$;
 (4) $(x-y^{-1})(x+y^{-1})(x^2+y^{-2})(x^4+y^{-4}).$
85. 1. 87. $5(x^2+y^2+z^2+yz+zx+xy).$ 88. $y^2+\frac{2}{y}.$
89. $\frac{x}{a}-\frac{2a}{x}-2$; $a\left(1+\frac{1}{2b}\right)-\frac{c}{a}.$ 90. $(a-2b)$ years hence.
92. $2(a+b+c)$ 93. $12a^2b^2(a^2+ab+b^2)(a^4-b^4)$; $(x^2+1)(x+1).$
94. (1) $(a+b+c)(ab+bc+ca)$; (2) $(\frac{2}{3}x^2+\frac{1}{2}xy+\frac{1}{3}y^2)(\frac{2}{3}x^4-\frac{1}{3}xy+\frac{1}{3}y^2)$;
 (3) $\frac{1}{2}(1-2)(x+2)(x^2-2x+4)(x^2+2x+4)$; (4) $\frac{1}{2}a(3a-b)(2a-3b).$
96. 2. 97. $5(a-b)(b-c)(c-a)(a^2+b^2+c^2-bc-ca-ab)$
98. $\frac{x}{y}+\frac{y}{x}.$ 99. $-20a^3$; $25a^3.$ 100. 16

Examples 76 Pages 224-225.

1. 4. 2. 12. 3. 4. 4. $12\frac{1}{2}.$ 5. -2. 6. $2\frac{1}{2}.$
 7. 2. 8. 2. 9. 11. 10. 2c. 11. $-5\frac{2}{3}.$ 12. 4.
 13. o. 14. $4\frac{1}{2}.$ 15. 18. 16. 17. $11\frac{5}{8}.$ 18. $-\frac{1}{2}.$
 19. $\frac{1}{3}.$ 20. $-2\frac{1}{2}.$ 21. 13. 22. 3. 23. $b-c$ 24. o.
 25. 1. 26. $a+b.$ 27. $a+b$ 28. -2. 29. 1. 30. $-2\frac{8}{10}.$
 31. o. 32. -3. 33. o. 34. $\frac{1}{2}.$

Examples 77. Page 226

1. -1. 2. $-5\frac{5}{6}.$ 3. -5. 4. $\frac{1}{2}$ 5. o. 6. -5.
 7. -1. 8. $-\frac{1}{2}b.$ 9. 2. 10. $1\frac{3}{4}.$

Examples 78. Pages 228-229.

1. $\frac{d-b}{a+1}.$ 2. $\frac{2(ab+cd)}{b-a}.$ 3. $\frac{m(n-m)}{2n}.$ 4. $\frac{a^2-b^2-c^2+d^2}{2(ac-bd)}.$
 5. $-\frac{l^2+m^2+n^2}{2(l+m+n)}.$ 6. $-\frac{c^2+a^2}{2c}.$ 7. $-\frac{a^3+b^3+c^3}{3(a^2+b^2+c^2)}.$ 8. 6.
 9. $4\frac{2}{3}.$ 10. $\frac{1}{3}.$ 11. 106. 12. 111. 13. 6. 14. 4.

15. $y=7$. 16. 7. 17. 7. 18. 12. 19. $\frac{1}{2}$. 20. 5.
 21. 5. 22. 10. 23. $1\frac{1}{2}$. 24. $2\frac{1}{2}$. 25. $5\frac{1}{2}$. 26. $-2\frac{3}{8}$.
 27. $-2\frac{1}{3}$. 28. 144. 29. $2\frac{2}{3}$. 30. abc . 31. $\frac{ab^2+bc^2+ca^3}{abc}$.
 32. $\frac{4abc}{ab+bc+ca}$. 33. $\frac{ac+b^2}{b^2+c^2}$. 34. abc . 35. $\frac{ab}{a+b}$. 36. $\frac{ab}{a+b}$.
 37. $\frac{2}{3}$. 38. $2\frac{1}{3}$. 39. 1. 40. $m+n$. 41. $1\frac{7}{8}$.
 42. $\frac{1}{n}\left(\frac{l}{a}+\frac{m}{b}\right)-a^2$.

Examples 79. Page 231

1. $19\frac{1}{2}$. 2. $-3\frac{1}{2}$. 3. $-2\frac{3}{4}$. 4. $-\frac{1}{18}$. 5. $\frac{ac-bd}{ad-bc}$.
 6. $\frac{ab}{3b-2a}$. 7. $\frac{7}{2}$. 8. -1 . 9. $\frac{1}{6}$. 10. $\frac{2}{3}$.
 11. $\frac{kr+n^2}{2mn-(l+m)r-(l-m)k}$. 12. $\frac{pc-qd}{aq+pd-bp}$. 13. 1.

Examples 80. Page 232.

1. 64. 2. 14. 3. 3. 4. 7. 5. $12\frac{9}{17}$. 6. $1\frac{1}{2}$.
 7. 1. 8. $\frac{1}{4}$. 9. $\frac{1}{2}$. 10. 4. 11. $\frac{1}{2}$. 12. $\frac{ab-cd}{c^2-a^2}$.

Examples 81. Pages 236-237.

1. $4\frac{2}{3}$. 2. $\frac{1}{2}(a+b)$. 3. $-\frac{7}{4}$. 4. 0. 5. $1\frac{5}{8}$. 6. 4.
 7. 8. 8. $\frac{7}{108}$. 9. -12 . 10. $1\frac{1}{2}$. 11. -2 .
 12. $\frac{c^2(a+b)-ab(c+d)}{ab-cd}$. 13. 2. 14. $\frac{1}{3}$. 15. $-1\frac{1}{2}$.
 16. $\frac{(m+n)pq-(p+q)mn}{l(mn-pq)}$. 17. $-2\frac{1}{4}$. 18. $-\frac{2ab}{a+b}$. 19. $-2\frac{5}{4}$.
 20. $\frac{c^2-ab}{m(a+b-2c)}$.

Examples 82. Pages 237-238.

1. $-\frac{2}{3}$. 2. $-\frac{1}{3}$. 3. -3 . 4. $-\frac{7}{2}$.
 5. $-\frac{1}{2}$. 6. -5 . 7. $\frac{ab-cd}{a+b-c-d}$. 8. $5\frac{3}{8}$.

Examples 83. Page 239.

1. -4 . 2. $-\frac{1}{14}$. 3. $11\frac{1}{2}$. 4. $-\frac{5}{2}$. 5. $\frac{1}{18}$.
6. $\frac{cl(d-b)-bm(c-d)}{a\{l(b-d)+m(c-d)\}}$. 7. $\frac{(d-b)c^2+(b-a)d^2}{(a-b)d+(b-d)c}$.
8. $\frac{nbp-an}{mcp-bn}$. 9. $\frac{ab}{b-c-a}$. 10. $-\frac{3}{2}$. 11. $\frac{2lbd-m(bc+ad)}{2ma-l(bc+ad)}$.
12. $2\frac{1}{3}$. 13. 3 . 14. 1 . 15. $\frac{3abc}{ab+bc-ca}$.
16. $-\frac{6abc}{ab+bc+ca}$.

Examples 84. Page 242.

1. -7 2. -10 . 3. -2 4. -3 . 5. 8
6. 8 . 7. 4 . 8. $\frac{be-cd}{c^2-ae}$ 9. 1 . 10. -1

Examples 85. Page 244

1. -6 . 2. $\frac{2}{3}$. 3. -4 . 4. $6\frac{1}{2}$. 5. $-\frac{1}{2}(a+b)$.
6. $a+b+c$. 7. $-\frac{2}{3}$. 8. $-a$. 9. 8 . 10. 15 .

Examples 86. Page 245.

1. -4 . 2. 6 3. $\frac{8}{27}$. 4. $1\frac{9}{13}$.
5. -14 6. 3 . 7. $1\frac{1}{18}$. 8. 3 .

Examples 87. Page 247

1. a 2. 1 . 3. a . 4. 0 , or ± 1 .
5. 0 . 6. ± 2 . 7. -3 . 8. $\frac{2}{3}\sqrt{5}$ or $-\frac{2}{3}\sqrt{5}$.

Examples 88. Pages 249-250.

1. 3 . 2. $2\frac{1}{2}$ 3. $-\frac{1}{3}$ 4. $\frac{a^2-c^2}{2(d-ab)}$ 5. $\pm\sqrt{\frac{8}{3}}$
6. 12 . 7. 2 . 8. $\frac{2}{3}$. 9. $\frac{(a-b)^2}{4b}$. 10. 15 .
11. $-\frac{ab}{a+b}$ 12. $\frac{1}{3}$ 13. $-\frac{5a^2+5b^2+6ab}{4(a+b)}$ 14. $\frac{(b+1)^2}{a}$.
15. 2 16. b . 17. $\frac{a}{2}$. 18. $\frac{b}{a}$. 19. $\frac{1}{2}l^2$. 20. 0 . 21. 2

22. 3. 23. 4. 24. 5. 25. 4. 26. 15. 27. 8.
 28. $\left\{ \frac{4bc - (m-b-c)^2}{4a(2m-3c-6b)} \right\}^{\frac{1}{2}}$ 29. 1. 30. 24. 31. $(d-c)^{\frac{1}{n+1}} - a$.
 32. $\frac{3}{2}$. 33. $-\frac{1}{2}$. 34. $-\frac{17}{18}$ 35. 3. 36. $\frac{3}{10}$.
 37. $\frac{ab}{a+b}$ 38. 4. 39. a 40. $\frac{b}{a}$.

Examples 89. Page 251.

1. 3. 2. 4. 3. 5. 4. $\frac{1}{a} \left\{ \left(\frac{b-a+c^2}{2c} \right)^2 - b \right\}$. 5. -5.
 6. 4. 7. $-\frac{13}{8}$ 8. 1. 9. $-\frac{23}{8}$ 10. $-\frac{13}{8}$ 11. $-\frac{45}{4}$ 12. $\frac{13}{2}$.

Examples 90. Page 253.

1. $\sqrt{a^2}$ 2. $\left(\frac{l-m}{\sqrt{l}-\sqrt{m}} \right)^2$ 3. $\frac{4b^2+c^2}{4ac}$ 4. $1\frac{5}{8}$ 5. $6\frac{2}{3}$.
 6. $-\frac{3}{2}$ 7. 1. 8. 1. 9. $\sqrt[4]{\left(\frac{3}{4}\right)a}$ 10. 5.

Examples 91. Pages 253-254.

1. $\pm \frac{4}{3}$ 2. $\pm \sqrt{\left(\frac{2a^2+3a^2-1}{2a+3} \right)}$ 3. $\pm \sqrt{\left\{ \frac{m^2(8m+15r)+r^2(6m-r)}{27n^2} \right\}}$
 4. 1, 4 01 - 2 5. $\frac{216+135a^2+18a^4-a^6}{108}$ 6. 0 01 $\pm \frac{b^2}{a}$.
 7. $\frac{5}{2}$ 01 - $\frac{3}{2}$ 8. $\pm \sqrt{\left(\frac{8b^6+15b^4c^2+6b^2c^4-c^6}{27a^2c^2} \right)}$.

Examples 92. Pages 257-258.

1. 4. 2. $\frac{b(c^2+1)}{2ac}$ 3. $\frac{mc^2-b}{n}$ 4. $\frac{5}{8}$ 5. $\frac{2ab}{b^2+1}$ 6. $\frac{4}{3}$.
 7. $\frac{a(b-1)^2}{2b}$ 8. 2. 9. $\frac{b(c^m+1)}{2ac^m}$ 10. $\pm \frac{1}{m} \sqrt{\left(\frac{2m-n}{n} \right)}$.
 11. $\pm b \sqrt{\left(\frac{a}{2b-a} \right)}$ 12. 2a. 13. 0 14. $\frac{4a}{(1+a)^2}$ 15. $-\frac{1}{4}$.
 16. $\frac{a(b-c)}{b+c}$ 17. $-\frac{1}{2}$ 18. $4(p-1)$ 19. $\frac{65b}{63a}$ 20. $\frac{\sqrt{b^2}-\sqrt{a^2}}{\sqrt{b^2}+\sqrt{a^2}}$.
 21. $\frac{ab^{\frac{2}{3}}}{a^{\frac{2}{3}}-b^{\frac{2}{3}}}$ 22. 1. 23. 2. 24. 2. 25. $\pm a$.

Examples 93. Pages 259—260

1. 7. 2. 4. 3. $\frac{c}{b}$ 4. $\frac{n}{m}$ 5. 6.
 6. $\frac{b+d}{a+c}$ 7. $\frac{3}{2}$ 8. $\frac{5}{2}$ 9. $\frac{n}{3}$ 10. 2.
 11. 2. 12. 3 13. b 14. -4. 15. 0

Examples 94. Pages 260—262.

1. $-\frac{1}{9}$. 2. $\frac{ab+c^2}{2a-b-c}$ 3. $\frac{1}{b}$ 4. $\frac{x-1}{3}$
 5. -35. 6. 2. 7. $-\frac{1}{10}$ 8. $\frac{1}{2}(d-c)$ 9. 3 10. 1.
 11. $\frac{7}{9}$ 12. 1. 13. $-\frac{3}{4}$ 14. $\frac{1}{10}$ 15. 5 16. -2 17. $-\frac{1}{4}$.
 18. $\frac{a}{c-2\sqrt{a}}-b$ 19. $\frac{1}{5}$ 20. $\frac{ad-bc}{b-d}$ 21. $8\frac{3}{5}$ 22. a^2+b+2c .
 23. 3. 24. $\frac{2}{17}$ 25. 9 26. 4 27. 1. 28. $5\frac{1}{2}$ 29. $\frac{1}{2}a$
 30. $-\frac{a+2b}{b}$ 31. $\frac{1}{a+1}$ 32. $\frac{4}{7}$ 33. $\frac{1}{3}(a+b+c)$ 34. -a
 35. $\pm\sqrt{(2a-1)}$ 36. $\pm 4a\sqrt{(a-4a^2)}$ 37. 10 38. $\frac{1}{4}$.
 39. $-\frac{6}{a}$ 40. $\pm\sqrt{\left\{\frac{28b^3+24ab^2+3a^2b-a^3}{27a^2(a+b)}-a^3\right\}}$ 41. -5.
 42. 5. 43. $\frac{4ac^2}{b^2}$ 44. $\frac{3}{4}a^2$ 45. 4. 46. $8\frac{8}{9}$
 47. $\frac{2(\sqrt[3]{a}-1)^2}{\sqrt[3]{a^2+4}\sqrt[3]{a+1}}$ 48. $\frac{d^2-c^2}{d^2+c^2}$ 49. $\frac{a^2}{b}$ 50. 0 or $(2a-1)^{\frac{2a}{1-a}}$.

Examples 95. Pages 266—271

1. 120. 2. 11. 3. 7. 4. $\frac{1}{10}$ 5. $\frac{1}{10}$
 6. 8, 12. 7. 140. 8. A, Rs. 336; B, Rs. 168; C, Rs. 56.
 9. A, £24. 6s.; B, £16. 4s.; C, £12. 3s.
 10. A, £154. 4s.; B, £76. 2s.; C, £50. 8s.; D, £37. 11s.
 11. A, 20; B, 43; C, 132; D, 531. 12. 46 men, 104 women
 13. £1274. 14. £818. 8s 15. Rs. 90000, 4 p.c.; Rs. 73000, 5 p.c
 16. Father, 30; son, 10 yrs. 17. 20. 18. £1.
 19. Rs. 6 per head. 20. Re. 1. 13as. 21. 90; Re. 1.
 22. 15 hours. 23. A, Rs. 180; B, Rs. 175.
 24. 240 rupees, 360 two-anna pieces, 960 pysas.

25. 10 half-crowns, 25 shillings, 50 six-pences, 75 four-pences.

26. $\frac{7}{8}$. 27. $\frac{3}{8}$. 28. $\frac{a+3b}{3b}$.
29. 1710. 30. 16, 18 31. 9s. 32. 1836.
33. Near bank, 20 ft.; at mid-stream, 40 ft. 34. *A*, Rs. 90, *B*, Rs. 30.
35. £100. 36. Rs. 762 37. £1800. 38. 680. 39. 10. 40. 500.
41. 8400. 42. 91, 225. 43. £6660. 44. 3 mds. 20 srs.
45. Gentleman, 59. $4\frac{1}{2}d$. ; servant, 6s. $4\frac{1}{2}d$. 46. 120.
47. Rs. 23. 48. 2s. 49. 30. 50. Rs. 3. 51. £4800; $2\frac{1}{2}d$.
52. £5600. 53. 420. 54. 36. 55. £2. 56. 32.
57. 240 ; 360. 58. 2as ; 16 59. 26. 60. $\frac{1}{2}(a+b+3c)$.

Examples 96. Pages 272—273.

1. 160. 2. 8 days. 3. $\frac{1}{2}th$. 4. 200. 5. 80.
6. 40 women, 110 men. 7. 18 days. 8. 12.

Examples 97. Pages 275—277.

1. 12 days 2. 12 days. 3. 18 days. 4. $22\frac{2}{3}$ days. 5. 40.
6. $4\frac{2}{3} wks.$; £353. 7s. 4d. nearly. 7. $\frac{4}{5}$ day. 8. $2\frac{1}{2}$ days.
9. 12 min. 10. 2 hrs. 48 min 11. $4\frac{2}{3}$ hrs. 12. 24.
13. After $12\frac{1}{2}$ min. 14. $11\frac{2}{3}$ min.

Examples 98. Pages 279—280.

1. 320, 300. 2. 225. 3. Rs. 120000
4. Cost, Rs. 75, Rs. 100 ; selling price, Rs. 72, Rs. 103.
5. Horse, £105 ; carriage, £54. 6. 2080. 7. 2 mds. 8. £2 $\frac{1}{2}$.
9. 10, 12. 10. 16. 11. $\frac{3}{4}d$. ; 512. 12. £89. 8s. 9d. 13. $33\frac{1}{3}$.
14. Rs. 500. 15. Rs. $8\frac{1}{2}$. 16. 1500000. 17. $4\frac{1}{8}$; $\frac{1}{8}th$.
18. $37\frac{1}{2}$ days.

Examples 99. Pages 283—286.

1. 3 hours after start ; 60 miles from Bristol.
2. 5 P. M., 95 miles from Howrah.
3. 4.30 P. M., 180 miles from Howrah.
4. 5 miles. 5. $4\frac{1}{2}$ min., $5\frac{1}{2}$ min. 6. 6 sec 7. 30 sec.
8. $\frac{3}{4}$ of a mile. 9. 360. 10. $\frac{5}{12}$ mile 11. $5\frac{1}{2}$ miles per hour.

12. $4\frac{7}{8}$ miles 13. $2\frac{5}{8}$ miles from P. 14. 3 miles per hour.
 15. $\frac{ac}{a-b}$ ft. per sec. 16. After $\frac{3c}{a+b}$ hrs, at $\frac{(2b-a)c}{a+b}$ miles from A;
 after $\frac{c}{a-b}$ hours, at a distance of $\frac{bc}{a-b}$ miles from B ($a > b$).

Examples 100. Pages 286—287.

1. 10 miles per hour, 8 miles per hour 2. 6 miles per hour.
 3. $48\frac{1}{2}$ miles. 4. 3 hrs 9 min. 5. $\frac{7}{8}$. 6. 8 miles
 7. 7 hrs 12 min. 8. 45 min.

Examples 101. Pages 291—292

- 1 (a) $21\frac{9}{11}$ min past 4; (b) $49\frac{1}{11}$ min past 9.
 2 (a) $42\frac{7}{11}$ min. past 2; (b) $51\frac{5}{11}$ min past 7.
 3 $51\frac{5}{11}$ min. (coincident) and $38\frac{2}{11}$ min. (opposite) past 1.
 4 (a) $21\frac{9}{11}$ min and $54\frac{6}{11}$ min past 1; (b) $10\frac{1}{11}$ min. and $13\frac{7}{11}$ min.
 past 5; (c) $27\frac{8}{11}$ min. past 8, and at 9; (d) $10\frac{1}{11}$ min and
 $43\frac{7}{11}$ min. past 11.
 5. $10\frac{1}{11}$ min. and $32\frac{8}{11}$ min past 4; $51\frac{5}{11}$ min and $49\frac{1}{11}$ min.
 past 11.
 6 (1) $32\frac{8}{11}$ min.; (2) $21\frac{9}{11}$ min 7. $10\frac{1}{11}$ min. after 8. $22\frac{1}{2}$ min.
 9. At $34\frac{6}{11}$ min, again at $58\frac{6}{11}$ past 10; 40 min past 11.
 10. $17\frac{2}{11}$ min. past 5. 11. $2\frac{9}{11}$ min.
 12 $36\frac{1}{11}$ min. past 4; 2 hrs. $46\frac{2}{11}$ min.

Examples 102. Page 294.

1. 81. 2. 120 3. 3072. 4. $4(ab+bc+ca)$.

Examples 103. Pages 296—297.

- 1 4 gal. 2. 24, 8. 3. 27, 18. 4. 8, 7; 7, 8.
 6. Wine, $\frac{a(a^2-b)^2}{a^2+(a-b)^2}$; water, $\frac{a^2}{a^2+(a-b)^2}$ 7. 2 gal.
 8. Vol. of cork = 2; vol. of metal = 8 9. 76 lbs gold; 30 lbs. silver.
 10. $1\frac{2}{3}$ parts of first and 3 parts of second.

Examples 104. Pages 298—299.

1. Height = $1\frac{1}{2}$ ft, base = $4\frac{1}{2}$ ft. 2. Sides = 60 and 61 ft. 3. 10.
 4. 5, 8. 5. 5. 6. 32 rt. angles 7. Perp. = $3\frac{1}{2}$; hyp. = $8\frac{1}{2}$.
 8. 20 ft. 9. $5\frac{1}{2}$ ft. 10. Base = 10 ft.; each side = 13 ft.
 11. 8 ft., 11 ft. 12. $1\frac{7}{8}$ ft.

Examples 105. Pages 304—305.

1. $x=10, y=3$. 2. $x=9, y=6$. 3. $x=3, y=-4$. 4. $x=7, y=2$.
 5. $x=6, y=4$. 6. $x=10, y=8$. 7. $x=2, y=3$. 8. $x=10, y=20$.
 9. $x=\frac{3}{2}, y=\frac{3}{2}$. 10. $x=5, y=10$. 11. $x=13, y=11$.
 12. $x=\frac{1}{2}, y=2$. 13. $x=5, y=2$. 14. $x=1, y=6$.
 15. $x=5, y=-2$. 16. $x=7, y=9$. 17. $x=2, y=4$.
 18. $x=-\frac{1}{6}, y=\frac{1}{6}$. 19. $x=10, y=2$. 20. $x=1, y=-1$.
 21. $x=-\frac{1}{3}, y=\frac{1}{3}$. 22. $x=2, y=3$. 23. $x=3, y=7$.
 24. $x=-\frac{2}{3}, y=\frac{1}{3}$. 25. $x=4, y=2$. 26. $x=\frac{2}{3}b, y=-\frac{1}{3}a$.
 27. $x=c, y=c$. 28. $x=6, y=15$. 29. $x=4, y=5$.
 30. $x=3a, y=-2b$. 31. $x=\frac{ab}{a^2-b^2}(an-bm), y=\frac{ab}{a^2-b^2}(am-bn)$.
 32. $x=a, y=a+b$. 33. $x=\frac{(a+b)(a+c)(b+c-a)}{2abc},$
 $y=\frac{(a-b)(a-c)(a+b+c)}{2abc}.$
 34. $x=ab, y=ac$ 35. $x=a/b, y=b/a$. 36. $x=6, y=-5$.
 37. $x=ac, y=a^2-c^2$. 38. $x=b(a+c), y=a(b+c)$. 39. $x=1, y=10$.
 40. $x=y=5$.

Examples 106. Pages 307—308.

1. $x=2, y=-1$. 2. $x=1, y=6$. 3. $x=3, y=-1$.
 4. $x=\frac{2(b-1)}{2ab-a-b}, y=\frac{2(a-1)}{2ab-a-b}$. 5. $x=\frac{1}{11}, y=2\frac{1}{11}$.
 6. $x=1, y=3$. 7. $x=15, y=25$. 8. $x=7, y=5$.
 9. $x=6, y=5$. 10. $x=3, y=-1$. 11. $x=4, y=1$.
 12. $x=4, y=7$. 13. $x=6, y=12$. 14. $x=2, y=6$.
 15. $x=5, y=2\frac{1}{2}$. 16. $x=3, y=7$. 17. $x=1\frac{2}{3}, y=-\frac{2}{3}$.
 18. $x=2(a+b), y=b+c$. 19. $x=\frac{ac(c-d)}{d(ad-bc)}, y=\frac{bc(c-d)}{d(ad-bc)}$.
 20. $x=2, y=2$. 21. $x=2, y=1$. 22. $x=y=1$.
 23. $x=8\frac{1}{2}, y=7\frac{1}{2}$. 24. $x=8, y=2$.

Examples 107. Page 309.

1. $x = \frac{1}{2}, y = \frac{1}{8}$. 2. $x = 8, y = 15$. 3. $x = \frac{2a}{c+d}, y = \frac{2b}{c-d}$.
4. $x = y = ab$. 5. $x = (pm - nq)/ap, y = (pm - nq)/aq$.
6. $x = 7, y = -\frac{1}{15}$. 7. $x = (ac - bd - c)/c, y = (ac - bd - d)/d$.
8. $x = \frac{a^2 - b^2}{m(ac - b)}, y = \frac{a^2 - b^2}{n(a - bc)}$. 9. $x = \frac{a+b}{2}, y = \frac{a-b}{2}$.
10. $x = 28\frac{1}{2}, y = -19\frac{1}{2}$.

Examples 108. Page 310

1. $x = 2, y = 3$. 2. $x = 1, y = 3$. 3. $x = 2, y = -1$. 4. $x = 1, y = \frac{1}{2}$.
5. $x = -2, y = -\frac{1}{2}$. 6. $x = 1, y = 2$. 7. $x = y = 2$. 8. $x = y = 2$.
9. $x = \left(\frac{b}{a}\right)^{b-a}, y = \left(\frac{b}{a}\right)^{a-b}$. 10. $x = 2, y = 4$.

Examples 109. Pages 312—313.

1. $x = 2, y = 2, z = 1$. 2. $x = 2, y = 3, z = 4$. 3. $x = 2, y = 6, z = 5$.
4. $x = \frac{3}{2}, y = 3, z = \frac{5}{2}$. 5. $x = -4, y = 2, z = -1$.
6. $x = -1, y = 2, z = 4$. 7. $x = 2, y = 3, z = 4$. 8. $x = 14, y = 17, z = 21$.
9. $x = 1, y = 1, z = 1$. 10. $x = 2, y = 3, z = \frac{1}{2}$. 11. $x = 1, y = 1, z = 1$.
12. $x = \frac{1}{4}, y = \frac{1}{3}, z = \frac{1}{5}$. 13. $x = 1, y = -1, z = 1$.
14. $x = (b+c)/2a, y = (c+a)/2b, z = (a+b)/2c$.
15. $x = \frac{1}{2}a(b+c), y = \frac{1}{2}b(c+a), z = \frac{1}{2}c(a+b)$. 16. $x = ab, y = bc, z = ca$.
17. $x = 1, y = 3, z = 5, u = 2$. 18. $x = 2, y = 5, z = 3, u = -2$.
19. $x = 2, y = 3, z = 4, u = 1, v = -1$.
20. $x = 2, y = 3, z = 3, u = -4, v = -1$.

Examples 110. Pages 314—315.

1. $x = 2, y = 4, z = 6$. 2. $x = 1, y = 2, z = 3$.
3. $x = \frac{1}{2}(a+b), y = \frac{1}{2}(c+a), z = \frac{1}{2}(b+c)$. 4. $x = 2, y = 3, z = 4$.
5. $x = \frac{2bc}{b+c}, y = \frac{2ca}{c+a}, z = \frac{2ab}{a+b}$. 6. $x = 2, y = 4, z = 6$.
7. $x = y = z = 1$. 8. $x = y = z = 1$. 9. $x = y = z = \frac{a+b}{c}$.
10. $x = y = z = a(b-c)$. 11. $x = 4, y = 3, z = 4, u = -1$.
12. $x = 5\frac{1}{2}, y = 2\frac{1}{2}, z = \frac{1}{2}, u = 4\frac{1}{2}$.

Examples 111. Pages 318—320.

1. $x=1, y=-2, z=1$.
2. $x=1, y=-3, z=2$.
3. $x=-14, y=-2, z=10$.
4. $x=\frac{7}{2}, y=3\frac{1}{2}, z=-2\frac{1}{2}$.
5. $x=22, y=-15, z=16$.
6. $x=7, y=8, z=9$.
7. $x=-11\frac{1}{2}, y=13\frac{1}{2}, z=6\frac{1}{2}$.
8. $x=-12, y=15, z=-9$.
9. $x=7, y=-3, z=1$.
10. $x=-5\frac{1}{2}, y=2\frac{1}{2}, z=3\frac{1}{2}$.
11. $x=3, y=4, z=5$.
12. $x=8, y=-10, z=12$.
13. $x=8, y=9, z=-12$.
14. $x=\frac{3}{2}, y=-\frac{1}{2}, z=\frac{5}{2}$.
15. $x=1, y=-1$.
16. $x=y=2$.
17. $x=b, y=c-a, z=x-b$.
18. As in Ex. 17.
19. $x=bc(b-c), y=ca(c-a), z=ab(a-b)$.
20. $x=k(b+c-2a), y=k(c+a-2b), z=k(a+b-2c), [k=\frac{1}{2}(a+b+c)]$
21. $x=a(b-a)d/k, y=b(a^2b-1)d/k, z=(1-ab^2)d/k$, where
 $k=1-(ab^2+bc^2+ca^2)+abc+a^2b^2c^2$.
22. $x=y=z=0$.
23. $x=a/bc, y=b/ca, z=c/ab$.
24. $x=a/(b+c), y=b/(a+c), z=c/(a+b)$.
25. $x=a, y=b, z=c$.
26. $x=1, y=2, z=3$.
27. $x=a, y=b, z=c$.
28. $x=2, y=3, z=4$.
29. $x=1/a^2, y=1/b^2, z=1/c^2$.
30. $x=a, y=b, z=c$.

Examples 112. Page 322.

1. $x=1$.
2. $x=92$.
3. $x=10\frac{1}{2}$.
4. $x=1$.
5. $y=-\frac{7}{18}$.
6. $y=abc$.
7. $x=3, y=4, z=5$.
8. $x=3, y=2, z=1$.
9. $x=-\frac{4}{11}, y=3\frac{1}{11}, z=\frac{6}{11}$.
10. $x=\frac{1}{8}, y=\frac{1}{8}, z=\frac{1}{8}$.

Examples 113. Page 325.

1. No.
2. No.
3. No.
4. Yes.
5. No.
6. Yes.
7. Yes.
8. No.
9. 5.
10. -2.

Examples 114. Pages 327—328.

1. $x=1, y=\frac{1}{2}$.
2. $x=\frac{5}{2}, y=2\frac{1}{2}$.
3. $x=\frac{7}{2}, y=\frac{3}{2}, z=\frac{1}{2}$.
4. $x=\frac{3}{2}, y=\frac{3}{2}, z=\frac{3}{2}$.
5. $x=2, y=3, z=4$.
6. $x=a, y=\frac{1}{2}b, z=\frac{1}{2}c$.
7. Either $x=5, y=1, z=4$, or $x=-5, y=-1, z=-4$.
8. Either $x=2, y=5, z=6$, or $x=-2, y=-5, z=-6$.
9. Either $x=2, y=3, z=4$, or $x=-2, y=-3, z=-4$.
10. $x=(a^2+b^2)/2b, y=(a^2-b^2)/2b$.

Examples 115. Pages 328—329.

1. $x = pr/(p^2 - q^2), y = qr/(q^2 - p^2)$. 2. $x = 9, y = 20$.
3. $x = 1, y = 2, z = 3$. 4. $x = -6, y = -4, z = 5$. 5. $x = 1, y = \frac{1}{3}$.
6. $x = 2a, y = 3b, z = 4c$. 7. $x = \pm a, y = \pm b, z = \pm c$.
8. $x = 1, y = 2, z = -3$. 9. $x = 4, y = 2$.
10. $x = 9, y = 4$. 11. $x = 3\frac{3}{8}, y = 1\frac{1}{2}$. 12. $x = \frac{7}{3}, y = 1, z = \frac{1}{3}$.
13. $x = \frac{a}{bc}, y = \frac{b}{ca}, z = \frac{c}{ab}$. 14. $x = \frac{9}{8}, y = \frac{2}{8}$. 15. $x = \frac{8}{9}, y = -\frac{1}{9}$.
16. $x = a + b, y = b + c, z = -(c + a)$. 17. $x = 8, y = 5$.
18. $x = -2, y = -4$ 19. $x = 3, y = 2, z = 1$. 20. $x = 3, y = 4, z = 6$.
21. $x = \frac{1}{2}a, y = \frac{1}{3}b, z = \frac{1}{4}c$. 22. $x = 15, y = 12, z = 1$.

Examples 116. Pages 337—344

1. 9, 12. 2. Tea, 1s. 6d.; coffee, 4s. 6d. 3. 14. 4. Rs. 400.
5. $\frac{1}{2}$. 6. $\frac{1}{18}$. 7. A—30, B—20. 8. A cow, Rs. 20; a sheep Rs. 2
9. No. of 8-anna pieces = 14, that of 4-anna pieces = 52.
10. Rs. 7. 13 as., Rs. 4. 11 as. 11. Horse, Rs. 240; cow, Rs. 120.
12. Gold, 70 oz., silver 30 oz. 13. 15 rich, 85 poor.
14. A, Rs. 45; B, Rs. 48. 15. 40 at Rs. 10, 50 at the other rate.
16. Rs. 46. 8 as. 17. Rs. 325, A; Rs. 150, B; Rs. 275, C.
18. A, £96; B, £60; C, £70 19. 10 as. 20. 20; Rs. 7½.
21. 6; 8 as. 22. Horse, Rs. 300; carriage, Rs. 200.
23. $\frac{(9a-16b)c}{12(a-b)}$ and $\frac{(16a-9b)c}{17(a-b)}$ shillings. 24. 6, 8, 10. 25. 11; 6½.
26. £6 6s. 27. 7s., 5s. 6d. 28. A, 15 qts.; B, 2 qts.
29. 5½. 30. 6. 31. Father, 44 years; wife, 36; child, 20.
32. 4. 33. 3 34. 10 yds., 7 yds. 35. 63 sq. ft.
36. Length = 18 ft, breadth = 12 ft, height = 15 ft.
37. Officer, 4 cwt.; assistant, 10 cwt.; servant, 6 cwt.
38. A, 12s.; B, £1. 8s. 39. 184 and 816. 40. A, 21; B, 42 days.
41. $\frac{(a+b)cd}{c+d}$ days. 42. $\frac{3}{4}$ of a day. 43. 1 hour.
44. 5 miles per hour; 15 miles. 45. 480 miles.
46. $\frac{(ab-bc)cd}{d^2-c^2}$ miles and $\frac{(b^2-ac)cd}{d^2-c^2}$ miles.

47. $2\frac{1}{2}$ miles and $\frac{1}{2}$ mile per hour.
 48. A, $1\frac{1}{3}$ miles per hour; B, 4 miles.
 49. 25 hours. 50. 16 miles per hour; 36 miles.
 51. 3 miles and 4 miles per hr. 52. $7\frac{1}{2}$ and 2 miles per hour.
 53. 600 ft. 54. $64\frac{1}{8}$ ft.; 5 miles per hour.
 55. 10 yds. and 2 yds. per sec. 56. 2 miles. 57. $7\frac{1}{2}$ miles.
 58. 2 miles per hour. 59. 3 miles per hour. 60. A, 20; B, 24 sec.
 61. A, 15 yds. per sec.; B, 12 yds. per sec.
 62. $66\frac{2}{3}$ yds.; A's rate = 8 yds., B's = 5 yds. per sec.

Examples 117. Page 345.

1. 13. 2. 72. 3. 45. 4. 54. 5. 213. 6. 424. 7. 640. 8. 32.

Examples 118. Page 348.

1. 1 : 1. 2. 1 : 1. 3. 3 : 1. 4. $a^2 : c^2; a - 2c : a + 3c$.
 5. 3 : 2 : 3. 6. 1 : 2 : 3; 3 : 5 : 4. 7. 1 : 2.
 8. 8 : 27; 4 : 9. 9. First.

Examples 119. Pages 351—352.

11. 59 : 41. 12. 1 : 1.

Examples 120. Pages 353—354.

11. 3. 12. 12. 13. 18. 14. 4, 16. 15. 6, 36, 216.
 16. $(a^3 - ab + b^3)^2$.

Examples 121. Pages 358—361.

58. $x = 1$. 59. $x = \frac{3ad - 4bc}{3(6a - c) - 2(8b - d)}$ 60. $x = -1$.
 61. $x = 6a - 2b - 2c, y = 6b - 2a - 2c, z = 6c - 2a - 2b$.
 62. $x = b - c, y = c - a, z = a - b$.
 63. $x = 1 / \{2(c + a)(a + b)\}, y = 1 / \{2(a + b)(b + c)\}, z = 1 / \{2(b + c)(c + a)\}$
 64. $x = \frac{a^2 b^2 + c^2 - a^2}{2a^2 + b^2 + c^2}, y = \frac{b^2 c^2 + a^2 - b^2}{2a^2 + b^2 + c^2}, z = \frac{c^2 a^2 + b^2 - c^2}{2a^2 + b^2 + c^2}$.
 65. 1 : 2. 66. 20 and 30 miles from the stations.
 67. 3000; 5000. 68. 5. 69. 7, 8, 9, 10. 70. 40; 10.
 71. 8; 14. 72. $\frac{2}{3}(307)$ in. 73. $17\frac{1}{2}$ ft., $12\frac{1}{2}$ ft.
 74. 15, 18, 27; Rs. 10, Rs. $6\frac{2}{3}$, Rs. $3\frac{1}{2}$. 75. A, £25; B, £20.

Examples 123. *Pages 371—372.*

1. $b^2 = ac$. 2. $a^2 a^2 = b^2 c^2$. 3. $4ab = c^2 - a^2$. 4. $a^2 - b^2 = 3a$.
5. $(b_1 c_2 - b_2 c_1)(a_1 b_2 - a_2 b_1) = (c_1 a_2 - c_2 a_1)^2$.
6. $(b c_1 - b_1 c)^2 (ab_1 - a_1 b) = (ca_1 - c_1 a)^4$.
7. $a_2(b_1 c_2 - b_2 c_1) + b_3(c_1 a_2 - c_2 a_1) + c_3(a_1 b_2 - a_2 b_1) = 0$. 8. $a^2 + b^2 = 1$.
9. $a^3 + 2c^3 = 3ab^2$. 10. $(a+b)^n + (a-b)^n = c^n$. 11. $bc = a + 2$.
12. $a^2 + b^2 + c^2 = 4 + abc$. 13. $a^3 + b^3 + c^3 = 3abc$.
14. $lm + mn + nl + 1 = 0$. 15. $l^3 + 3p^3 = n^3 + 3lm^2$.
16. $abc = a^2(a+b+c+2d)$. 17. $(a+b+c-4)^2 = abc$. 18. $a^3 = b^2 + 3c^2$.
19. $\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} = 1$. 20. $\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} = 1$.

Examples 127. *Page 384.*

6. Second. 7. Second. 8. First. 9. First.

Examples 128. *Page 386.*

1. $x^4 + 2x^3 + 9$. 2. $x^6 - 2x^5 - 4x^4 + 10x^3 - x^2 - 8x + 4$
3. $x^4 - 5x^2y + 14x^2y^2 - 19xy^3 + 15y^4$. 4. $x^8 - x^6y - 2xy^5 - 4y^4$.
5. $256a^7 + 64a^6 + 64a^5 + 1$. 6. $a^6 - b^6$.
7. $a^4 - 2a^3 + 4a^2 - 8a + 17$. 8. $2a^2 + 3a^3 + 10a + 4$
9. $x^3 - 2xy + 4y^3$; $16x^4 - 8x^3y + 4x^2y^2 - 2xy^3 + y^4$.
10. $a^4 - 5a^2b + 4b^2$; $a^8 + a^6b + a^4b^2 + a^2b^3 + b^4$.

Examples 129. *Pages 387—388.*

1. $2(x^2 + y^2 + z^2) + 5(xy + yz + zx)$. 2. $(a-b)(b-c)(c-a)(a+b+c)$.
3. $3ab(a+b)$. 4. $(b-c)(c-a)(a-b)(bc+ca+ab)$.
5. $5(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)$.
6. $-(a-b)(b-c)(c-a)$. 7. $(a+b)(b+c)(c+a)$.
8. $3(a+b)(b+c)(c+a)$. 9. $24xyz$.
10. $4xyz$. 11. $2xyz(x+y+z)$.

Miscellaneous Examples III. *Pages 388—404.*

1. $\frac{1+a^2b^2+4b^2c^2+9c^2a^2}{a^2+b^2+c^2}$. 2. $2x^{\frac{1}{2}} - 8x^{\frac{1}{4}} - 1$. 3. $a=6, b=7$.
4. (1) $(x-1)\{a(b-c)x - c(a-b)\}$; (2) $(a-b)(b+c)(c+a)(a+b-c)$;
(3) $(2a+1)(a^2-2a+4)$.

9. (1) $3\frac{1}{2}$; (2) 5; (3) $x=\frac{1}{2}, y=1, z=\frac{1}{2}$.
10. $4a(a+b)$. 11. $(a+b)(b+c)(c+a)$. 13. 8; $\frac{x-4}{x-5}$.
14. (1) $(a-b)(a+b)^3$; (2) $(a-b)(a+b)(c-a)(c+a)(b^3+c^3)$;
(3) $(2x+1)(x^2-2x+1)$. 15. abc .
19. (1) $3a$; (2) $x=\frac{1}{2}, y=\frac{1}{2}, z=\frac{1}{2}$; (3) $x=ac, y=bc$. 20. $3\frac{7}{8}$.
21. x^2+3y^2 . 22. $a^2b^2x^4-a^2b^2c(n+b)x^3+c^3(a+b)x-c^4$; $\left(\frac{a-b}{a+b}\right)^4 \cdot c^4$.
24. (i) $(a^{x-y}+1)(a^{x+y}+1)$;
(ii) $(a^{-1}-b^{-1})(a^{-2}+a^{-1}b^{-1}+b^{-2})(a^{-6}+a^{-3}b^{-3}+b^{-6})$;
(iii) $(4x^{\frac{1}{6}}+1)(2x^{\frac{1}{6}}-1)$. 25. 0. 26. $2a^{\frac{1}{2}}b^{\frac{1}{2}}-2b^{\frac{3}{2}}/3c^{\frac{1}{2}}+4b^{\frac{5}{2}}/c^{\frac{3}{2}}$.
29. (1) $1/a$; (2) $x=b+c-a, y=c+a-b, z=a+b-c$. 30. $3\frac{3}{4}$ stone.
32. x^2+x^2+x+1 . 33. (i) $(a^x+b^y)(a^y+b^x)$; (ii) $(2x^{\frac{1}{3}}-3)(3x^{\frac{1}{3}}-2)$.
39. (1) m^2-4m+3 ; (2) $x=\frac{1}{2}, y=\frac{1}{2}, z=\frac{1}{2}$; (3) $x=y=z=0$.
40. 4 hrs. 41. $(1+a+a^2)^4+2(1+a^2b^{\frac{1}{2}})^3/m+1\{3+(a+b)$
 $-2a^{\frac{1}{2}}b^{\frac{1}{2}}+3ab\}/2m^2+2(1+a^{\frac{1}{2}}b^{\frac{3}{2}})/m^3+(1+b+b^3)m^4$.
43. (i) $(a^2-2ab+4b^2)^2-(2ab)^2$; (ii) $(3x^2+4yx)^2-4x^2(3y+x)^2$.
44. (i) $(e^x-1)^2(e^x+1)$; (ii) $(ax-x-1)(ax-x+1)(ax+x-1)$
 $(ax+x+1)$.
45. $a^{2x-1}-a^{2x-2}b+a^{2x-3}b^2-\dots-a^{2x-2}b^{2x-2}+b^{2x-1}$.
49. (1) $\frac{1}{2}a$; (2) $x=2, y=1, z=0$. 50. Gold, £6; silver, 10s.
51. $2a^3+3a^2-5$. 52. (i) $5(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)$;
(ii) $(x-a)\{(a-b)x-b(2a+b)\}$.
55. 1. 56. $a^x c^{y-1}-4ab^{\frac{2}{3}}-\frac{3d}{c^{y+1}}$.
59. (1) $a\sqrt{\left(\frac{b-2a}{4a+b}\right)}$; (2) $x=b-c, y=c-a, z=a-b$.
60. At first, no. of males; no. of females = 5; 4.
61. $(a/b+b/a)^3$. 62. $(ax+by)(ay-bx)$. 63. $a+b+c$.
64. (i) $(2x^2-3a^2)(2x^2+3a^2)(4x^4+9a^4)$; (ii) $(a^{x+y}+b^{x-y})(a^{x-y}+b^{x+y})$;
(iii) $(ax+by+c)(a'x+b'y+c')$. 66. 3.
69. (1) 1; (2) $x=5, y=2$; (3) $x=a, y=b, z=c$.
70. $9\frac{1}{2}$ lbs., $11\frac{1}{2}$ lbs. 71. $3(b^3+bc+c^3)(c^3+ca+a^3)(a^3+ab+b^3)$.
72. 3; $2x+3y$. 73. 4^6 ; 1. 74. $(a^3-a^4+1)(a^3+a^4+1)$.

78. $a^2 + b^2 + c^2 + 2abc = 1$. 79. (i) 1; (2) $x = y = 4$, $z = 2$. 80. 230.
81. $1 + x^2 + x^4 + \dots + x^{4n}$. 82. (i) $-(a - \frac{1}{2}b)(b - c)(c - a)$;
(ii) $(x - 1)(x - 2)(x - 3)(x - 4)$; (iii) $(2x + 3y - 5z)(x + y + 1)$.
84. $16x^5/(1 - x^{10})$. 85. $x^2\sqrt[3]{16 + 2x - x^3}\sqrt[3]{4 + \frac{2}{3}}\sqrt[3]{2 + 1}$.
87. $3a^{\frac{2}{3}} - \frac{1}{8}ab^{\frac{1}{3}} + \frac{1}{8}a^{\frac{1}{3}}b$. 88. 1; $(\sqrt[3]{a} \cdot \sqrt[3]{b})^6 \cdot (\sqrt[3]{b} \cdot \sqrt[3]{c})^8 \cdot (\sqrt[3]{c} \cdot \sqrt[3]{a})^{12}$.
89. (1) $\frac{1}{4}(a + b)^2 - c$; (2) $x = 4$, $y = 2$; (3) $x = 2$, $y = 3$, $z = 4$.
90. 6 miles. 91. $a^4 - b^4 + 4b^3c - 6b^2c^2 + 4b^2c^3 - c^4$.
93. (i) $(x^2 - 2x^{-2})(x^2 + 2x^{-2})(x^2 + 2x^{-2} + 2)(x^2 + 2x^{-2} - 2)$;
(ii) $(x^n - 2)(1 + 2x^{-n})(x^n + 3x^{-n})$;
(iii) $(a + b + 1)(a + b - 1)(1 + a - b)(1 - a + b)$. 94. $(x^2 - 1)(x^{\frac{2}{3}} + 1)$.
97. 4 : 2 : 3. 99. (1) 4; (2) $x = 10$, $y = 6$. 100. £2000. 101. 0
102. (i) $(a + b)^2 + (1 - ab)^2$, or $(a - b)^2 + (1 + ab)^2$;
(ii) $(dc + bd)^2 + (bc - ad)^2$, or $(ac - bd)^2 + (bc + ad)^2$.
103. (i) $(a + 2b - 2)(a - 2b - 2)$; (ii) $(a + 2b + 2)(a - 2b + 2)$;
(iii) $(5x^m + 7a^n)(2x^m - 3a^n)$.
104. $z - \frac{z}{xy} - xy + 1$. 105. $\frac{1}{abc}$.
109. (1) $\frac{1}{3}$; (2) $x = \frac{a(b + c + 3a)}{b + c - a}$, $y = \frac{b(c + a + 3b)}{c + a - b}$, $z = \frac{c(a + b + 3c)}{a + b - c}$.
110. Number of males = $\frac{a - c}{b - c} \cdot p$; number of females = $\frac{b - a}{b - c} \cdot p$.
112. (i) $(x - 1)(x - 3)(x + 5)$; (ii) $(x - 1)^2(x + 1)(x - 3)$;
(iii) $(2x + 5y - b)(2x - 5y + b)$. 113. $3x(10x + 7y)$; $-9xy$.
114. $(a^6 + b^6)^{\frac{2}{3}} + (a^6 - b^6)^{\frac{2}{3}} = 2c^2$. 115. $x^{\frac{1}{3}} + 3x^{\frac{1}{3}}y^{\frac{2}{3}} - 9y^{\frac{2}{3}}$.
116. $\frac{1}{2}(ab + a^{-1}b^{-1})$. 119. (1) $\frac{1}{2}b\sqrt[3]{4a - b^3}$; (2) $x = 1$, $y = a$, $z = a^2$.
120. 38 sovereigns, 19 half-sovereigns, 14 half-crowns, 5 shillings.
121. 8. 123. $\left(\frac{c + d}{a + b}\right)^2 + \left(\frac{c - d}{a - b}\right)^2 = 2l^2$. 125. $z - y$.
128. $x^2 - 2x + 1 - 2x^{-1}$. 129. (1) 8; (2) $x = 4$, $y = 5$, $z = 6$.
130. 15 men, 18 women, 27 children
131. Quotient = $2a - 4b + c$; product = $2a^2 + ac - 8b^2 + 2bc$.
132. $a^{22} + a^{20}b^2 + a^{28}b^4 + \dots + b^{32}$.
134. (i) $ab(2 + a)(2 + b)$; (ii) $(5a^{\frac{1}{3}} + 2)(3a^{\frac{1}{3}} + 4)$;
(iii) $(a - b)(a + b + 1)(a + b - a^2 - b^2)$. 135. $(a + b + c)^2$.

139. (1) $\frac{b}{c} \cdot \sqrt{(b^2 + c^2)}$; (2) $x = \frac{2abc}{bc + ca - ab}$, $y = \frac{2abc}{ab + bc - ca}$,
 $z = \frac{2abc}{ca + ab - bc}$; (3) $x = 104$, $y = 209$, $z = 419$.
140. $293\frac{1}{2}$ ft. 143. (i) $(1-x)(1-y)(1+x)(1+y)$;
(ii) $(x-1)(x-2)^2(x-3)$; (iii) $(3x+5y-9)(4x+7y-6)$.
144. 1. 145. $bc + ca + ab$. 146. $a^2(a^3 + b^3 + c^3 + 2a^2b) = a^2b^3c^3$.
149. (1) $a + b + c$; (2) $x = \frac{1}{2}(2a + b + c)$, $y = \frac{1}{2}(a + 2b + c)$, $z = \frac{1}{2}(a + b + 2c)$.
150. $12\frac{1}{2}$ miles. 151. $\frac{1}{2}ab$. 152. $(a^7 + 6a^5 + 10a^3 + 4a)\sqrt{(a^2 + 4)}$.
153. (i) $(1+a)(1+b)(1-a)(1-b)(1+a^2)(1+b^2)$; (ii) $3(a+b)(b-c)(c+a)$;
(iii) $(x-1)(x+2)(x^2+x+1)(x^2-2x+4)$.
159. (1) $x = 4b + 2a$, $y = \frac{5}{3}b$; (2) $x = 69$, $y = 40$, $z = -2$.
160. $16 : 6 : 2 : 5$. 161. $3(x+y)^2 + z^2$.
162. (i) $(a-b)(b-c)(a-c)$, (ii) $(x-1)^2(x^2+x-3)^2$,
(iii) $(ax - by + 1)(bx - ay + 2)$.
164. $2\sqrt[3]{(x/y)} - 1 + 2\sqrt[3]{(y/x)}$ 165. $\sqrt{(a^2 + b^2 - a^2)}$.
169. (1) $3a$; (2) $x = y = z = 1$ 170. 4. 171. 5.
172. $1 + \frac{x}{a} + \frac{1}{2}\frac{x^2}{a^2} + \frac{1}{6}\frac{x^3}{a^3}$; $\frac{ab}{c} - \frac{2c^2}{ab}$ 173. $\frac{3x-8}{x^2-7}$ 175. $a-b$.
176. (i) $(2a + 2b + c)^2 - (a - 2c)^2$; (ii) $(x^2 + 6ax - \frac{1}{2}a^2)^2 - (\frac{3}{2}a^2)^2$.
179. (1) 1; (2) $x = 2$, $y = 3$, $z = -1$. 180. 1 hour. 182. $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}$.
184. 1. 189. (1) $3\cdot5$; (2) 3, 2. 190. 99 yds., 77 yds.
193. $a + b + c$. 194. $\frac{1}{2}$. 197. $a^2 + b^2 + c^2 = 2(ab + bc + ca) + abc$.
199. (1) $\frac{a^2 + b^2 + c^2 + ab + bc + ca}{a + b + c}$; (2) $x = 14$, $y = 13$;
(3) $\frac{b(c' - c) + d(a' - a)}{ac' - a'c}$. 200. $9\frac{9}{10}$ miles per hour.

UNIVERSITY PAPERS.

SELECTED QUESTIONS FROM INDIAN UNIVERSITIES.

Find the value of—

$$1. \frac{4y}{5} (y-x) - 35 \left[\frac{3x-4y}{5} - \frac{1}{10} \left\{ 3x - \frac{5}{7} (7x-4y) \right\} \right],$$

when $x = -\frac{1}{2}$ and $y = 2$.

[Ans. 94.]

$$2. \text{ Add together } x^2 - (x-y+z)(x+y-z), y^2 - (y-x+z)(y+x-z), \\ \text{ and } z^2 - (z-x+y)(z+x-y). \quad [\text{Ans. } 2x^2 + y^2 + z^2 - yz - zx - xy].$$

Multiply :—

$$3. x^{\frac{1}{2}}y + y^{\frac{2}{3}} \text{ by } x^{\frac{1}{3}} - y^{\frac{1}{2}}. \quad [\text{Ans. } x^{\frac{1}{2}}y + x^{\frac{1}{3}}y^{\frac{2}{3}} - x^{\frac{1}{2}}y^{\frac{5}{6}} - y.]$$

$$4. x+y+z - \sqrt{(yz)} - \sqrt{(xz)} - \sqrt{(xy)} \text{ by } \sqrt{x} + \sqrt{y} + \sqrt{z} \\ [\text{Ans. } \sqrt{x^3} + \sqrt{y^3} + \sqrt{z^3} - 3\sqrt{(xyz)}.]$$

$$5. x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \text{ by } x^{\frac{1}{3}} - y^{\frac{1}{3}}. \quad [\text{Ans. } x - y.]$$

$$6. a^{2n} - a^n x^n + x^{2n} \text{ by } a^n + x^n. \quad [\text{Ans. } a^{3n} + x^{3n}.]$$

$$7. \text{ Square } a^{\frac{1}{2}} - b^{\frac{1}{2}} + c^{\frac{1}{2}}. \quad [\text{Ans. } a^2 + b^2 + c^2 - 2a^{\frac{1}{2}}b^{\frac{1}{2}} + 2a^{\frac{1}{2}}c^{\frac{1}{2}} - 2b^{\frac{1}{2}}c^{\frac{1}{2}}.]$$

$$8. \text{ Add together the squares of} \\ 2[\sqrt{ab(1+a)(1+b)}] + \sqrt{ab(1-a)(1-b)} \text{ and} \\ \{a + \sqrt{(1-a^2)}\}\{b - \sqrt{(1-b^2)}\} - \{a - \sqrt{(1-a^2)}\}\{b + \sqrt{(1-b^2)}\}, \\ \text{and simplify the result.} \quad [\text{Ans. } 4(a+b)^2.]$$

Simplify the following expressions :—

$$9. (x-y)' + (x+y)'' + 3(x-y)^2(x+y) + 3(x+y)^2(x-y). \quad [\text{Ans. } 8x^3.]$$

$$10. \left\{ \frac{(x^m)^{\frac{1}{p}} \cdot (x^n)^{\frac{1}{q}}}{\sqrt[n]{x^p} \sqrt[m]{x^q}} \right\}^{pq} \quad [\text{Ans. } x^{\frac{(mp+qn)(mn-qp)}{mn}}.]$$

$$11. (16x^5 - 20x^3 + 5x)^2 + (1-x^2)\{16(1-x^2)^2 - 20(1-x^2) + 5\}^2. \quad [\text{Ans. } 1.]$$

Divide :—

$$12. x^6 + 2x^3y^3 + y^6 \text{ by } (x+y)^2. \quad [\text{Ans. } (x^2 - xy + y^2)^3.]$$

$$13. x^{\frac{3}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}} + y \text{ by } x^{\frac{3}{2}} - x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{1}{2}}. \quad [\text{Ans. } x^{\frac{3}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{1}{2}}.]$$

$$14. a^7 - a^6 \text{ by } a - x. \\ [\text{Ans. } a^7 - a^6x + a^5x^2 - a^4x^3 + a^3x^4 - a^2x^5 + ax^6 - x^7.]$$

$$15. x^3 + a^4x^4 + a^5 \text{ by } x^2 - ax + a^2. \quad [\text{Ans. } (x^4 - a^2x^2 + a^4)(x^2 + ax + a^2).]$$

$$16. (x+y+z)(xy+xz+yz) - xyz \text{ by } x+y. \quad [\text{Ans. } (y+z)(z+x).]$$

- ✓ 17. $x^8 + x^6y^2 + x^4y^4 + x^2y^6 + y^8$ by $x^4 - x^2y + x^2y^2 - xy^3 + y^4$.
[Ans. $x^4 + x^2y + x^2y^2 + xy^3 + y^4$.]
18. $\frac{a^3}{b} + \frac{b^3}{a}$ by $\frac{a}{b} + \frac{b}{a}$. [Ans. $\frac{a^3}{b^2} - 1 + \frac{b^3}{a^2}$.]
- ✓ 19. $x^{2n} - y^{2n}$ by $x^{n-1} + y^{n-1}$. [Ans. $x^{n-1} - y^{n-1}$.]
20. $(a_1 + by)^4 + (a_1 - by)^4 - (ay - bx)^2 + (ay + bx)^2$
by $(a + b)^2x^2 - 3ab(x^2 - y^2)$. [Ans. $2(a + b)x$.]
- 21. $x - 6a^{\frac{1}{2}}x^{\frac{3}{2}} + 6a^{\frac{3}{2}}x^{\frac{5}{2}} + a + 5a^{\frac{3}{2}}x^{\frac{3}{2}} + 7a^{\frac{5}{2}}x^{\frac{5}{2}}$ by $x^{\frac{1}{2}} - a^{\frac{1}{2}}$.
[Ans. $x^{\frac{1}{2}} - 5a^{\frac{1}{2}}x^{\frac{3}{2}} + 6a^{\frac{3}{2}}x^{\frac{5}{2}} + a^{\frac{1}{2}}$.]
22. $(x^2 - yz) + 8yz^3$ by $x^2 + yz$. [Ans. $x^4 - 4x^2yz + 7y^2z^2$.]
23. $(4x^3 - 3a^2x)^2 + (4y^3 - 3a^2y)^2 - a^4$ by $x^2 + y^2 - a^2$.
[Ans. $16(x^4 - x^2y^2 + y^4) - 8a^2(x^2 + y^2) + a^4$.]
24. $1 + a + a^2 + a + a^3 + a^6 + a^7 + a^8 + a^{10} + a^{11}$ by $1 - a^5 + a^6$.
[Ans. $1 + a + a^2 + a^3 + a^4 + a^5 + a^6 + a^7 + a^8 + a^9$.]
25. $a + 8b^3 + 27c^3 - 18abc$ by $a^2 + 4b^2 + 9c^2 - 6bc - 3a - 2ab$.
[Ans. $a + 2b + 3c$.]
26. 1 by $(a + b)$ giving three terms of the quotient.
[Ans. $a^{-2} - 2a^{-1}b + 3a^{-2}b^2$.]
27. $x^6 + y^6$ by a number which is greater than x by y .
[Ans. $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.]
28. The difference of $(x^2 - bx + b^2)(x + c - b)$ and $(x^2 - ax + a^2)(x - a - b)$
by $a - b$. [Ans. $3x^2 - 2(a + b)x + a^2 - b^2$.]
29. The product of $y^3 - 12y + 16$ and $y^3 - 12y - 16$ by $y^3 - 16$
[Ans. $(y^3 - 4)$.]
30. The product of $(b + c)^2 - a^2$ and $a^2 - b^2 - c^2 + 2bc$ by $b^2 - (c - a)^2$.
[Ans. $c^2 + a^2 + 2ac - b^2$.]
31. The continued product of $1 + x + y$, $1 + x - y$, $1 - x + y$ and $1 + y - 1$
by $1 + 2xy - x^2 - y^2$. [Ans. $(x + y)^2 - 1$.]
- Resolve into factors—
32. $x^{12} - a^{12}$.
[Ans. $(x - a)(x^2 + ax + a^2)(x + a)(x^2 - ax + a^2)(x^2 + a^2)(x^6 - a^2x^3 + a^4)$.]
33. $4(uz - xy)^2 - (u^2 - x^2 - y^2 + z^2)^2$.
[Ans. $(u + x + y + z)(u + x - y - z)(u - x - y + z)(x - y + z - u)$.]
34. $(1 - c^2)(1 + a)^2 - (1 - a^2)(1 + c)^2$. [Ans. $2(a - c)(1 + a)(1 + c)$.]
35. $8x^3 + 729y^3$. [Ans. $(2x + 9y^3)(4x^3 - 18xy^3 + 81y^6)$.]
36. $x^5 + 7x^3 - 5x^2 - 35$. [Ans. $(x^2 + 7)(x^3 - 5)$.]
37. $(x^2 + 4x)^2 - 2(x^2 + 4x) - 15$. [Ans. $(x - 1)(x + 1)(x + 3)(x + 5)$.]

38. $a^2 - 4b^2 - c^2 + 9d^2 - 2(ad - 2bc).$

[Ans. $(a + 2b - c - 3d)(a - 2b + c - 3d).$]

39. $x^{16} - y^{16}.$

[Ans. $(x - y)(x + y)(x^2 + y^2)(x^4 + y^4)(x^8 + y^8)$]

40. $x^4 - (p^2 + 2)x^2y^2 + y^4.$

[Ans. $(x^2 + pxy - y^2)(x^2 - pxy - y^2).$]

41. $(2a + 2b - ab)^2 - (b^2 - 4a)(a^2 - 4b).$

[Ans. $4(a - b)^2(a + b + 1).$]

Resolve into three factors—

42. $x^3 - 2x^2 - 23x + 60$

[Ans. $(x - 3)(x - 4)(x + 5).$]

43. $a(b - c)^2 + b(c - a)^2 + c(a - b)^2 + 8abc.$

[Ans. $(b + c)(c + a)(a + b).$]

44. $(x + 1)(x + 3)(x + 5)(x + 7) + 15.$

[Ans. $(x + 2)(x + 6)(x^2 + 8x + 10).$]

Resolve into four factors—

45. $a(b^2 - c^2) + bc(c^2 - b^2) - a^2(c - b).$

[Ans. $(b - c)(c - a)(a - b)(a + b + c).$]

46. $(a^2 - b^2)^2 + (c^2 - d^2)^2 - (a + b)^2(c - d)^2 - (a - b)^2(c + d)^2.$

[Ans. $(a - b + c + d)(a + b - c - d)(a - b - c + d)(a - b + c - d).$]

47. $a^2x^2 - \frac{8a^2}{y} - x^2 + \frac{8}{y}.$

[Ans. $(a + 1)(a - 1)\left(x - \frac{2}{y}\right)\left(x + \frac{2}{y} + \frac{4}{y^2}\right).$]

48. Find the G.C.M. of $20a^4 - 3a^3b + b^4$ and $64a^4 - 3ab^3 + 5b^4.$

[Ans. $4a^2 - 3ab + b^2.$]

And prove the rule for finding the G.C.M. of two numbers a and $b.$

49. Find the L.C.M. of

$x^5 - 5x^3 + x^2 + 4x - 4$ and $x^4 + x^3 - 6x^2 - 4x + 8.$

[Ans. $(x^2 - 4)(x^2 + x - 2)(x - x + 1).$]

50. $2(x - 2)^2, 2x^2 - 8, x^3 + 2x$ and $2x^2 + 4x.$

[Ans. $2x(x - 2)(x^2 - 4)(x^2 + 2).$]

51. $x^6 - 4x^4 + 4x^2 - 16x^4$ and $x^6 + 2x^4 - 8x^2 - 16x^4.$

[Ans. $x^6 + 4x^4 - 16x^2 - 32x^4 + 64x^6$]

52. Find the expression of lowest dimensions which is exactly divisible by $a^2b - b(b - c)^2, ac - a(a - b)^2$ and $(a + c)^2c - b^2c.$

[Ans. $abc(a + b + c)(b + c - a)(c + a - b)(a + b - c).$]

Reduce each to its lowest terms—

53. $\frac{2x^4 - x^3 - 9x^2 + 13x - 5}{7x^3 - 19x^2 + 17x - 5}.$

[Ans. $\frac{2x^2 + 3x - 5}{7x - 5}.$]

54. $\frac{10x^2 + 10x^2 - 9}{25x^3 - 19x + 6}.$

[Ans. $\frac{2x + 3}{5x - 2}.$]

Simplify :—

55. $\frac{1}{x - 1} - \frac{x - 5}{x^2 - 7x + 10} + \frac{x - 6}{x^2 - 9x + 18}.$

[Ans. $\frac{1}{(x - 1)(x - 2)(x - 3)}.$]

56. $\frac{x^3}{(x - y)(x - z)} + \frac{y^3}{(y - z)(y - x)} + \frac{z^3}{(z - x)(z - y)}.$

[Ans. $\frac{x + y + z}{x - y - z}.$]

$$57. \frac{1}{x(x-y)(z-y)} + \frac{1}{y(y-z)(x-y)} + \frac{1}{z(z-x)(x-y)} \quad [Ans. \frac{1}{xyz}]$$

$$58. \sqrt{\frac{x^2-yz}{(x-y)(x-z)} + \frac{y^2+zx}{(y-z)(y-x)} + \frac{z^2+xy}{(z-x)(z+y)}} \quad [Ans. 0]$$

$$59. \frac{x+3y}{4(x+y)(z+2y)} + \frac{z+2y}{(x+y)(z+3y)} - \frac{z+y}{4(z+2y)(z+3y)} \quad [Ans. \frac{1}{z+y}]$$

$$60. \left(\frac{a+b}{a-b} + \frac{a^2+b^2}{a^2-b^2} \right) - \left(\frac{a-b}{a+b} - \frac{a^3-b^3}{a^3+b^3} \right) \quad [Ans. -\frac{(a^4+a^2b^2+b^4)}{ab(a-b)^2}]$$

$$61. \frac{x}{x-a} - \frac{1}{x+a} - \frac{z-a}{x+a} - \frac{x+a}{z-a} \quad [Ans. \frac{4a^2z}{x^4-a^4}]$$

$$62. \frac{x^a+b, x^a-b, x^{a-2b}}{x^{a-b}} \quad [Ans. 1^a]$$

$$63. \left(1 - \frac{1}{1+x} \right) \left(x + \frac{1}{2+x} \right) \times \frac{1}{1+\frac{1}{x}} \div \left(1 + 1 + \frac{1}{x} \right) \quad [Ans. \frac{x(1-x)}{2+x}]$$

$$64. \frac{\frac{1+x}{1-x} + \frac{4x}{1+x^2} + \frac{8x}{1-x^2} - \frac{1-x}{1+x}}{\frac{1+x^2}{1-x^2} + \frac{4x^2}{1+x^4} - \frac{1-x^2}{1+x^2}} \quad [Ans. \frac{(2+x^2)(1+x^4)}{x}]$$

$$65. \frac{x^{m+2n} \cdot x^{3m-8n}}{x^{5m-8n}} \quad [Ans. 1^{-m}]$$

$$66. \frac{\frac{a^3}{b^3} - \frac{b^3}{a^3}}{\left(\frac{a}{b} - \frac{b}{a} \right) \left(\frac{a}{b} + \frac{b}{a} - 1 \right)} \times \frac{\frac{1}{b} - \frac{1}{a}}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{ab}} \quad [Ans. a-b]$$

$$67. \frac{x+2}{1+x+x^2} - \frac{x-2}{1-x+x^2} - \frac{2x^2-4}{1-x^2+x^4} \quad [Ans. \frac{4x^4+b}{x^8+x^4+1}]$$

$$68. \frac{1}{(4x^2-3x)^2} - \left\{ \frac{3 \sqrt{1-x^2} - \frac{(1-x^2)^2}{x^3}}{1-3\left(\frac{1-x^2}{x^2}\right)} \right\}^2 \quad [Ans. 1]$$

$$69. \frac{x-y}{x-z} + \frac{x-z}{z-y} - \frac{(y-z)^2}{(x-y)(z-y)} \quad [Ans. 2]$$

$$70. \frac{1}{abx} + \frac{1}{a(a-b)(x-a)} + \frac{1}{b(b-a)(x-b)} \quad [Ans. \frac{1}{x(z-a)(x-b)}]$$

$$71. \left\{ \sqrt{\left(\frac{a+x}{x}\right)} - \sqrt{\left(\frac{x}{a+x}\right)} \right\}^2 - \left\{ \sqrt{\left(\frac{x}{a}\right)} - \sqrt{\left(\frac{a}{x}\right)} \right\}^2 + \frac{1^2}{a(a+x)^2} \quad [\text{Ans. 1.}]$$

$$72. \frac{(b+c)(x^2+a^2)}{(c-a)(a-b)} + \frac{(c+a)(x^2+b^2)}{(a-b)(b-c)} + \frac{(a+b)(x^2+c^2)}{(b-c)(c-a)}. \quad [\text{Ans. 0.}]$$

$$73. \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\left(\frac{x}{z} + \frac{z}{y} + \frac{y}{x}\right) - \left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{z}{x} + \frac{x}{y}\right)\left(\frac{y}{z} + \frac{z}{x}\right). \quad [\text{Ans. 1.}]$$

$$74. \frac{\left(p + \frac{1}{q}\right)^m \cdot \left(p - \frac{1}{q}\right)^n}{\left(q + \frac{1}{p}\right)^m \cdot \left(q - \frac{1}{p}\right)^n}. \quad [\text{Ans. } \left(\frac{p}{q}\right)^{m+n}.]$$

$$75. \left\{ \frac{(1-a)(y-b)\sqrt{(x^2-1)}}{\sqrt{(b^2-1)}\sqrt{(a+b)(x-y)}} \right\}^0 \cdot \frac{(1+y)(x-1)(x^2+y^2)}{x^4-y^4} \cdot \sqrt{(a^2+2ab+b^2)} \quad [\text{Ans. } (a+b).]$$

$$76. \frac{ac-1}{(a-b)(1+ax)} + \frac{bc-1}{(b-a)(1+bx)}. \quad [\text{Ans. } \frac{c+x}{(1+ax)(1+bx)}.]$$

$$77. \frac{2^{2n+1} - 2^{n+1} + 2}{2^{2n+1} - 2^{n+1}}. \quad [\text{Ans. } \frac{2^n - 1}{2^n}.]$$

$$78. \frac{\sqrt{(ax)}}{\sqrt{a} + \sqrt{a} - \sqrt{(a+x)}} - \frac{\sqrt{(ax)}}{\sqrt{a} + \sqrt{a} - \sqrt{(a+x)}}. \quad [\text{Ans. } \sqrt{(a+x)}.]$$

$$79. \frac{x^3 + 3x^2 + 5x + 15}{x^3 + 2x + 5x + 10} + \frac{x^4 + x^3 + 3x^2 + x - 2}{x^4 + 2x^3 + 3x^2 + 4x - 4}. \quad [\text{Ans. 2.}]$$

$$80. \frac{(a+b)^2}{(x-a)(x+a+b)} - \frac{a+2b+x}{2(x-a)} + \frac{(a-b)x}{x-b-a-a-ab-x}. \quad [\text{Ans. 0.}]$$

$$81. \frac{(a^4-b^4)^2 + 2a^4b^2 + 5a^4b^4 + 2a^2b^6}{(a^4+ab+b^2)^2(a^2-ab+b^2)^2}. \quad [\text{Ans. 1.}]$$

$$82. \frac{a^4 + b^4 + ab(a^2+b^2)}{(a+b)^2} - \frac{a^4+b^4-ab(a^2+b^2)}{(a-b)^2} + \frac{12a-b}{(a-b)-(a-b)^2}. \quad [\text{Ans. } ab]$$

$$83. \left(\frac{ap^2 - aq^2 + 2bpq}{p^2 + q^2}\right)^2 + \left(\frac{bq^2 - bp^2 + 2apq}{p^2 + q^2}\right)^2. \quad [\text{Ans. } a^2 + b^2.]$$

$$84. \frac{xy + 2xz - 3y^2 + 4yz + xz - z^2}{2x^2 - 9x - 5xyz + 4z^2 - 8yz - 12z}. \quad [\text{Ans. } \frac{x-y+z}{x-4y-4z}.]$$

$$85. \frac{b^2 + c^2 - 2a^2}{(a-b)(a-c)} + \frac{c^2 + a^2 - 2b^2}{(b-c)(b-a)} + \frac{a^2 + b^2 - 2c^2}{(c-a)(c-b)}. \quad [\text{Ans. } -3.]$$

$$86. \frac{a^2 - (b-c)^2}{(a-b)(a-c)} + \frac{b^2 - (c-a)^2}{(b-c)(b-a)} + \frac{c^2 - (a-b)^2}{(c-a)(c-b)}. \quad [\text{Ans. 4.}]$$

$$87. \frac{x^2 - (y-z)^2}{(x+z)^2 - 1} + \frac{y^2 - (z-x)^2}{(z+y)^2 - 1} + \frac{z^2 - (x-y)^2}{(x+y)^2 - 1}. \quad [\text{Ans. 1.}]$$

Reduce

$$88. \frac{1}{4a^2(a+x)} + \frac{1}{4a^2(a-x)} + \frac{1}{2a^2(a^2+x^2)} \text{ to the form } \frac{1}{a^2-x^2}$$

$$89. \frac{1}{1+\frac{1}{a+x}} + \frac{1}{1-\frac{1}{a-x}} + \frac{2}{1+\frac{1}{a^2+x^2}} \text{ to the form } \frac{4a^2}{a^4-x^4}; \text{ and show}$$

that the notation $\frac{a}{b}$ is of ambiguous meaning.

Solve the following equations:—

$$> 90. \left(\frac{a^2}{x} + b\right)^{\frac{1}{2}} - \left(\frac{a^2}{x} - b\right)^{\frac{1}{2}} = c^{\frac{1}{2}}. \quad [\text{Ans. } \frac{4a^2c}{4b^2+c^2}]$$

$$91. \frac{\sqrt{x+a}}{(\sqrt{x-b})(\sqrt{x-c})} + \frac{\sqrt{x+b}}{(\sqrt{x-a})(\sqrt{x-c})} + \frac{\sqrt{x+c}}{(\sqrt{x-a})(\sqrt{x-b})} = 0. \quad [\text{Ans. } \frac{1}{2}(a^2+b^2+c^2)]$$

$$92. x - k + \sqrt{k^2 + x^2} = m. \quad [\text{Ans. } \frac{m(m+2k)}{2(m+k)}]$$

$$93. \frac{1}{1-a} - \frac{1}{x-a+c} - \frac{1}{x-b-c} - \frac{1}{x-b} = 0. \quad [\text{Ans. } \frac{1}{2}(a+b)]$$

$$94. \frac{1}{(a-b)(x-a)} - \frac{1}{(c-d)(x-c)} = \frac{1}{(a-b)(x-b)} - \frac{1}{(c-d)(x-d)} \quad [\text{Ans. } \frac{ab-cd}{a+b-c-d}]$$

$$95. 120x - 4[5x - 2\{6x + 7(x-8)\}] = 16 - 4[3x - 2\{x - 6(x-1)\}] \quad [\text{Ans. } 2]$$

$$96. \frac{75-x}{3(x+1)} + \frac{80x+21}{5(3x+2)} = \frac{23}{x+1} + 5. \quad [\text{Ans. } 3\frac{1}{2}]$$

$$97. \frac{a+x}{a^2+a^2x+x^2} + \frac{a-x}{a^2-a^2x+x^2} = \frac{2a^4}{x(a^4+a^2x^2+x^4)} \quad [\text{Ans. } a]$$

$$> 98. \frac{2x+3}{x+1} = \frac{4x+5}{4x+4} + \frac{3x+3}{3x+1} \quad [\text{Ans. } 5]$$

$$> 99. \frac{1}{ab-ax} + \frac{1}{bc-bx} = \frac{1}{ac-ax} \quad [\text{Ans. } \frac{b}{a}(a-b+c)]$$

$$100. \sqrt{bx-a} - \sqrt{ax-b} = (\sqrt{a} - \sqrt{b})\sqrt{x-1}. \quad [\text{Ans. } 0]$$

$$101. \sqrt{\frac{x-a}{x-b}} + \frac{a}{x} = \sqrt{\frac{x-b}{x-a}} + \frac{b}{x} \quad [\text{Ans. } \frac{ab}{a+b}]$$

$$102. \frac{(x-1)(x-2)(x-6)}{(x-3)^3} = 1. \quad [\text{Ans. } 2\frac{1}{2}]$$

ALLAHABAD ENTRANCE.

3. Find the square root of—

$$x + \frac{1}{x} + \sqrt{2} \left(\sqrt{x + \frac{1}{x}} \right) + \frac{1}{2}. \quad [\text{Ans. } \sqrt{x} + \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{2}}.]$$

4. Solve the equations—

$$(i) \quad 69x - \frac{49}{y} = 182\frac{1}{2}, \quad 49x - \frac{69}{y} = 112\frac{1}{2}; \quad [\text{Ans. } 3, 2]$$

$$(ii) \quad 2y + z = 11, \quad 2z + x = 12, \quad 2x + y = 13. \quad [\text{Ans. } 4\frac{1}{2}, 3\frac{1}{2}, 3\frac{1}{2}]$$

5. Three numbers are in the ratios of 2 : 3 : 5, and the sum of their cubes is 4320. Find them. [Ans. 6, 9, 15.]

If four positive numbers are in continued proportion, show that the difference between the extremes is at least three times as great as the difference between the means.

ALLAHABAD ENTRANCE.

1891.

1. Define the following :—"term," "dimension of a term," "homogeneous terms."

2. Express in their simplest forms—

$$(i) \quad \left(1 - \frac{2xy}{x^2 + y^2} \right) \div \left(\frac{x^2 - y^2}{x - y} - 3xy \right). \quad [\text{Ans. } \frac{1}{x^2 + y^2}]$$

$$(ii) \quad (x - y + z)(x + y - z) - (x + y + z)(x - y - z) - 4yz. \quad [\text{Ans. } 0]$$

3. State and prove the two lemmas on which the proof of the rule for finding the G. C. M. depends.

Find the G. C. M. of

$$x^3 - 2ax^2 - 5a^2x - 12a^3 \text{ and } x^3 - 7ax^2 + 13a^2x - 4a^3. \quad [\text{Ans. } x - 4a.]$$

4. Solve—

$$(i) \quad \frac{x - \frac{1}{2}}{x - 1} - \frac{3}{5} \left(\frac{1}{x - 1} - \frac{1}{3} \right) = \frac{23}{10(x - 1)}. \quad [\text{Ans. } 3]$$

$$(ii) \quad (a + b)x + (a - b)y = 2a, \quad (a - b)x + (a + b)y = 2b.$$

$$[\text{Ans. } x = \frac{1}{2} \left(\frac{a}{b} + \frac{b}{a} \right), y = \frac{1}{2} \left(\frac{b}{a} - \frac{a}{b} \right) + 1.]$$

5. A farmer bought equal numbers of two kinds of sheep, one at £3 each, the other at £4 each. Had he expended his money equally in the two kinds, he would have had two more sheep than he did. How many did he buy? [See Ex. 3, page 263.]

6. Find the square root of

$$x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64.$$

$$[\text{Ans. } x^3 - 6x^2 + 12x - 8.]$$

1892.

1. Find the value of $\frac{x+2a}{x-2a} + \frac{x+2b}{x-2b}$, when $x = \frac{4ab}{a+b}$. [Ans. 2.]

2. (a) Find the H. C. F. of $x^3 - x^2 - 8x + 12$ and $3x^2 - 2x - 8$.

(b) Extract the square root of

$$\left(x + \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right). \quad [\text{Ans. (a) } x-2. \text{ (b) } x - \frac{1}{x} - 2.]$$

3. Simplify:—

$$\frac{a(a+1)+1}{(a-b)(a-c)} + \frac{b(b+1)+1}{(b-a)(b-c)} + \frac{c(c+1)+1}{(c-a)(c-b)}. \quad [\text{Ans. 1.}]$$

4. Solve the following equations:—

(i) $\frac{(x+a)(x+b)}{x+a+b} = \frac{(x+c)(x+d)}{x+c+d}$. [Ans. $\frac{ab(c+d)-cd(a+b)}{cd-ab}$.]

(ii) $\frac{m}{x} + \frac{n}{y} = a, \quad \frac{n}{x} + \frac{m}{y} = b$. [Ans. $\frac{m^2 - n^2}{am - bn}, \frac{m^2 - n^2}{bm - an}$.]

5. If $a : b :: c : d$, prove that $\frac{2a+3b}{4a+5b} = \frac{2c+3d}{4c+5d}$.

1893.

1. Simplify:—

(a) $\left(\frac{x-y}{x+y} - \frac{x^3-y}{x^3+y^3}\right) \left(\frac{x+y}{x-y} + \frac{x^3+y^3}{x^3-y^3}\right)$. [Ans. $-\frac{4xy(x^2+y^2)}{x^4+x^2y^2+y^4}$.]

(b) $\frac{2}{b-c} + \frac{b-c}{(c-a)(a-b)} + \frac{2}{c-a} + \frac{c-a}{(a-b)(b-c)} + \frac{2}{a-b} + \frac{a-b}{(b-c)(c-a)}$.

2. Find the H. C. D. of

$$2x^4 - 2x^3 + x^2 + 3x - 6 \text{ and } 4x^4 - 2x^3 + 3x - 9. \quad [\text{Ans. } 2x^2 - 3.]$$

3. Solve:—

(a) $5 + \frac{2}{3 - \frac{1}{4-x}} = \frac{29}{5}$. (b) $\frac{x-a}{b-a} + \frac{x-c}{b-c} = 2$. [Ans. (a) 2; (b) b.]

(c) $\frac{x-a}{c-a} + \frac{y-b}{c-b} = 1, \quad \frac{x+a}{c} + \frac{y-a}{a-b} = \frac{a}{c}$. [Ans. (c) $x=c, y=b$.]

4. If I subtract from the double of my present age the triple of my age 6 years ago, the result is my present age. What is my present age?

[Ans. 9 years.]

5. What is Involution? Find the square root of

$$1 - 4x + 10x^2 - 20x^3 + 25x^4 - 24x^5 + 16x^6. \quad [\text{Ans. } 1 - 2x + 3x^2 - 4x^3.]$$

6. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, each of these ratios = $\left(\frac{pa^3 + qc^3 + re^3}{pb^3 + qd^3 + rf^3}\right)^{\frac{1}{3}}$.

1894.

1. (a) Divide $(x+y)^3 - 8x^3$ by $x+y-2x$.

[Ans. $(x+y)^3 + 2x(x+y) + 4x^2$.

- (b) Shew that $(x+2)(x+3)(x+4)(x+5)+1$ is a perfect square.

2. Resolve into elementary factors

$39x^3 - 7x - 22, x^4 + 2x^2 + 9, a^3 + b^3 + c^3 - 3abc$ and $(a+b-3c)^2 - a - b + 3c$.

[Ans. $(13x-11)(3x+2)$; $(x^2+2x+3)(x^2-2x+3)$;

see page 93; $(a+b-3c)(a+b-3c-1)$.

3. Simplify

$$\left(\frac{ax}{x^2-y^2} - \frac{b}{y-x} - \frac{a}{x+y} \right) \div \left(\frac{ax}{a^2-b^2} - \frac{y}{b-a} - \frac{x}{a+b} \right). \quad [\text{Ans. } \frac{a^2-b^2}{x^2-y^2}.$$

4. Solve :—

(i) $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{x+a+b} + \frac{1}{x}. \quad [\text{Ans. } -\frac{a+b}{2}.$

(ii) $\frac{a+b}{x} - 5b = \frac{a-b}{y} - a, \frac{a}{x} - 2a = \frac{b}{y} - 3b. \quad [\text{Ans. } x = \frac{1}{2}, y = \frac{1}{2}.$

5. A says to B : Two fifths of my salary is $\frac{1}{8}$ of yours, and the difference between our salaries is Rs. 600. What is A's salary ? [Rs. 400.

6. If $a : b = c : d$, prove that $a : a+c = a+b : a+b+c+d$.

1895.

1. Resolve into factors :—

(i) $x^3 + 4x^2 + 4x$. (ii) $x^3 + x^2 - x - 1$. (iii) $a^2b^2 - a^2 - b^2 + 1$.

[Ans. $x(x+2)(x+2)$; $(x+1)^2(x-1)$; $(a+1)(a-1)(b+1)(b-1)$

2. Simplify $\frac{x}{(x-y)(x-z)} + \frac{y}{(y-z)(y-x)} + \frac{z}{(z-x)(z-y)}$. [Ans. 0.

3. Solve :—(i) $\frac{1}{\sqrt{x-1}} + \frac{1}{\sqrt{x-2}} = \frac{3}{\sqrt{x-3}}$. [Ans. 3.

(ii) $(a+b)x + (a-b)y = 2ac, (b+c)x + (b-c)y = 2bc. [\text{Ans. } x=y=c.$

4. If $a : b = b : c$, shew that $a^2 + ab + b^2 : b^2 + bc + c^2 = a : c$.

5. Two sums of money are together equal to £54. 12s., and there are as many pounds in the one as shillings in the other. What are the sums ?

[Ans. £52; 52s.

1896.

1. (a) Factorise (i) $x^{12} - a^{12}$. (ii) $x^4 + 2x^2 + 9$. (iii) $8x^2 + 6x - 27$.

[Ans. $(x-a)(x+a)(x^2+a^2)(x^2-ax+a^2)(x^2+ax+a^2)(x^4-a^2x^2+a^4)$;

$(x+2x+3)(x^2-2x+3)$; $(4x+9)(2x-3)$.

- (b) Find the H. C. F. of $x^3 - 1$ and $x^{10} - 1$. [Ans. $x - 1$.

2. Simplify $\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-c)(b-a)} + \frac{c^2}{(c-a)(c-b)}$, [Ans. $a+b+c$.

3. Solve the following equations:—

(a) $\frac{7x+1}{x-1} = \frac{35}{9} \left(\frac{x+4}{x+2} \right) + \frac{28}{9}$. (b) $\frac{a_1}{x} + \frac{b_1}{y} = c_1$, $\frac{a_2}{x} + \frac{b_2}{y} = c_2$.

[Ans. (a) -107 ; (b) $\frac{a_1b_2 - a_2b_1}{c_1b_2 - c_2b_1}$, $\frac{b_1a_2 - b_2a_1}{c_1a_2 - c_2a_1}$.

4. If $a : b = c : d$, and $p : q = r : s$, prove that

$$ap + cr : bq + ds = \sqrt{acpr} : \sqrt{bdqs}.$$

5. Two towns X and Y , on a railway, are 64 miles apart. Coals at X cost 18s. per ton and at Y 16s. per ton; they cost two pence per ton per mile to carry on the line. Find the distance from X of the place at which it is immaterial to the consumer whether he buys coals from X or from Y . [Ans. 26 miles.

1897.

1. Shew that $(ay - bx)^2 + (bz - cy)^2 - (cx - az)^2 + (ax + by + cz)^2$ is divisible by $a^2 + b^2 + c^2$ and $x^2 + y^2 + z^2$.

2. Find the H. C. F. of $4x^4 - 9x^3 + 6x - 1$ and $6x^3 - 7x^2 + 1$.

[Ans. $(x-1)(2x-1)$.

3. Simplify the expressions:—

(i) $\frac{1}{a^2 - 3b^2 + 2ab} + \frac{1}{b^2 - 3a^2 + 2ab} - \frac{2}{3a^2 + 10ab + 3b^2}$. [Ans. 0.

(ii) $\frac{a^4 + x^4 + ax(a^2 + x^2) + a^2x^2}{a^5 - x^5} \div \frac{a^2 + x^2 + ax}{a^3 - x^3}$. [Ans. 1.

4. A merchant buys goods at 24 guineas the cwt., and by retailing them at 5s. 3d. the lb. makes 10 per cent. more profit than if he had sold the whole for £240. What weight did he buy? [Ans. 1000 lbs.

5. If $a : b :: b : c :: c : d$, prove that $a^2 : d^2 :: a^3 : c^3$.

1898.

1. Find the H. C. F. of $2x^5 - 11x^3 - 9$ and $4x^5 + 11x^4 + 81$.

[Ans. $x^2 + 2x + 3$.

2. Simplify (a) $\frac{m-n}{(x-m)(x-n)} + \frac{n-p}{(x-n)(x-p)} + \frac{p-m}{(x-p)(x-m)}$. [Ans. 0.

(b) $\frac{a^3}{(x-a)^n} + \frac{2a}{(x-a)^{n-1}} + \frac{1}{(x-a)^{n-2}}$. [Ans. $\frac{x^2}{(x-a)^n}$.

3. Solve (a) $2x + \frac{3}{y} = 4$, $3x + \frac{2}{y} = 5$. (b) $\frac{1}{x+5} + \frac{1}{x+10} = \frac{2}{x}$.

[Ans. (a) $1\frac{1}{2}$, $2\frac{1}{3}$; (b) $-6\frac{1}{2}$.

4. If $a : b = c : d$, shew that $a(a+b+c+d) = (a+b)(a+c)$.

5. The number of months in the age of a man, on his birth-day in the year 1875, was exactly half of the number denoting the year in which he was born. In what year was he born? [Ans. 1800.]

1899.

1. Find the difference between $(1+x)^3 + (1+x)^2y + (1+x)y^2 + y^3$ and $3x(x+1) + y(y+1) + 2xy + 1$, and show by what expression this difference must be multiplied that the product may be $y^4 - x^4$.

[Ans. $x^3 + x^2y + xy^2 + y^3$; $y - x$.]

2. Find the H.C.D. of $x^5 - 4x^3 - x^2 + 2x + 2$ and $x^5 - x^3 - 2x + 2$, and find such a value of x as will make both the expressions vanish. [$x = 1$; 1.]

3. Reduce the following expressions into factors :

(i) $x^4 - 10x^2 + 9$; (ii) $a^3(a+b-c)^2 - c^3(b+c-a)^2$.

[Ans. (i) $(x^2 + 2x - 3)(x^2 - 2x - 3)$;

(ii) $\{(a-c)^2 + b(a+c)\}(a-c)(a+b+c)$.

4. Solve the equations :—

(i) $\frac{x+a}{x-a} - \frac{x-b}{x+b} = \frac{2(a+b)}{x}$. [Ans. $\frac{ab}{b-a}$.]

(ii) $\frac{1}{x} + \frac{1}{y} = \frac{5}{6}$, $3x + 2y = 2xy$. [Ans. $x = 2$, $y = 3$.]

5. Simplify $\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-c)(b-a)} + \frac{1}{c(c-a)(c-b)}$. [Ans. $\frac{1}{abc}$.]

6. If $\frac{x}{a} = \frac{y}{b}$, prove that $\frac{x^2 + a^2}{x+a} + \frac{y^2 + b^2}{y+b} = \frac{(x+y)^2 + (a+b)^2}{(x+y) + (a+b)}$.

• 1900.

1. Divide $(1-a^2)(1-b^2)(1-c^2) - (a+bc)(b+ca)(c+ab)$

by $1-a^2-b^2-c^2-2abc$,

and extract the square root of $1+(x+1)(x+2)(x+3)(x+4)$.

[Ans. $1+abc$; $x^2 + 5x + 5$.]

2. Simplify $\frac{(x-a)^2}{(a-b)(a-c)} + \frac{(x-b)^2}{(b-a)(b-c)} + \frac{(x-c)^2}{(c-a)(c-b)}$. [Ans. 1.]

3. The expression $ax - by$ is equal to 10 when $x=2$ and $y=3$, and it is equal to 25 when $x=3$ and $y=2$, a and b being constants; find a and b .

[Ans. $a = 11$, $b = 4$.]

Solve $\frac{(x+a)(x+b)}{(x+c)(x+d)} = \frac{x-c-d}{x-a-b}$. [Ans. $x = \frac{cd(c+d) - ab(a+b)}{(a^2 + ab + b^2) - (c^2 + cd + d^2)}$.]

4. $a : b$, $c : d$, $e : f$, &c. are m equal ratios; prove that each of them is equal to $\sqrt[n]{\frac{pa^n + qc^n + re^n + \dots}{pb^n + qd^n + rf^n + \dots}}$ and to $\sqrt[n]{\frac{ace\dots}{bdf\dots}}$, where n , p , q , r , &c. are any quantities whatever.

5. A's present age is to B's present age as 8 : 7; 27 years ago their ages were as 5 : 4. Find their present ages. [Ans. A, 72; B, 63 yrs.]

1901.

1. (a) Find the H. C. F. of $x^3 - 2x^2 + 1$ and $2x^5 + x^3 + 4x - 7$.
[Ans. $x - 1$.

(b) Extract the square root of

$$(a-b)^4 - 2(a^2 + b^2)(a-b)^2 + 2(a^4 + b^4). \quad [\text{Ans. } a^2 + b^2.]$$

2. Simplify (i) $(a-b+c)^3 + (a+b-c)^3 + 6a\{a^2 - (b-c)^2\}$. [Ans. $8a^3$.

(ii) $\frac{1}{1+x+x^2} - \frac{1}{1-x+x^2} + \frac{2x}{1-x^2+x^4}$. [Ans. $\frac{4x}{1+x^4+x^8}$.

3. Solve the equations:—

(i) $\frac{3x+1}{4} - 2(6-x) = \frac{5x-4}{7} - \frac{x-2}{3}$. [Ans. 5.

(ii) $\frac{2}{x-1} + \frac{3}{y+1} = 2$, $\frac{3}{x-1} + \frac{2}{y+1} = \frac{13}{16}$. [Ans. $-2\frac{1}{2}$, $\frac{1}{16}$.

4. A number has three digits which increase by 1 from left to right. The quotient of the number divided by the sum of the digits is 26. What is the number? [Ans. 234

5. If $\frac{x-1}{x+y} = a$, $\frac{y-z}{y+z} = b$, $\frac{z-x}{z+x} = c$,

$$\text{shew that } (1-a)(1-b)(1-c) = (1+a)(1+b)(1+c)$$

1902.

1. Simplify (i) $\frac{a^5 - a^4b - ab^4 + b^5}{a^4 - a^3b - a^2b^2 + ab^3}$. [Ans. $\frac{a^2 + b}{a}$.

and (ii) $\frac{a^3 + b^3 + c^3 - 3abc}{(a-b)^2 + (b-c)^2 + (c-a)^2}$. [Ans. $\frac{a+b+c}{2}$.

2. Extract the square root of $16x^3(x-2) - 8x(1-3x) + 1$.

[Ans. $4x^2 - 4x + 1$.

3. Solve $\frac{4x+17}{x+4} - \frac{5x+26}{x+7} = \frac{2x+7}{x+3} - \frac{3x+19}{x+6}$. [Ans. -5 .

4. Find x and y from the two equations

$$a(x+y) + b(x-y) = 2a, \quad y(a+b) - x(a-b) = 2b. \quad [\text{Ans. } x=y=1]$$

5. I wished to give a certain number of old men 1 anna 8 pies each, and I found that I had not money enough in my purse by 11 annas; so I gave them 1 anna 5 pies each, and then I had money enough and 3 annas 3 pies to spare. Find the number of old men. [Ans. 57.

6. $a : b = c : d$, prove that

$$a^2b - 3ac^2 : b^3 - 3ad^2 = a^2 + 5c^2 : b^2 + 5d^2.$$

APPENDIX.

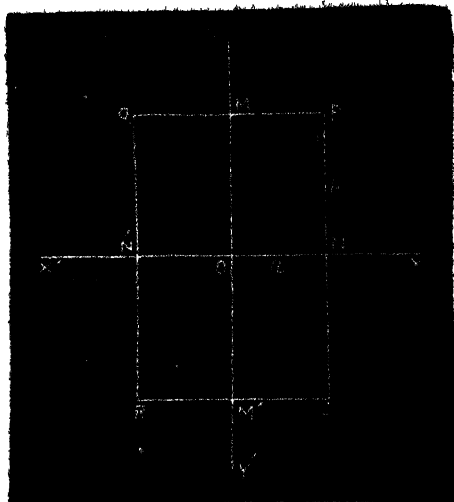
GRAPHS OF SIMPLE EQUATIONS.

1 Co-ordinates Suppose that XOX' and YOY' are two fixed lines in a plane. Let P be any point in this plane.

How are we to know its position ?

Draw PN parallel to YOY' to meet XOX' , and draw PM parallel to XOX' to meet YOY' .

If we know the lengths of the lines PM and PN , we know the position of the point P . For, we have only to measure off ON and OM equal to the given lengths PM and PN , and then to draw NP and MP parallel respectively to OY and OX .



The lengths MP and NP or ON and OM , which define the position of the point P with reference to the lines OX and OY , are called the co-ordinates of the point P , and the lines OX and OY are called the axes of co-ordinates ; O , the point of intersection of the axes is called the origin of co-ordinates.

$ON (=MP)$ is generally called the abscissa, and NP the ordinate of the point P .

It is usual to denote the abscissa by the symbol x , and the ordinate by y . If, in the diagram, $ON = a$ (units of length), and $NP = b$ (units of length), then at the point P , $x = a$, and $y = b$, and the point is briefly denoted as (a, b) .

A convenient scale for the representation of lengths must be first agreed upon. Unless the contrary is stated, the scale units for x and y are the same.

Sometimes, however, it is convenient to have them different.

The axes OX and OY are generally, though not necessarily, chosen at right angles to each other, and we shall henceforth suppose them so chosen.

2. **Convention of signs.** On OX' take ON' equal to ON , and on OY' take OM' equal to OM . Through M' and N' draw the lines SR and QR parallel to OX and OY respectively. $\{$

Since $ON' = ON$, and $OM' = OM$, the co-ordinates of R are equal in magnitude to those of P . Will then the point R be denoted as (a, b) ? The same question will arise with respect to the points Q and S .

How are we to distinguish the points P, Q, R and S from one another by means of their co-ordinates? To do this, we consider all lines measured in the directions OX and OY to be positive, and therefore those in the directions OX' and OY' to be negative.

| | | | |
|---------------|-----------|-----------|--------------|
| Thus at P , | at Q , | at R , | and at S , |
| $x = a,$ | $x = -a,$ | $x = -a,$ | $x = a,$ |
| $y = b;$ | $y = b;$ | $y = -b;$ | $y = -b.$ |

The four regions $XOY, YOX, X'OP$ and $Y'OX$ are spoken of as the first, second, third and fourth quadrants.

It is easy to see that—

| | | | |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| in the 1st quadrant, | in the 2nd, | in the 3rd, | in the 4th. |
| x is +, }
y is + ; } | x is -, }
y is + ; } | x is -, }
y is - ; } | x is +, }
y is - . } |

To find the point $(-2, -3)$, proceed thus :—

On any convenient scale set off along OX' a length ON' equal to 2, and then from N' set off a length $N'R$ in the direction of OY' and equal to 3; R is the required point. It is usual to mark its position by a small dot or cross.

The operation of marking in the diagram described above any point whose co-ordinates are given is called **plotting the point**.

It is useful to notice that the point $(0, 0)$ denotes the origin.

3. **Squared Paper.** It is convenient for graphical representation to use paper ruled twice over by two sets of equidistant parallel straight lines, the one set being perpendicular to the other. Every tenth line is a little thicker, and every fiftieth or every hundredth a little thicker still. The axes of reference are chosen parallel to the ruled lines, so that lengths containing any number of units, tenths and hundredths can be readily set off by merely counting the divisions on the paper. Such paper is usually had at shops where drawing materials are sold.

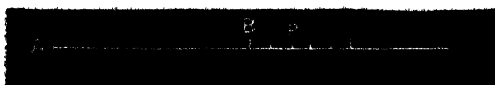
When it is not available, ordinary paper will do, if we are provided with a scale, a set square and a dividing compass.

4. Ready way of estimating lengths

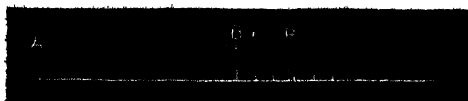
The eye should be trained to readily subdivide any unit of length into tenths without actual measurement.

As *one-half-five-tenths*, fix your eye on the middle point, and mentally divide each half into five equal parts.

Thus, BP is $\cdot 2$ in., and AP is $1\cdot 2$ in.



When AP is more than $1\cdot 2$ in., but less than $1\cdot 3$ in., we may mentally subdivide the tenth in which P falls into ten equal parts, i.e., into *hundredths of an inch*, and judge as nearly as we can how many hundredths are to be added to $1\cdot 2$. In the diagram given below, AP is $1\cdot 27$ in. nearly.



EXAMPLES I.

Plot the following points :—

- | | | | |
|-------------|-------------|-------------|---------------|
| 1. (2, 5). | 2. (-1, 2). | 3. (7, -1). | 4. (-5, -42). |
| 5. (86, 0). | 6. (0, 59). | 7. (-3, 0). | 8. (0, -67). |

5. Definitions. A function of x is any expression involving the variable quantity x , and is denoted by the symbol $f(x)$ [read as—function x].

Thus, $ax + b$, $2x^3 + 3x - 5$, $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$ are functions of x .

If we put $y = a_0x^n + a_1x^{n-1} + \dots + a_n$,

then y is a function of x , and varies in value as x varies. Here x is called the independent variable, and y the dependent variable.

6. Graph of a function.—Let us take any function of x , say $y = x^2 - 2x$. Put $y = x^2 - 2x$.

For different values of x , the values of y , i.e., of the function will in general be different. Putting $x = 1$, we have $y = -1$. Plot the point $(1, -1)$. Give to x other values, and plot the points determined by the simultaneous values of x and y . The line, straight, or curved, which passes through the plotted points is called the graph of the function.

Since x may be given any value, y is also capable of an infinite number of values.

In practice it is not, however, possible to plot all the infinity of points on the graph. It is usual to plot a sufficiently large number of them, and then to draw a curve passing through them either with a free-hand or by means of a flexible strip of metal or wood. This gives a pretty good idea of the graph, at least between the extreme values of x chosen. Of course, the larger the number of plotted points, the more accurate the graph. In drawing the curve $y = x^2 - 2x$, we proceed thus:—

$$\text{when } x = -4, y = (-4)^2 - 2(-4) = 24;$$

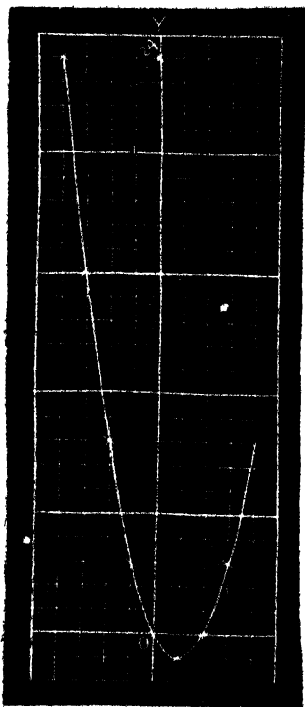
$$,, \quad x = -3, y = (-3)^2 - 2(-3) = 15;$$

and so on.

we thus get the following table of values:

| | | | | | | | | | |
|-----|----|----|----|----|---|----|---|---|---|
| x | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
| y | 24 | 15 | 8 | 3 | 0 | -1 | 0 | 3 | 8 |

Now plot the points $(-4, 24)$, $(-3, 15)$, &c., and draw a curve through them.



7. Importance of graphs. The graph of a function suggests at a glance the leading features of its variation.

Tables of calculated values, as exemplified in the last article, hardly give an idea of the variation as readily as the graph does. As it can be made more or less accurate according to the requirements of practice, it has come extensively into use.

8. Graph of MX . Put $y = mx$. Let us suppose as a particular case that $m = 1$. We then have $y = x$.

Hence

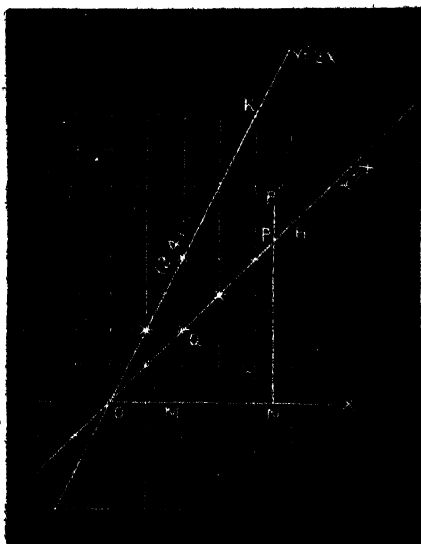
| | | | | | |
|-----|---|---|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 |
| y | 0 | 1 | 2 | 3 | 4 |

Joining successively the points (0, 0), (1, 1), (2, 2) &c., we get a straight line OH. Also we can readily see that every point on this line satisfies the condition $y = x$, i.e., the ordinate of any point on it is equal to the abscissa; for instance, in the diagram $PN = ON$.

This is also geometrically evident. For let one of the plotted points (2, 2) be denoted by Q, and let QM be its ordinate. Then since $QM = OM$ (constr.) OQM is an isosceles right-angled triangle. It is now easy to see that $\triangle OPN$ also must be so.

Conversely, no point other than one lying on the line found satisfies the condition $y = x$.

If possible, let P' do so. Draw $P'N$, the ordinate of P' , so that ON is its abscissa. Then should $P'N = ON$.



Let $P'N$ meet OP in P .

We already have $PN = ON$.

But $P'N = ON$;

Hyp.

$\therefore PN = P'N$, which is impossible.

\therefore the straight line OH , and OH only, is the graph of $y = x$.

Next suppose $m = 2$. We have now $y = 2x$.

Hence

| | | | | | |
|-----|---|---|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 |
| y | 0 | 2 | 4 | 6 | 8 |

Joining successively the points $(0,0)$, $(1,2)$, $(2,4)$, &c., we get a straight line, OK . It will be seen that every point on OK satisfies the condition $y = 2x$, (i.e., the ordinate = double the abscissa), and no point other than one lying on OK satisfies the same condition.*

\therefore the straight line OK , and OK only, is the graph of $y = 2x$.

Similarly, $y = 3x$, $y = 4x$, &c., will be found to give straight lines passing through O .

Hence, more generally,

the graph of mx (or the graph of the simple equation $y = mx$) is a straight line passing through the origin.

EX. 3.—The line is of unlimited length. It is determined when two points on it are known, since no two straight lines can pass through the same two points. Hence to draw the graph, we have simply to join the origin with one other point on it.

A. Graph of $y = 0$ and that of $x = 0$. For any point of the x -axis $y = 0$, whatever x may be.

$\therefore y = 0$ denotes the x -axis.

Similarly, $x = 0$ denotes the y -axis.

It will be readily seen that

(1) $y = b$ is a straight line parallel to the x -axis at a distance b from it.

(2) $x = a$ is a straight line parallel to the y -axis at a distance a from it.

* This can be not only seen by actual measurement, but readily follows from the well-known property of similar triangles.

EX. 1. Draw the graph of $y = -4.5x$, and find the value of y when $x = 3.8$

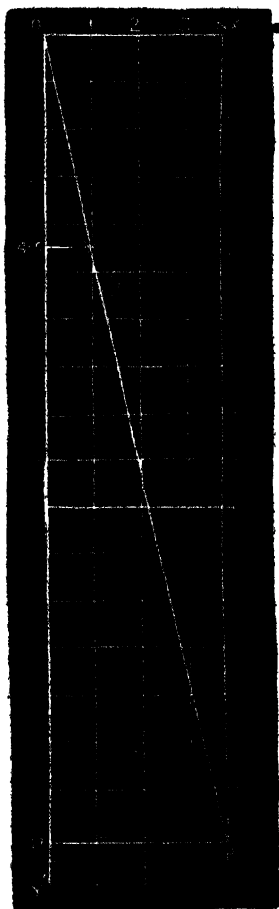
The graph must be a straight line passing through the origin. We have to plot one other point.

Putting $x = 1$, we have $y = -4.5$. Join $(1, -4.5)$ with the origin. The straight line thus obtained is the required graph.

Now set off ON along OX equal to 3.8 . Draw NP parallel to OY to meet the graph. On measuring NP , it is found equal to 17.1 , and is negative.

$\therefore y = -17.1$, when $x = 3.8$.

N.B.—The graph here lies in the 2nd and 4th quadrants.



Ex. 2. Find by graphic method the value of x to 2 places of decimals when $y = 2.34$, being given that $2.74x + 1.23y = 0$.

First find the graph of $2.74x + 1.23y = 0$

It is a straight line passing through the origin, O . [It is readily seen that $y = 0$, when $x = 0$.]

$$\text{We have } \frac{x}{y} = \frac{-1.23}{2.74}$$

Hence when

$$x = -1.23, y = 2.74.$$

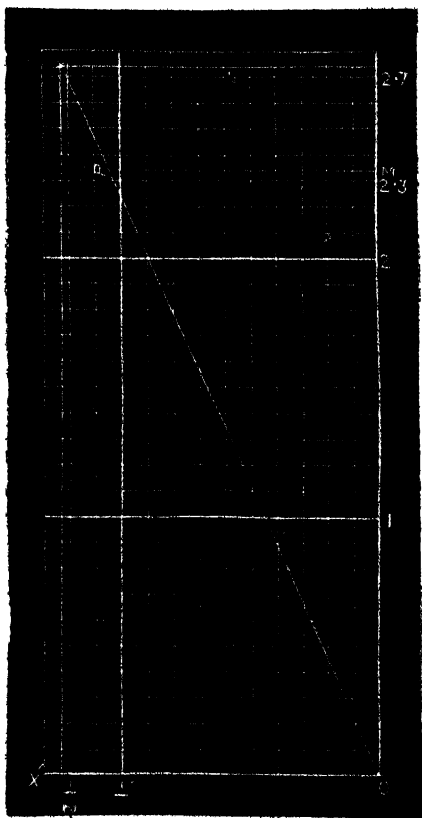
Join the point $(-1.23, 2.74)$ with the origin.

The straight line thus obtained is the graph sought.

In the y -axis set off $ON = 2.34$ in the positive direction.

Through N draw MP parallel to the x -axis to meet the graph at P .

Measure the length of MP , and it is the required value of x . This is -1.06 nearly.
Ans.



The student should find out the result by actual calculation in order to see how the graphic method saves much trouble.

EXAMPLES 2.

Find the graphs of the following :—

1. $5x$. 2. $-2x$. 3. $y=2.7x$. 4. $3y=7.35x$.
 5. $4x+11y=0$. 6. $2.5x+3.7y=0$.

7. Find graphically to 2 places of decimals the value of $2.37x$, when $x=7.49$.

8. Given $1.791x+2.435y=0$, find to 3 places of decimals by graphic method the value of x , when $y=2.087$.

10. Graph of $mx+b$. Draw the graph of $y=mx$.

It is a straight line passing through the origin. Move this line parallel to itself through a distance b , measured parallel to the y -axis. This increases the ordinate of every point on the line $y=mx$ by b . For instance, in the diagram,

$$QN=PN+b$$

But $PN=m \cdot ON$.

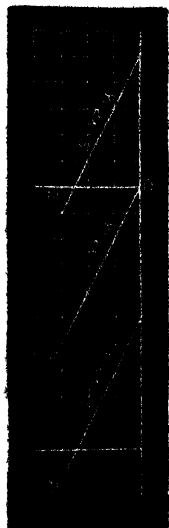
$$\therefore QN=m \cdot ON+b,$$

i.e., new ordinate = m times abscissa + b , or symbolically, $y=mx+b$.

Thus the graph of $y=mx+b$ (or, simply, of the function $mx+b$) is a straight line parallel to the straight line $y=mx$ and at a distance b from the latter measured parallel to the y -axis.

To draw the graph of $y=mx+b$, notice that when $x=0$, $y=b$. Hence the line intersects the y -axis at a point whose distance from the origin is b . We have to plot one more point.

N.B.—The diagram gives the graph when $m=2$, and $b=5$. The graph of $2x-5$ has been added for comparison.



11. Graph of $ax+by+c=0$ The graph is evidently a straight line, for we may put

$$y = -\frac{a}{b}x - \frac{c}{b} = mx + B, \text{ suppose,}$$

which form comes under the last article.

To draw the graph, observe that

$$y=0, \text{ when } x = -\frac{c}{a}, \text{ and } x=0, \text{ when } y = -\frac{c}{b}.$$

Plot the point $(-\frac{c}{a}, 0)$ on the x -axis and the point $(0, -\frac{c}{b})$ on the y -axis, and join them.

N. B.—These points may not always be convenient. More convenient ones have sometimes to be plotted, as will be illustrated by examples.

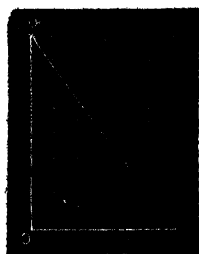
Cor. The graph of any simple equation in x and y is a straight line.

EX. 1. Draw the graph of $4x+3y=24$.

When $y=0$, we have $4x+3 \times 0=24$, and
 $\therefore x=6$;

When $x=0$, we have $4 \times 0+3y=24$, and
 $\therefore y=8$.

Plot the points $(6,0)$ and $(0,8)$; the joining straight line is the graph sought.
 {Draw it}



EX. 2. Draw the graph of the function $\frac{1}{3}(7x+5)$, and find its value when $x=85$. What value of x makes the function equal

$$\text{Ans. } y = \frac{7x+5}{3} = 2x+1+\frac{x-1}{3}. \quad (1)$$

From (1), putting $x=1$, we have $y=4$; and putting $x=-5$, we have $y=5$.

Plot the points (1, 4) and (-5, 5); the joining line is the required graph [Draw it].

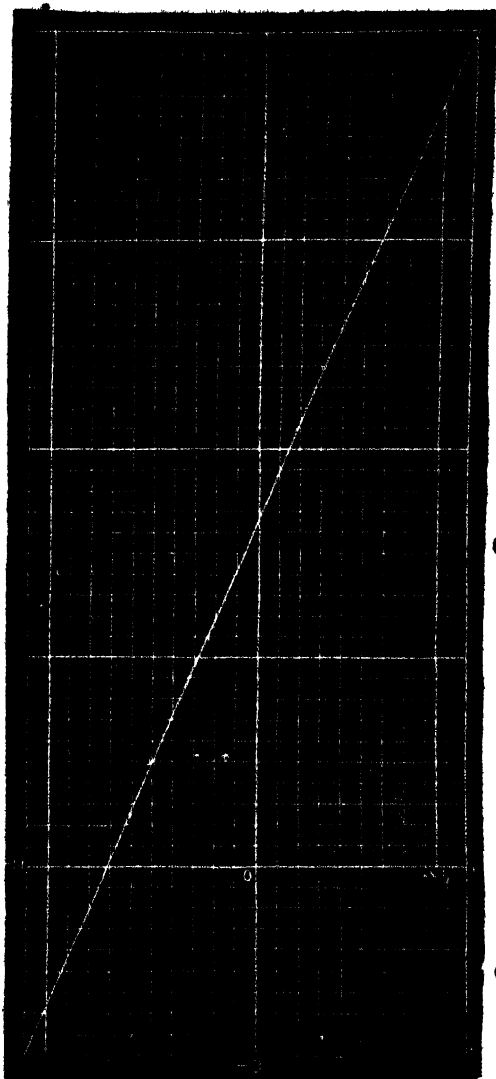
To find the value of the function when $x=85$, set off on the x -axis a length = 85 in the positive direction; at its extremity draw a parallel to the y -axis to meet the graph. The length of this ordinate is the value sought. It will be found to be 3.65.

To find x when the value of the function (i.e., y) is -9, set off on the y -axis a length = 9 in the negative direction (i.e., along OP); at its extremity draw a parallel to the x -axis to meet the graph. The length of this parallel (=abscissa) is the value of x sought.

It will be found to be -11.

N. B.—We have here $3y=7x+5$. If we put $x=0$, we have $3y=5$, and $\therefore y=\frac{5}{3}=1\frac{2}{3}$.

If we put $y=0$, we have $7x+5=0$, and $\therefore x=-\frac{5}{7}=-.71428\frac{5}{7}$.



Hence the points where the line intersects the axes are $(-1428\frac{1}{2}, 0)$ and $(0, 1\frac{1}{6})$. But these cannot be conveniently plotted. See the remarks under Art. 11. The student should note the division performed in (1).

12. Collection of important results.

- (1) The pt. $(0,0)$ is the origin.
- (2) The *Ordinate* of every point on the *x-axis* is zero.
- (3) The *Abcissa* of every point on the *y-axis* is zero.
- (4) The ordinate is + in the 1st. and 2nd quadrants only.
- (5) The abscissa is + in the 1st and 4th quadrants only.
- (6) The graph of every simple equation in x and y is a straight line:
- (7) $y=b$ is a st. line parallel to the x -axis and at a distance b from it; $y=0$ is the x -axis.
- (8) $x=b$ is a st. line parallel to the y -axis and at a distance b from it.
- $x=0$ is the y -axis.

EXAMPLES 3

Draw the graphs of the following equations :

1. $2x+3y=6$. 2. $8x-3y=36$. 3. $3x+5y+7=0$
4. $2x-3y=25$. 5. $x/4+y/5+2=0$. 6. $12x-7y=44$.

Draw the graphs of the following functions :

7. $4x+7$. 8. $\frac{1}{2}(18-8x)$. 9. $1\cdot23x+0\cdot7$.

Find graphically the value of

10. $\frac{1}{2}(15-x)$, when $x=9$.
11. $1+3\cdot25y$, when $y=1\cdot5$.
12. y , when $x=7$, and $5y=2x-3$.
13. y , when $x=1\cdot5$, and $1\cdot2x+1\cdot7y+2\cdot4=0$.
14. Given $x+2\cdot4y=3\cdot5$, for what value of x is $y=1\cdot5$?
15. Find correct to 3 places of decimals the value of x when $y=2\cdot045$, being given that $1\cdot125x=2\cdot01y+2\cdot40$.

Miscellaneous Examples Worked out.

Ex. 1. Solve graphically the equations,

$$2x - 3y = 19, \quad 3x + 4y = 3.$$

First draw the graph of $2x - 3y = 19$.

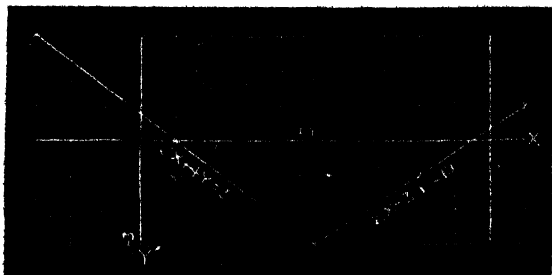
Here, $x = \frac{3y + 19}{2} = y + 9 + \frac{y + 1}{2}$.

$$\therefore \text{when } y = -1, x = 8; \quad \left. \begin{array}{l} \text{and } y = 1, x = 11. \end{array} \right\}$$

Plot these pts. ; the st. line joining them is the graph of $2x - 3y = 19$.

Next draw the graph of $3x + 4y = 3$.

Here $y = \frac{3(1-x)}{4}$.



$$\therefore \text{when } x = 1, y = 0; \quad \left. \begin{array}{l} \text{and } x = -3, y = 3. \end{array} \right\}$$

Plot these pts. ; the joining line is the graph of $3x + 4y = 3$.

The co-ordinates of the common point of the two st. lines must satisfy both the equations.

From the diagram it will be seen that the lines meet at $(5, -3)$.

Hence $x = 5, y = -3$, is the required solution.

N.B.—Problems leading to simultaneous equations can be solved by means of the graphs of those equations.

Ex. 2 Assuming 1 inch = 2.54 centimetres, construct a graph to convert centimetres into inches. Read off the value of 500 centimetres in inches.

We have

$$255 \text{ cm.} = 1 \text{ in.};$$

$$\therefore 255 \text{ cm.} = 100 \text{ in.};$$

$$\therefore 51 \text{ cm.} = 20 \text{ in.}$$

Hence, if generally x cm., $= y$ in.,

$$\frac{x}{51} = \frac{y}{20}$$

Find the graph of this equation on squared paper.

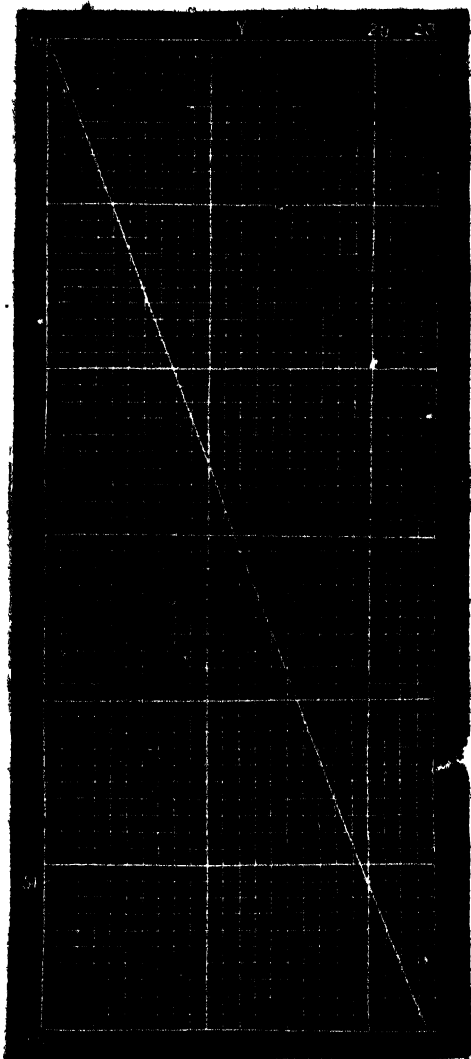
It is a straight line passing through the origin and (51, 20)

Suppose now each division of the x -axis to represent 10 centimetres, and each division of the y -axis to represent 10 inches.

Then 600 cm. = 60 divisions on the x -axis.

Measure off the ordinate of the graph at the 60th division. This is 23.5 nearly.

\therefore the reqd. value)
 $= 23.5 \times 10 \text{ in. nearly}$
 $= 235 \text{ in. nearly.}$



Ex. 3. The census returns of a town show the population to be 20,000 in 1900, 24,000 in 1901, 30,000 in 1905, 34,000 in 1907, 40,000 in 1911, and 44,000 in 1912. Find by a graphic method the population in 1910.

Let the origin represent the year 1900, and the pts. $(1,0)$, $(3,0)$, $(5,0)$, represent respectively the years 1901, 1903, 1905, &c.

Let each division of the y -axis represent 2,000 souls.

Thus x will represent the year after 1900, and y the population in thousands.

We have thus the following table :

| | | | | | | |
|-----|----|----|----|----|----|----|
| x | 0 | 2 | 5 | 7 | 11 | 12 |
| y | 10 | 12 | 15 | 17 | 21 | 22 |

The graph constructed from this table is found to be a st line.

Assuming that there is a fixed law governing the population, we must take it to be the one indicated by the graph.

The y for 1910 is found to be 20.

\therefore the reqd. population = $20 \times 2,000 = 40,000$.

Ex. 4. A shop-keeper finds that there is a linear relation between the daily nett profit and the number of days passed since the opening date of the shop ; on the fifth day there is actually a loss of £3. 10s., but on the fourteenth day there is just a gain of £1. Find the relation referred to. When is there neither profit nor loss ? What will be the profit on the 20th day ?

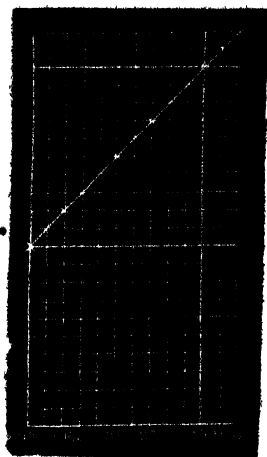
Let y denote the profit generally,
and x " the no. of days passed since the opening date.

Since the profit and the no. of days are linearly connected,
[given]

we must have $y = mx + b$(1)

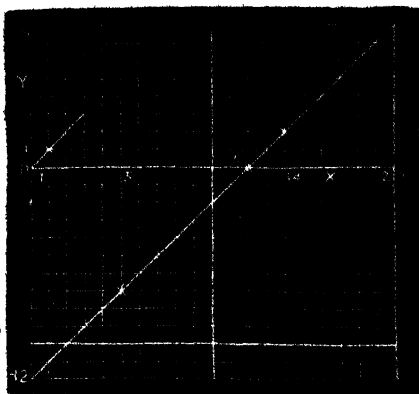
We have now to draw the graph.

Let two divisions on the y -axis represent £1 and one division on the x -axis represent 1 day.



Hence £3. 10s. will be represented by 7 divisions on the y -axis. Considering profit as a positive quantity, and consequently loss as a negative one, we are given that

when $x=5$, $y=-7$,
and when $x=14$, $y=2$.
Plot these pts.; the joining st. line must be the graph of (1).



At the pt. where (1) intersects the y -axis ($x=0$), $y=b$.
We thus find by measurement $b=-12$.

(1) Now stands as $y=mx-12$(2)

To find m , draw a parallel to the graph through the origin.

This is the line $y=mx$, or $\frac{y}{x}=m$.

This line is seen from the diagram to pass through the point (1, 1).

\therefore by substitution, $1=m$.

Hence, finally (2) becomes $y=x-12$. *Ans.*

To find when there is no profit, find where the first graph intersects the x -axis ($y=0$). x is found to be 12.

\therefore there is no profit on the 12th day. *Ans.*

The profit on the 20th day is found to be represented by 8, every two of which by supposition represents £1.

\therefore the profit reqd. = £4. *Ans.*

N.B.—Two quantities are said to be linearly connected with one another when there is an equation involving them in the first degree.

Ex. 5. Two towns, A and B , are 50 miles apart. A cyclist starts from A towards B at the rate of 8 miles per hour, and another from B towards A just at the same time and at the rate of 4 miles per hour. When and where do they meet? How far apart will they be at the end of 3 hours from starting? When are they first at a distance of 5 miles from each other? When are they again at this distance?

Let the distance AB on the squared paper represent 50 miles

If x represents the distance gone through in y hours by the first man, whose rate is 6 miles per hour, we have $x = 6y$,

supposing x to be measured from A .

It represents a st. line passing through A and through the point $(6, 1)$.

Draw the graph AP .

To find the graph of B 's motion, take B to be the origin.

Since B 's rate is 4 miles per hour,

we have to draw the st. line $x = 4y$,

supposing x now to be measured from B along BA . It is easy to see that the st. line passes through B and through $(4, 1)$.

Draw this graph, *vis.*, BP .

PN , the ordinate of the common point of the two graphs, gives the time when they meet, while AN and BN give respectively the distances walked by the two.

Thus they meet at the end of 5 hours, at a distance of 30 miles from A . *Ans.*

To find how far apart they are at the end of 3 hours, take AM on the y -axis = 3; let Q and R be the points where the parallel to the x -axis through M intersects AP and BP .

Then QR or HL represents the distance between the men at the end of the common time (3 hours) represented by the ordinate QH or RL .

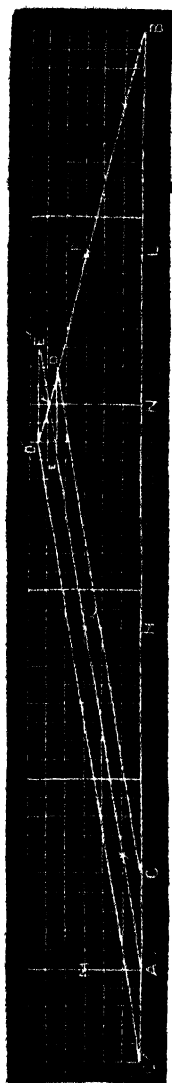
\therefore the reqd. distance = 20 miles. *Ans.*

Next, to find when they are at a distance of 5 miles for the first time.

Along AB take $AC = 5$; through C draw CD parallel to AP to meet BP in D ; through D draw DE parallel to the x -axis to meet AP .

Then $DE = AC = 5$, and the ordinate of D or E gives the required time.

Thus the time reqd = $4\frac{1}{2}$ hours. *Ans.*



Lastly, to find when they are *again* 5 miles apart.

Along BA produced take $AO' = 5$; draw OD' parallel to AP , and $D'E'$ parallel to the x -axis.

Then $D'E' = 5$, the ordinate of D' or E' gives the required time.

Thus, the time sought = $5\frac{1}{2}$ hours.

Ans.

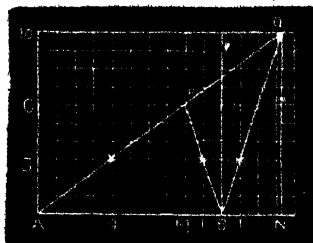
EX. 8: A cistern can be filled by one pipe in $7\frac{1}{2}$ hours, and by another in 30 hours. How long will it take in filling when both are open? What is the answer when the second is an emptying pipe?

Let AB represent the capacity of the cistern.

Let each division of the y -axis denote 1 hour.

If y be the quantity of water poured in in x hours by the 1st pipe,

then $\frac{x}{y} = \frac{AB}{7\frac{1}{2}} = \frac{10}{7\frac{1}{2}}$, representing AB by 10, $-\frac{1}{3}$.



Draw the graph AP , passing through $(0, 0)$ and $(4, 3)$.

Similarly, for the 2nd pipe, taking B as origin and BA as the positive direction of the x -axis, draw the graph, BQ , of

$$\frac{x}{y} = \frac{10}{30} = \frac{1}{3}.$$

The ordinate of the common pt P gives the time in which the cistern is filled, the first pipe supplying a quantity equal to AM , and the second supplying a quantity equal to BM .

This time is from the figure = 6 hours.

Ans.

If the second be an emptying pipe, then with B as origin and AB produced as the positive direction of the x -axis draw the graph, BQ , of $\frac{x}{y} = \frac{1}{3}$.

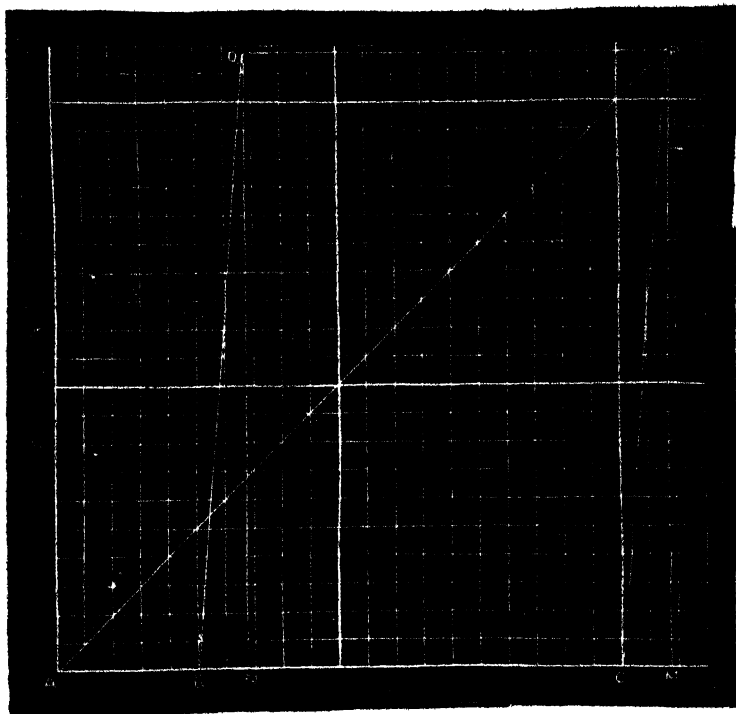
The ordinate QN of the common pt. of AP and BQ gives the reqd. time, the first pipe pouring in a quantity AN , and the second letting out BN , so that the result is $AN - BN = AB$.

From the figure, the time sought = 10 hours.

Ans.

N.B. The solution of problems on Work is similar to the above.

Ex. 7. Find the time between 1 and 2 o'clock when the hour and minute hands are at right angles to one another for the first time.



Let $AB (= 5 \text{ minute-spaces})$ denote the distance between the hands at 1 O'clock.

Let each division on the x -axis represent a *one minute space*, and that on the y -axis *one minute of time*.

Let y denote the time in which x minute-spaces are gone over by a hand.

Since in 60 minutes the minute-hand goes over 60 minute-spaces, we have for its motion' $\frac{y}{x} = \frac{60}{60} = 1$;

$$\text{i.e., } y = x.$$

With A as origin draw the graph, AP .

To represent the motion of the hour-hand take B as origin and AB produced as axis. Observe that the hour-hand passes 5 minute-spaces in 60 minutes of time.

We have then the equation $\frac{y}{x} = \frac{60}{5} = 12$.

Draw the graph, BQ .

Take BC equal to 15 minute-spaces

Draw CP parallel to BQ to meet AP .

The ordinate of P gives the time when the hands are at rt. \angle s to one another.

For, draw PQ parallel to AC . Draw the ordinates PM and QN . In time PM the minute-hand gets separated from the 12 o'clock mark (i.e., A) by AM , and in time QN ($= PM$) the hour-hand moves over space BN , so that it is separated from the same mark by $AB + BN$ or AN .

$$\begin{aligned}\text{Hence the space between the hands} &= AM - AN \\ &= MN = PQ = BC \\ &= 15 \text{ minute-spaces.}\end{aligned}$$

\therefore the hands are at rt. \angle s to each other.

From the figure $PM = 22$ nearly

\therefore approximate time = 22 minutes past one. *Ans.*

Ex. 8. An examiner has marked a set of papers; the highest number of marks is 80, the lowest 20. He wishes to alter the marks, making 100 the maximum and 10 the minimum; show how this may be done graphically, and read off the altered mark corresponding to the old mark 60, and the old mark corresponding to the altered mark 25.

The dif. between any boy and the last
the whole range of marks

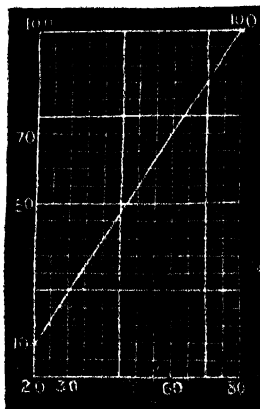
must be the same in the two scales.

Let the marks x in the old scale correspond to the marks y in the new scale.

Then, from the above principle,

$$\frac{x-20}{80-20} = \frac{y-10}{100-10},$$

$$\text{i.e., } \frac{x-20}{60} = \frac{y-10}{90}.$$



This is the equation of a straight line, evidently passing through the points (20, 10) and (80, 100).

From the graph, we easily find

60 old marks = 70 altered marks,

25 altered marks = 30 old marks.

MISCELLANEOUS EXAMPLES.

1. $x + y = 13$, 2. $2x + 3y = 36$, 3. $\frac{1}{2}(x-3) = \frac{1}{3}(y-7)$,

$x - y = 7$. $3x + y = 33$. $11x = 13y$.

4. $2x + 1.3 = 0$, 5. $x/2 + y/3 = 8$, 6. $1.29x - 2.43y$
 $x + 2y = 35$, $x/3 + y/5 = 5$. $= 4.45y - 2.15x = 3$.

7. Find approximately to two places of decimals the values of x and y from the following equations :—

$$x + 2.37y = 4.74, \quad 1.3x - 2y = 1.43.$$

8. Find two numbers such that twice one of them plus thrice the other will yield 54, and thrice the first plus twice the second number will yield 51.

9. 4 lbs. of tea and 3 lbs. of coffee together cost 19s. 6d., and 3 lbs. of tea and 2 lbs. of coffee together cost 13s. 6d.; find the prices per lb. of tea and coffee.

10. The expression $ax - 5b$ is equal to 3 when x is 7, and to 27 when x is 13; what is its value when x is 9, and for what value of x is it zero?

11. Given that one kilometre = $\frac{5}{8}$ of a mile, construct a graph to convert miles into kilometres and kilometres into miles. How many kilometres are there in 210 miles? And how many miles, in 840 kilometres?

12. Given that 1 centimetre = $\frac{1}{2.54}$ inches, find the value of 2.8 centimetres in inches and that of 4.5 inches in centimetres as accurately as you can.

13. Given that 1 kilogramme = 2.2 lbs., find by a graph the values of 15 and 27 kilogrammes in lbs., and of 20.5 and 38 lbs. in kilogrammes.

14. Given that 1 cubic inch = 16.4 cubic centimetres, construct a graph to convert cubic centimetres into cubic inches; find the values of 40 and 30 cubic centimetres in cubic inches and of 1.5 and 3.2 cubic inches in cubic centimetres.

15. A man starts business and is worth Rs. 400 + 30π at the end of π years. What is he worth at the end of 2.5 and 4 years? Find the time when his value is Rs. 550.

16. In the five years from 1801 to 1805 the population of one town increases from 38200 to 50200, and in the five years from 1800 to 1804 the population of another town decreases from 51300 to 33300, supposing the rates of increase and decrease to be uniform; find the year and month when the two towns had the same number of souls. Find also this number.

17. A labourer is paid Rs. 100 a year; draw a graph giving his weekly wages to the nearest anna; determine his wages for 25 weeks.

18. 20 articles cost 2s. 8d.; find graphically the cost of 13 articles to the nearest half-penny, and the number of articles for 3s. 4d.

19. A farmer sold to one person 9 horses and 7 cows for Rs. 3000, and to another 6 horses and 13 cows at the same prices and for the same sum; what was the price of a horse and of a cow?

20. The first thousand copies of a book cost £3 to print, but every one thousand in excess of the first cost 12s. Draw a graph showing the cost of any number up to 6000, and read off the cost of 4600 copies. How many copies do you get for £4. 7s.?

21. The initial cost of starting a manufacture is £100 and the subsequent cost of each article produced is 4s. Draw a diagram on the scale of one inch to one hundred copies and one inch to £10, showing the total cost of any number up to 800. Read off the cost of 240 articles, and the number of articles costing £224.

22. Plot the points (2, 3), (4, 6), (6, 9), (8, 12), (10, 15), and find the equation of the graph.

23. Plot the following points, and deduce the equation of the graph:

| | | | | | |
|-----|----|----|----|----|----|
| x | -5 | 1 | 15 | 18 | 23 |
| y | -8 | -2 | 0 | 12 | 28 |

when is $x=y$?

24. The income of a man was Rs 3000 in the year 1800, Rs 3400 in 1802, Rs. 4000 in 1805, Rs. 5000 in 1810. What was his income in 1803?

25. A can do in 20 days a piece of work which B can do in 30 days; how long would they take to do it working together?

26. A and B together can reap a field in 4 days, while B alone can reap it in 6 days; in what time can A alone reap it?

27. A cistern is filled by two pipes in 20 and 30 minutes respectively; how long will it be in filling, when both are open? What would be the answer if the last pipe were a waste-pipe.

28. Two pipes P and Q fill a cistern in 25 and 30 minutes respectively. Both pipes being opened, find when the first must be closed that the cistern may be just filled in 15 minutes.

29. A train going 20 miles an hour leaves Bristol for London, and another going 30 miles per hour leaves London for Bristol at the same time; when and where will they meet? [London to Bristol is 150 miles] How far apart are they at the end of two hours? When are they first at a distance of 25 miles from each other? When are they again at the same distance?

30. A mail train leaves Howrah for Jubbulpore at 10.30 A.M., and travels 30 miles per hour; an express leaves Howrah for Jubbulpore at noon and travels 40 miles per hour; when and where will it overtake the mail?

31. A starts on a cycle at the rate of 500 yds in 6 secs, and B starts from the same place 1.5 secs later at the rate of 100 yds. per sec. Find by a graph when and where they meet?

32. Find the time when the hour and minute hands of a watch are exactly in the same direction between 9 and 10 o'clock.

33. Find the time when the hour and minute-hands of a watch are exactly opposite each other, between 7 and 8 o'clock.

34. When are the hour and minute-hands of a watch at right angles to one another, between 8 and 9 o'clock.

35. The relation between two variable quantities P and Q is given by $P = lQ + m$; when Q is 3, P is 10.9, and when Q is 2, P is 8.4. Find the values of l and m .

36. The temperature of a mixture rises in proportion to the number of hours it is exposed to the sun. On 3 hours exposure it is found to be 36 degrees, and in another two hours it is found to be 40 degrees. Show by a graph how to determine the temperature at any time? What was the initial temperature? What is the equation connecting the temperature and the time of exposure?

37. In a class examination the top boy gets 72 marks, and the last boy 24. You are to scale these, so that the top boy gets 80, and the last boy zero. Draw a graph for this purpose and read off as accurately as you can the new marks of boys who get 28, 33 and 40. What mark remains unaltered.

38. If n denote the number of guests in a hotel in a day, rupees e the total daily expenditure, and rupees r the total daily receipts, the following numbers are obtained from the entries on several days in the books of the proprietor.

| n | 2 | 3 | 5 | 6 |
|-----|------|------|------|------|
| e | 5.5 | 10 | 19 | 23.5 |
| r | 10.6 | 14.8 | 23.2 | 27.4 |

What law seems to connect s with r and the profit? find s , r and the profit on the day when he has 11 guests? how many guests just bring in no profit?

39. 0° and 100° of the Centigrade thermometer correspond respectively to 32° and 212° of Fahrenheit. Construct a graph to convert degrees Fahrenheit into degrees Centigrade, and vice versa. Read off the Fahrenheit equivalent of 10° Centigrade and the Centigrade equivalent of 86° Fahrenheit. What reading coincides in the two scales?

40. 0° and 80° of the Reaumur scale correspond respectively to 0° and 100° of the Centigrade scale; construct a graph which will enable you to convert R degrees into C degrees, and vice versa. Read off $36^{\circ} R$. in C , and $22.5^{\circ} C$. in R . degrees.

ANSWERS TO THE EXAMPLES ON GRAPHS.

Examples 2. Page 9.

7. 17.75.

8. 2837.

Examples 3. Page 12.

10. 1.5.

11. 5.9 nearly.

12. 22.

13. -2.5 nearly.

14. -1.

15. 3.867.

Miscellaneous Examples Pages 21.

1. $x=10, y=3$.

2. $x=9, y=6$.

3. $x=13, y=11$.

4. $x=-6.5, y=5$.

5. $x=6, y=15$.

6. $x=4, y=2$.

7. $x=1.2, y=1.44$.

8. 9, 12.

9. Tea, 1s. 6d; coffee, 4s. 6d.

10. 1.625. 11. 339 nearly; 52 nearly. 12. 1.09 in nearly;

11.5 cent nearly. 13. 33 lbs, 59, 1bs.; 9.3 kilo, 17.3 kilo. nearly.

14. 2.4, 1.8 nearly; 24.6, 52.5 nearly.

15. Rs. 475, Rs. 520; at the end of 5 years.

16. End of July, 1802; 42,000

17. Re 1.15as, Rs. 48 1a, nearly.

18. 1.8 1/2d. nearly; 25.

19. Horse, Rs. 240; cow, Rs. 120.

20. £5 3s. nearly; 3250.

21. £148; 620.

22. $3x=2y$.

23. $4x-6; x=y-2$.

24. Rs. 3600.

25. 12 days.

26. 12 days

27. 12 min.; 1 hour.

28. After 12 1/2 min.

29. 3 hours after start, 60 miles from Bristol; 50 mi; at the end of 2 1/2 hours; at the end of 3 1/2 hours.

30. 4.30 P. M., 180 miles from Howrah.

31. 9 secs after start; 750 yds from the starting point.

32. 9h. 49 min nearly.

33. 7h. 5.5 min. nearly.

34. 8h. 27 1/2 min nearly, and 9h.

35. $l=25, m=34$.

36. $30^\circ; y=2.30$.

37. 6.7 nearly, 15, 26.7 nearly; 60.

38. $e=4.5; r=4.2n+2.2; p=5.7-3n; e=46, r=48.4,$

$p=2; 19.$

39. $50^\circ F, 10^\circ C.$

40. $45^\circ C, 18^\circ R.$

THE END.

